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Virtual character modules of semisimple Lie groups and representations of Weyl groups

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Introduction.

Let G be a connected semisimple Lie group with finite centre and $\mathfrak{g}$ its Lie algebra. We call G acceptable if there exists a connected complex Lie group G_C with Lie algebra $\mathfrak{g}_C=\mathfrak{g}\otimes_R C$ which has the following two properties. (1) The canonical injection from $\mathfrak{g}$ into $\mathfrak{g}_C$ can be lifted up to a homomorphism of G into G_C. (2) For a Cartan subalgebra $\mathfrak{h}_C$ of $\mathfrak{g}_C$, let ρ be half the sum of positive roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$. Then $\xi_\rho(\exp X)=\exp(\rho(X))$ $(X\in\mathfrak{h}_C)$ defines a character of H_C into C^*.

We assume that G is acceptable throughout this paper.

For an irreducible quasi-simple representation π of G, we can associate π with an infinitesimal character $\lambda\in\mathfrak{h}_C^*$, where $\mathfrak{h}_C^*$ is the complex dual of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Also a distribution character $\Theta(\pi)$ of an irreducible quasi-simple representation π can be defined. We call $\Theta(\pi)$ an irreducible character of π which has an infinitesimal character λ. Let $V(\lambda)$ be the virtual character module of G whose element has an infinitesimal character λ.

In many papers, representations of the Weyl group $W=W(\mathfrak{h}_C)$ on the space $V(\lambda)$ are considered under the assumption that λ is regular and integral for G_C, i.e., λ is regular and is a differential of a character of H_C. G. Lustig and D. Vogan [**15**] considered W-module structure of $V(\lambda)$, using so-called "Springer representations". G. Zuckerman [**12**] also defined a representation of W on $V(\lambda)$, taking advantage of tensor products with finite dimensional representations of G. After his work, D. Barbasch and D. Vogan [**1**] restated his definition of the representation of W by means of "coherent continuation" and determined the W-module structure in the case that G is a connected reductive group with all the Cartan subgroups connected and that G has a compact Cartan subgroup. On the other hand, representations of the Weyl group W on the space of so-called Goldie rank polynomials are considered by A. Joseph [**10**], D. R. King [**11**] and others. It seems that these representations on the space of Goldie rank polynomials or the character polynomials can be realized as subrepresentations of the representation on a virtual character module $V(\lambda)$.

If λ is not integral for G_C, the above definitions of representations of the Weyl group W do not work. But similar representations on $V(\lambda)$ are not defined, so far as we know. In this paper, we assume λ to be regular and define representations of "*integral Weyl groups for* λ" as explained below. If λ is not integral for G_C, the full Weyl group W cannot act on $V(\lambda)$. So we choose a certain subgroup $W_H(\lambda)$ of $W \cong W(\mathfrak{h}_C)$ for each Cartan subgroup H of G and also choose a suitable subspace $V_H(\lambda)$ of $V(\lambda)$. We can define $W_H(\lambda)$-module structure of $V_H(\lambda)$, using the results of T. Hirai [**6, 7, 8**]. We believe $W_H(\lambda)$ is the most natural among the subgroups of W which act on $V(\lambda)$, and call it an integral Weyl group for λ. In the case that λ is regular and integral for G_C, our representations are canonically identified with Zuckerman's one. Roughly speaking, this is a consequence of the fact that Zuckerman's representation and Hirai's method $\boldsymbol{T}$ are "commutative" (see Theorem 4.3). Since we know the precise structure of the space of invariant eigendistributions (IEDs) due to T. Hirai, we can clarify the $W_H(\lambda)$-module structure of $V_H(\lambda)$ (Theorem 5.1). If λ is regular and integral for G_C, a generalization of the result in [**1**] is obtained as a corollary of Theorem 5.1 (Theorem 5.2).

We remark here that the results in this paper remain valid for a connected reductive group whose semisimple part has finite centre.

We now describe the contents of this paper, explaining each section briefly. In §§1 and 2, we state some main results of T. Hirai [**6, 7, 8**] about IEDs on G for the sake of self-containedness. In §1, we clarify the structure of $V(\lambda)$ and define $W_H(\lambda)$ for each Cartan subgroup H of G. §2 is devoted to explaining Hirai's method $\boldsymbol{T}$ constructing IEDs. The definition of representations of the integral Weyl groups $W_H(\lambda)$ on $V_H(\lambda)$ is given in §3 (Definition 3.1). This definition looks very natural and when λ is integral for G_C, it is essentially the same as Zuckerman's definition (Corollary to Theorem 4.3). We prove this in §4. In §5, we clarify the $W_H(\lambda)$-module structure of $V_H(\lambda)$ (Theorem 5.1). If λ is integral for G_C, we get a generalization of the result in [**1**] without any additional assumption on G (Theorem 5.2). In the last section §6, we describe some interesting examples for the groups $U(n, 1)$ and $SL(2, \boldsymbol{R})$.

Main results of this paper have been reported in [**17**].

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§1. Preliminaries on virtual character modules.

1.1. Basic definitions. Let G be a connected semisimple Lie group with finite centre and $\mathfrak{g}$ its Lie algebra. We always denote the Lie algebra of a Lie group H by corresponding German small letter $\mathfrak{h}$, and its complexification by $\mathfrak{h}_C$. We call G acceptable if there exists a complex Lie group G_C with Lie algebra $\mathfrak{g}_C$ which has the following two properties. (1) The canonical injection from $\mathfrak{g}$ into $\mathfrak{g}_C$ can be lifted up to a homomorphism j of G into G_C. (2) For a Cartan subalgebra $\mathfrak{h}_C$ of $\mathfrak{g}_C$, let ρ be half the sum of positive roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$. Then $\xi_\rho = \exp \rho$ is a well-defined character of $H_C = \exp \mathfrak{h}_C$ into $\boldsymbol{C}^*$. We assume G acceptable throughout this paper and fix a group G_C in the following.

Choose a Cartan subgroup H of G. By H_C we denote the analytic subgroup of G_C corresponding to $\mathfrak{h}_C$. Let $\Delta = \Delta(\mathfrak{g}_C, \mathfrak{h}_C)$ be the root system and $W = W(\mathfrak{h}_C)$ the Weyl group of $(\mathfrak{g}_C, \mathfrak{h}_C)$. We fix an order on Δ and write Δ^+ for the set of positive roots with respect to this order and Π for the simple system in Δ^+. Moreover, we define real roots Δ^R and imaginary roots Δ^I as follows.

$$\Delta^R = \{\alpha \in \Delta \mid \alpha \text{ takes real values on } \mathfrak{h}\},$$

$$\Delta^I = \{\alpha \in \Delta \mid \alpha \text{ takes purely imaginary values on } \mathfrak{h}\}.$$

Here we give a brief survey of admissible representations and give some definitions. Let $G = KAN$ be an Iwasawa decomposition, where K is a maximal compact subgroup of G.

DEFINITION 1.1. If $(\mathfrak{g}_C, K)$-module V satisfies the following conditions 0)–3), we call V *admissible.*

0) Every vector $v \in V$ is K-smooth and generates a finite dimensional K-stable subspace.

1) The representation of $\mathfrak{k} \subset \mathfrak{g}_C$ and the differential of that of K are compatible, i.e.,

$$\lim_{t \to 0} \frac{1}{t}(\exp(tX)v - v) = Xv \qquad \text{for} \quad v \in V,\ X \in \mathfrak{k}.$$

2) The adjoint representation of K on $\mathfrak{g}_C$ is compatible with $(\mathfrak{g}_C, K)$-module structure, i.e.,

$$(\mathrm{Ad}(k)X)v = k^{-1}(X(kv)) \qquad \text{for} \quad k \in K,\ X \in \mathfrak{g}_C,\ v \in V.$$

3) The multiplicity of any irreducible representation of K in V is finite.

Let π be a quasi-simple irreducible representation of G on a Hilbert space $\mathfrak{H}$ and $\mathfrak{H}_K$ the space of K-finite vectors. Any element of $\mathfrak{H}_K$ is differentiable and $\mathfrak{H}_K$ forms a $\mathfrak{g}_C$-invariant space. Thus we get the differential $(d\pi, \mathfrak{H}_K)$ of the representation π and $(d\pi, \mathfrak{H}_K)$ is an irreducible admissible $(\mathfrak{g}_C, K)$-module. Conversely, if an irreducible admissible $(\mathfrak{g}_C, K)$-module V is given, there exists a quasi-simple irreducible representation π of G on a Hilbert space $\mathfrak{H}$ such that $(d\pi, \mathfrak{H}_K)$ is isomorphic to V (see, for example, [**13**]). If two irreducible quasi-simple representations $(\pi_1, \mathfrak{H}_1)$ and $(\pi_2, \mathfrak{H}_2)$ give equivalent $(\mathfrak{g}_C, K)$-modules, then we have $\Theta(\pi_1) = \Theta(\pi_2)$, where $\Theta(\pi_i)$ $(i=1, 2)$ is the distribution character of π_i. As we consider the virtual character module, we identify irreducible quasi-simple representations of G with irreducible admissible representations of $(\mathfrak{g}_C, K)$ and sometimes we say irreducible admissible representations of G instead of $(\mathfrak{g}_C, K)$.

Let V be an irreducible $(\mathfrak{g}_C, K)$-module. An element of the centre $\mathfrak{Z}$ of $U(\mathfrak{g}_C)$ acts as a scalar operator on V, so we can define $\lambda \in \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{Z}, \boldsymbol{C})$ by the following equation

$$zv=\lambda(z)v \qquad (z \in \mathfrak{Z},\ v \in V).$$

We call this λ the *infinitesimal character* of V.

Put $\mathfrak{n}^+=\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$ and $\mathfrak{n}^-=\sum_{\alpha\in\Delta^+}\mathfrak{g}_{-\alpha}$, where $\mathfrak{g}_\alpha$ is the root space of α. Then by Poincaré-Birkhoff-Witt theorem,

$$U(\mathfrak{g}_C)=U(\mathfrak{h}_C)\oplus(\mathfrak{n}^+U(\mathfrak{g}_C)+U(\mathfrak{g}_C)\mathfrak{n}^-).$$

Let η be the projection from $U(\mathfrak{g}_C)$ to $U(\mathfrak{h}_C)$ with respect to the above decomposition. Since $\mathfrak{h}_C$ is abelian, we can canonically identify $U(\mathfrak{h}_C)$ with $S(\mathfrak{h}_C)$, the symmetric algebra of $\mathfrak{h}_C$. We define a linear map $\Gamma_\rho : U(\mathfrak{h}_C)\to U(\mathfrak{h}_C)$ by

$$\Gamma_\rho f(\lambda)=f(\lambda-\rho) \qquad \text{for} \quad \lambda\in\mathfrak{h}_C^*,$$

where we consider $f\in U(\mathfrak{h}_C)$ as a polynomial function on $\mathfrak{h}_C^*$, i.e., an element of $S(\mathfrak{h}_C)$.

THEOREM 1.2 ([**18**, p. 168]). (1) *The centre of $U(\mathfrak{g}_C)$ is isomorphic to $U(\mathfrak{h}_C)^W$ as an algebra. An isomorphism between $\mathfrak{Z}$ and $U(\mathfrak{h}_C)^W$ is given by $\Gamma_\rho\circ\eta : \mathfrak{Z}\to U(\mathfrak{h}_C)^W$.*

(2) *The set of algebra homomorphisms from $\mathfrak{Z}$ to $\boldsymbol{C}$ and the set of equivalence classes of $\mathfrak{h}_C^*$ with W-action can be identified by $\Gamma_\rho\circ\eta$, so-called Harish-Chandra map:*

$$\mathrm{Hom}_{\mathrm{alg}}(\mathfrak{Z}, \boldsymbol{C})\cong\mathrm{Hom}_{\mathrm{alg}}(U(\mathfrak{h}_C)^W, \boldsymbol{C})\cong\mathfrak{h}_C^*/W.$$

By the above theorem, we consider λ as an element of $\mathfrak{h}_C^*$. Assume that V is irreducible and has infinitesimal character λ. Denote the distribution character of V by $\Theta(V)$. Then $\Theta(V)$ can be expressed on a Cartan subgroup H as follows. Define the subset $H'(\boldsymbol{R})$ of H and the function $D(h)$ on H as

$$H'(\boldsymbol{R})=\{h\in H \mid \xi_\alpha(h)\neq 1 \text{ for any } \alpha\in\Delta^R\},$$
$$D(h)=\xi_\rho(h)\prod_{\alpha\in\Delta^+}(1-\xi_{-\alpha}(h)) \qquad (h\in H),$$

where ξ_α is a character of H defined by the equation $\mathrm{Ad}(h)X_\alpha=\xi_\alpha(h)X_\alpha$ (X_α is a non-zero root vector for α). For $h\exp X\in H'(\boldsymbol{R})$ ($h\in H$, $X\in\mathfrak{h}$), we have

$$\Theta(V)(h\exp X)=\frac{1}{D(h\exp X)}\sum_{s\in W}c(V, s\,; h)\exp s\lambda(X),$$

where $c(V, s; h)$ is a locally constant function on $H'(\boldsymbol{R})$. Of course the function $c(V, s; h)$ depends on the Cartan subgroup H and the order of Δ. In the next subsection 1.2, we write $\Theta(V)$ more explicitly after T. Hirai in the case that λ is regular.

Let Car(G) be the set of conjugacy classes of Cartan subgroups of G under inner automorphisms of G. We define a natural order on Car(G) as follows. Take $[A]\in$Car(G), where $[A]$ means the conjugacy class of a Cartan subgroup A. For $\alpha\in\Delta^R=\Delta^R(\mathfrak{g}_C, \mathfrak{a}_C)$, let H_α be the element of $\mathfrak{a}_C$ for which $\alpha(X)=B(H_\alpha, X)$, where $B(,)$ denotes the Killing form on $\mathfrak{g}_C$. Take root vectors X_α, $X_{-\alpha}$ from $\mathfrak{g}_C$ in such a way that $[X_\alpha, X_{-\alpha}]=H_\alpha$, and we put

$$H'_\alpha=\frac{2}{|\alpha|^2}H_\alpha, \qquad X'_{\pm\alpha}=\frac{\sqrt{2}}{|\alpha|}X_{\pm\alpha}.$$

Let $\nu=\nu_\alpha$ be the automorphism of $\mathfrak{g}_C$ defined by

$$\nu=\nu_\alpha=\exp\left\{-\sqrt{-1}\frac{\pi}{4}\mathrm{ad}(X'_\alpha+X'_{-\alpha})\right\},$$

so-called *Cayley transform* with respect to α. Then $\mathfrak{b}=\nu(\mathfrak{a}_C)\cap\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ not conjugate to $\mathfrak{a}$ under any automorphism of $\mathfrak{g}$, and $\beta=\nu(\alpha)$ is a singular imaginary root (see [7, p. 31]) of $\mathfrak{b}$. We have

$$\mathfrak{a}=\sigma_\alpha+\boldsymbol{R}H'_\alpha, \qquad \mathfrak{b}=\sigma_\alpha+\boldsymbol{R}H'_\beta,$$

where σ_α is the hyperplane of $\mathfrak{a}$ defined by $\alpha=0$ and

$$H'_\beta=\nu(H'_\alpha)=\sqrt{-1}(X'_\alpha-X'_{-\alpha}).$$

This relation between $\mathfrak{a}$ and $\mathfrak{b}$ is denoted by $(\mathfrak{a}, \alpha)\to(\mathfrak{b}, \beta)$ or simply by $\mathfrak{a}\to\mathfrak{b}$. We introduce the order $<$ in Car(G) by defining $[A]<[B]$ when $\mathfrak{a}\to\mathfrak{b}$ for an appropriate choice of representative B of $[B]$, and extend it transitively.

Let $V(\lambda)$ be the virtual character module of admissible representations of G which have an infinitesimal character λ. Here, we mean by a *virtual character* a complex linear combination of irreducible characters. An element of $V(\lambda)$ can be naturally considered to be an invariant eigendistribution (IED) on G with eigenvalue λ. We say a virtual character or an IED $\Theta\in V(\lambda)$ has a height $[H]\in$Car(G) if $\Theta|_H\not\equiv0$ and $\Theta|_J\equiv0$ for any $[J]\in$Car(G) such that $[J]>[H]$. We call Θ extremal if Θ has the unique height.

1.2. The structure of virtual character modules. We quote the results of T. Hirai [8] in this subsection. Let H be a Cartan subgroup of G. Let $S(\mathfrak{h}_C)$ be the symmetric algebra of $\mathfrak{h}_C$ and $I(\mathfrak{h}_C)=S(\mathfrak{h}_C)^W$ the space of Weyl group invariant elements in $S(\mathfrak{h}_C)$. For any subset B of G and a subgroup D of G, we write $W_D(B)=N_D(B)/Z_D(B)$, where $N_D(B)$ denotes the normalizer of B in D and $Z_D(B)$ the centralizer.

We denote by $\mathfrak{B}(H;\lambda)$ the set of analytic functions ζ on H satisfying the conditions (1) and (2).

(1) ζ is an eigenfunction of $I(\mathfrak{h}_C)$ with eigenvalue λ, where we identify canonically elements of $I(\mathfrak{h}_C)$ with differential operators of constant coefficients on H.

(2) ζ is ε-symmetric under $W_G(H)$, i.e.,

$$\zeta(wh)=\varepsilon(w, h)\zeta(h) \qquad (h\in H,\ w\in W_G(H)),$$

where $\varepsilon(w, h)$ is locally constant in h and is defined as follows. An element $w\in W_G(H)$ naturally induces an element $\tilde{w}$ of $W(\mathfrak{h}_C)$. Let $N_I(\tilde{w})$ be the number of imaginary roots $\alpha>0$ for which $\tilde{w}^{-1}\alpha<0$, and $S_R(\tilde{w})$ the set of real roots $\alpha>0$ for which $\tilde{w}^{-1}\alpha<0$. We put for $h\in H$ and $w\in W_G(H)$,

$$\varepsilon(w, h)=(-1)^{(N_I\tilde{w})}\prod_{\alpha\in S_R(\tilde{w})}\operatorname{sgn}(\xi_{\tilde{w}^{-1}\alpha}(h)). \tag{1.1}$$

THEOREM 1.3. *If λ is regular, $V(\lambda)$ is equal to the space of all the IEDs on G with eigenvalue λ.*

PROOF. This theorem is actually known. Here, we give a sketch of the proof. It is obvious that $V(\lambda)$ is contained in the space of IEDs with eigenvalue λ. Let P be a cuspidal parabolic subgroup of G and $P=M_PA_PN_P$ be a Levi decomposition of P. Take a discrete series representation D of M_P and a character ν of A_P. We mean by a generalized principal series representation an induced one $\mathrm{Ind}_P^G D\otimes\nu\otimes 1$. Then each IED with regular infinitesimal character is a linear combination of characters of generalized principal series representations induced from some cuspidal parabolic subgroups of G. Q.E.D.

THEOREM 1.4 (T. Hirai [8, p. 284, p. 302]). (1) *For an element ζ of $\mathfrak{B}(H;\lambda)$, we can construct an extremal IED $\boldsymbol{T}\zeta$ which has the height $[H]$ and on H it naturally provides ζ (see [8, p. 272]).*

(2) *Conversely, any IED with eigenvalue λ can be written as a linear combination of IEDs which are of the form $\boldsymbol{T}\zeta$ ($\zeta\in\mathfrak{B}(H;\lambda)$) for some H's.*

REMARK. We give in detail the method $\boldsymbol{T}$ of constructing IED in the next section.

We assume the following throughout this paper.

ASSUMPTION. The infinitesimal character λ is regular.

Because of Theorem 1.3, we identify virtual characters which have infinitesimal character λ with IEDs on G with eigenvalue λ.

Let $\widetilde{W}_H(\lambda)$ be the set of $w\in W(\mathfrak{h}_C)$ for which $\exp(w\lambda, X)$ ($X\in\mathfrak{h}$) defines an analytic function on H_0, the identity component of H. Let L be the kernel of the map $\exp:\mathfrak{h}\to H_0$. Then $\widetilde{W}_H(\lambda)=\{w\in W(\mathfrak{h}_C)\mid \langle w\lambda, L\rangle\subset 2\pi\sqrt{-1}\boldsymbol{Z}\}$, where

$\langle\,,\,\rangle$ is the pairing of $\mathfrak{h}_C^*\times\mathfrak{h}_C$. Put

$$L_\lambda=\sum_{w\in\tilde{W}_H(\lambda)} w^{-1}L, \qquad W_H(\lambda)=\{w\in W(\mathfrak{h}_C)\mid wL_\lambda=L_\lambda\}.$$

Let $W(H_i)=\{\tilde{w}\mid w\in W_G(H_i)\}$, where $\{H_i\mid 0\leqq i\leqq l\}$ is a set of representatives of connected components of H under the conjugation of $W_G(H)$. We get the following proposition.

PROPOSITION 1.5. (1) *The set $\tilde{W}_H(\lambda)$ is invariant under the left multiplication of $W(H_i)$.*

(2) *The set $\tilde{W}_H(\lambda)$ is invariant under the right multiplication of $W_H(\lambda)$. Moreover, the group $W_H(\lambda)$ is the largest subgroup of $W(\mathfrak{h}_C)$ which leaves $\tilde{W}_H(\lambda)$ invariant under the right multiplication.*

PROOF. (1) Let $\sigma\in W(H_i)$ and $w\in\tilde{W}_H(\lambda)$. Since L is the kernel of the map $\exp:\mathfrak{h}\to H_0$, σ preserves L. So, we have $\langle\sigma w\lambda, L\rangle=\langle w\lambda, \sigma^{-1}L\rangle=\langle w\lambda, L\rangle\subset 2\pi\sqrt{-1}\boldsymbol{Z}$. This means $\sigma w\in\tilde{W}_H(\lambda)$.

(2) Note that $W_H(\lambda)$ forms a subgroup of $W(\mathfrak{h}_C)$. Let $w\in\tilde{W}_H(\lambda)$ and $\sigma\in W_H(\lambda)$. Since $L_\lambda\supset w^{-1}L$, we get $L_\lambda=\sigma^{-1}L_\lambda\supset\sigma^{-1}w^{-1}L$. By the definition of L_λ, we see that

$$\langle\lambda, L_\lambda\rangle=\langle\lambda, \sum_{w\in\tilde{W}_H(\lambda)} w^{-1}L\rangle=\sum_{w\in\tilde{W}_H(\lambda)}\langle w\lambda, L\rangle\subset 2\pi\sqrt{-1}\boldsymbol{Z}.$$

Therefore we have $\langle w\sigma\lambda, L\rangle=\langle\lambda, \sigma^{-1}w^{-1}L\rangle\subset\langle\lambda, L_\lambda\rangle\subset 2\pi\sqrt{-1}\boldsymbol{Z}$. This means $w\sigma\in\tilde{W}_H(\lambda)$.

Conversely, let $\sigma\in W(\mathfrak{h}_C)$ be an element such that $\tilde{W}_H(\lambda)\sigma^{-1}=\tilde{W}_H(\lambda)$. Then we get

$$\sigma L_\lambda=\sum_{w\in\tilde{W}_H(\lambda)}\sigma(w^{-1}L)=\sum_{w\in\tilde{W}_H(\lambda)}(w\sigma^{-1})^{-1}L=\sum_{w'\in\tilde{W}_H(\lambda)}w'^{-1}L=L_\lambda.$$

Hence $\sigma\in W_H(\lambda)$. Q.E.D.

For each connected component H_i, take an element $a_i\in H_i$. Then we have

$$a_i^{-1}(sa_i)\in H_0 \qquad \text{for} \quad s\in W_G(H_i).$$

Therefore we can write $a_i^{-1}(sa_i)=\exp B_s$ for some $B_s\in\mathfrak{h}$.

ASSUMPTION ON λ. We assume that we can choose $\{a_i\mid a_i\in H_i,\ 0\leqq i\leqq l\}$ which satisfies the following condition. For any $t_1, t_2\in\tilde{W}_H(\lambda)$,

$$\exp(t_1\lambda, B_s)=\exp(t_2\lambda, B_s) \qquad (s\in W_G(H_i)).$$

Hereafter we fix these $\{a_i\mid 0\leqq i\leqq l\}$ and write

$$\xi^i(s)=\exp(t\lambda, B_s) \qquad (s\in W_G(H_i)),$$

which does not depend on $t\in\tilde{W}_H(\lambda)$ by assumption. We have the following lemma.

LEMMA 1.6. (1) *If* $G=SL(n, \boldsymbol{R})$, $Sp(2n, \boldsymbol{R})$ or $SO_0(p, q)$ $(p+q=2n)$, *then we can always choose* $\{a_i\}$ *which satisfies* $a_i=sa_i$ *for any* $s\in W_G(H_i)$. *So, in these cases, the assumption is trivially satisfied for any* λ.

(2) *If all the Cartan subgroups in G are connected, we can choose* $a_0=e$ *(the unit of G) and the assumption is satisfied.*

(3) *In particular, if G is a complex semisimple Lie group, then the assumption is satisfied for any* λ.

REMARK. For any G, there always exists a lattice in $\mathfrak{h}_C^*$ whose elements satisfy the above assumption. See also remark to the corollary to Theorem 4.3.

PROPOSITION 1.7 ([8, p. 319]). *The space* $\mathfrak{B}(H;\lambda)$ *has a base consisting of the element* $\{\zeta_{i,t}\}$ *of the following form. Take a complete system of representatives* $\{t\}\subset\widetilde{W}_H(\lambda)$ *for a left coset space* $W(H_i)\backslash\widetilde{W}_H(\lambda)$. *For* $0\leqq i\leqq l$ *and t, we put*

$$\zeta_{i,t}(wa_i\exp X)=\varepsilon(w, a_i)\sum_{s\in W_G(H_i)}\varepsilon(s, a_i)\xi^i(s)\exp(t\lambda, sX) \qquad (w\in W_G(H),\ X\in\mathfrak{h}),$$

and put $\zeta_{i,t}$ *zero outside the* $W_G(H)$*-orbit of* H_i.

REMARK. The formula listed in Proposition 1.7 is the corrected version of the formula (7.20) in [8].

Put $V_H(\lambda)=\boldsymbol{T}(\mathfrak{B}(H;\lambda))$. Then by the above proposition, we get a basis $\{\boldsymbol{T}\zeta_{i,t}\mid 0\leqq i\leqq l,\ t\in W(H_i)\backslash\widetilde{W}_H(\lambda)\}$ of $V_H(\lambda)$. Moreover, since $V(\lambda)=\sum_{[H]\in\mathrm{Car}(G)}^{\oplus}V_H(\lambda)$ by Theorem 1.4, we get a basis of $V(\lambda)$. This canonical basis plays an important role in the following sections.

§2. Hirai's method $\boldsymbol{T}$.

In this section we describe Hirai's method $\boldsymbol{T}$ in detail for later use. For simplicity we assume λ regular, but the argument here is valid too in the case that λ is singular. Notations and terms without explanations are refered to [8].

As is mentioned in former sections, $V(\lambda)$ is the space of all the IEDs on G with eigenvalue λ. Harish-Chandra [5] proved any IED Θ on G coincides essentially with a locally summable function on G which is analytic on the open dense subset G' of all the regular elements in G. Because G' is open dense in G and any element in G' is contained in a Cartan subgroup of G, Θ is determined by the values on Cartan subgroups $\{H\mid [H]\in\mathrm{Car}(G)\}$. Put

$$D^H(h)=\xi_\rho(h)\prod_{\alpha\in\Delta^+}(1-\xi_\alpha(h)^{-1}) \qquad (h\in H),$$

$$D_R^H(h)=\prod_{\alpha\in\Delta_R^+}(1-\xi_\alpha(h)^{-1}) \qquad (h\in H).$$

For a given IED Θ on G and a Cartan subgroup H of G, we put

$$C_H(\Theta)(h)=D^H(h)\Theta(h) \qquad (h\in H'=H\cap G'),$$
$$C'_H(\Theta)(h)=\varepsilon_R^H(h)D^H(h)\Theta(h) \qquad (h\in H'),$$

where $\varepsilon_R^H(h)=\mathrm{sgn}(D_R^H(h))$ $(h\in H'(\boldsymbol{R}))$.

THEOREM 3.1 ([**7**]). *Let Θ be an IED on G with eigenvalue λ. If Θ has a height $[H]\in\mathrm{Car}(G)$, then $C'_H(\Theta)$ can be extended to an analytic function on the whole group H. Moreover, it belongs to $\mathfrak{B}(H;\lambda)$.*

Hirai's method $\boldsymbol{T}$ is the method to construct an extremal IED from an element ζ of $\mathfrak{B}(H;\lambda)$. This is done by induction on the order of $\mathrm{Car}(G)$ and has two different steps $\boldsymbol{R}$ and $\boldsymbol{S}$. Roughly speaking, the step $\boldsymbol{R}$ corresponds to boundary conditions to be satisfied by IEDs, and the step $\boldsymbol{S}$ corresponds to Weyl group symmetricity which assures the invariance of IEDs. As is mentioned above, an IED Θ is determined by the system of functions $C_H(\Theta)$ ($[H]\in\mathrm{Car}(G)$). So, in order to give an IED $\boldsymbol{T}\zeta$ for $\zeta\in\mathfrak{B}(H;\lambda)$, it is sufficient to give functions $C_H(\boldsymbol{T}\zeta)$ for every $[H]\in\mathrm{Car}(G)$. T. Hirai studied what is necessary and sufficient for the system of functions $C_H(\Theta)$ ($[H]\in\mathrm{Car}(G)$) obtained from an IED Θ through the series of his works [**6**, **7**, **8**] and actually gave necessary and sufficient conditions. Using his results one can verify that constructed functions $C_H(\boldsymbol{T}\zeta)$ ($[H]\in\mathrm{Car}(G)$) really determine an IED $\boldsymbol{T}\zeta$.

Let us explain the construction in detail. Take an element $\zeta\in\mathfrak{B}(H;\lambda)$. We put

$$C_H(\boldsymbol{T}\zeta)=\varepsilon_R^H\cdot\zeta \qquad \text{for } H \text{ itself},$$
$$C_J(\boldsymbol{T}\zeta)\equiv 0 \qquad \text{for } [J]\not\leqq[H].$$

Let A be a Cartan subgroup of G and assume that we have already constructed $C_B(\boldsymbol{T}\zeta)$ for $[B]>[A]$. Let A_1 be a connected component of A and F a connected component of $A'_1(\boldsymbol{R})=A_1\cap A'(\boldsymbol{R})$. Denote by $\Sigma=\Sigma(A_1)$ the set of all real roots $\alpha\in\Delta(\mathfrak{g}_C, \mathfrak{a}_C)$ for which $\xi_\alpha(h)>0$ on A_1. Then Σ is a root system of a certain real semisimple Lie algebra. Let $S=S(A_1)$ be the subgroup of $W_G(A_1)$ generated by $\omega_\alpha|_{A_1}$ ($\alpha\in\Sigma$), where ω_α is the conjugation by an element $g_\alpha=\exp\pi(X'_\alpha-X'_{-\alpha})/2\in K$. We put $\omega_\alpha|_{A_1}=s_\alpha$. Let $P(F)$ be the set of $\alpha\in\Sigma$ for which $\xi_\alpha(F)>1$. Then $P(F)$ is the set of all the positive roots of Σ with respect to a certain order of roots. Let $\Pi=\Pi(F)=\{\alpha_1, \cdots, \alpha_r\}$ be the simple system in $P(F)$.

(I) *Step* $\boldsymbol{R}$. Denote by $\mathfrak{b}^m$ a Cartan subalgebra obtained from $\mathfrak{a}$ by the Cayley transform $\nu_{\alpha_m}=\nu_m$ with respect to the real root α_m ($1\leqq m\leqq r$). By assumption, the functions $C_{Bm}(\boldsymbol{T}\zeta)$ have been already determined. We write C_m instead of $C_{Bm}(\boldsymbol{T}\zeta)$ for brevity.

Recall the notations about Cayley transforms ν_m in § 1. We put

$$\Sigma_m=\{h\in A \mid \xi_{\alpha_m}(h)=1\},$$
$$\Sigma'_m=\{h\in\Sigma_m \mid \xi_\alpha(h)\neq 1 \quad \text{for any root } \alpha\neq\pm\alpha_m\}.$$

Then for $a\in\Sigma'_m\cap A_1$ and $X\in\mathfrak{a}$, we put

$$(\boldsymbol{R}_{\alpha_m}C_m)(a\exp X)=C_m(a\exp\nu_m X).$$

Here $\nu_m X$ may not be contained in $\mathfrak{b}^m$, but C_m is locally a linear combination of the functions of the form $\exp\mu(X)$ ($\mu\in(\mathfrak{b}_C^m)^*$), so $C_m(a\exp\nu_m X)$ has natural meaning.

(II) *Step S*. For a function f on A_1 and $s\in S$, we define sf as $(sf)(h)=f(s^{-1}h)$ ($h\in A_1$). For each $s_m=s_{\alpha_m}$ ($1\leqq m\leqq r$), we put

$$P_{s_m}=(1-s_m)(\boldsymbol{R}_{\alpha_m}C_m).$$

Each element $s\in S$ can be written in the form $s=s_{i_1}s_{i_2}\cdots s_{i_k}$ (see, for example, [3]). Then we put

$$P_s=P_{s_{i_1}}+s_{i_1}P_{s_{i_2}}+\cdots+s_{i_1}s_{i_2}\cdots s_{i_{k-1}}P_{s_{i_k}}.$$

It can be proved that P_s is independent of a choice of expressions for $s\in S$. Finally we put

$$Q=\boldsymbol{S}(P_{s_1}, P_{s_2}, \cdots, P_{s_r})=\frac{1}{|S|}\sum_{s\in S}P_s.$$

Denote by E_{A_1} the union of wA_1 over $w\in W_G(A)$. Define $C_A(\boldsymbol{T}\zeta)$ on $E_{A_1}\cap A'(\boldsymbol{R})$ by

$$C_A(\boldsymbol{T}\zeta)(wh)=\det(w)Q(h) \qquad (w\in W_G(A),\ h\in F).$$

Let $A_1, A_2, \cdots$ be a complete system of representatives of connected components of A under the conjugation of $W_G(A)$. Then A is the disjoint union of $E_{A_1}, E_{A_2}, \cdots$. Repeating the same argument for every A_i, we get $C_A(\boldsymbol{T}\zeta)$ on the whole A.

Thus we can define $C_H(\boldsymbol{T}\zeta)$ ($[H]\in\mathrm{Car}(G)$) inductively. We see that they altogether define an IED $\boldsymbol{T}\zeta$ ($\zeta\in\mathfrak{B}(H;\lambda)$) by Hirai's argument.

§ 3. Definition of representations of $W_H(\lambda)$.

In this section, we define a representation of $W_H(\lambda)$ on $V_H(\lambda)$ for each $[H]\in\mathrm{Car}(G)$. We assume λ regular and keep the notations in § 1.

At first we consider a representation $\mathcal{R}$ of $W_H(\lambda)$ on $\mathfrak{B}(H;\lambda)$. Take $[H]\in\mathrm{Car}(G)$ and let $\{H_i \mid 0\leqq i\leqq l\}$ be representatives of connected components of H under the conjugation of $W_G(H)$. Then $\mathfrak{B}(H;\lambda)$ is spanned by the set $\{\zeta_{i,t} \mid 0\leqq i\leqq l,\ t\in\widetilde{W}_H(\lambda)\}$, where

$$\zeta_{i,t}(wa_i\exp X)=\varepsilon(w,\ a_i)\sum_{s\in W_G(H_i)}\varepsilon(s,\ a_i)\xi^i(s)\exp(t\lambda,\ sX)$$
$$(w\in W_G(H),\ X\in\mathfrak{h}),$$

and $\zeta_{i,t}$ is zero outside the $W_G(H)$-orbit of H_i. We define the representation $\mathfrak{R}$ of $W_H(\lambda)$ on $\mathfrak{B}(H;\lambda)$ by

$$\mathfrak{R}_u\zeta_{i,t}=\zeta_{i,tu^{-1}} \quad \text{for} \quad u\in W_H(\lambda).$$

By the assumption in 1.2 and Proposition 1.5, this is well-defined.

DEFINITION 3.1. We define a representation τ of integral Weyl group $W_H(\lambda)$ on $V_H(\lambda)$ as follows. For $\zeta\in\mathfrak{B}(H;\lambda)$ and $u\in W_H(\lambda)$, put

$$\tau_u(\boldsymbol{T}\zeta)=\boldsymbol{T}(\mathfrak{R}_u\zeta).$$

We say λ is *integral for* G_C if λ is a differential of a character of H_C.

LEMMA 3.2. *If λ is integral for G_C, $\widetilde{W}_H(\lambda)=W_H(\lambda)=W(\mathfrak{h}_C)$.*

PROOF. By assumption, $\exp\lambda(X)$ $(X\in\mathfrak{h}_C)$ is a character of H_C. Then, for any $w\in W(\mathfrak{h}_C)$, $\exp w\lambda(X)=\exp\lambda(w^{-1}X)$ $(X\in\mathfrak{h})$ is well-defined on H_0. This proves $\widetilde{W}_H(\lambda)=W(\mathfrak{h}_C)$. Since $W_H(\lambda)$ is the largest subgroup of $W(\mathfrak{h}_C)$ which leaves $\widetilde{W}_H(\lambda)$ invariant under the right multiplication (Proposition 1.5), we have $W_H(\lambda)=W(\mathfrak{h}_C)$. Q.E.D.

By the above lemma, if λ is integral for G_C, we can consider $W(\mathfrak{h}_C)$-module structure of $V_H(\lambda)$. Since $V(\lambda)=\sum^{\oplus}_{[H]\in\mathrm{Car}(G)}V_H(\lambda)$, these $W(\mathfrak{h}_C)$-module structures of $V_H(\lambda)$'s naturally induce W-module structure of $V(\lambda)$. Here we identify all the Weyl groups $W(\mathfrak{h}_C)$'s by Cayley transforms and denote it by W. For integral λ, many people considered W-module structures of $V(\lambda)$. Among others, G. Zuckerman [**12**] defined W-module structure of $V(\lambda)$, using tensor products with finite dimensional representations of G. We show that his representation essentially coincides with ours in the next section §4. Then it is very likely that we can use the method of tensor products with finite dimensional representations for studying the W-module structure of $V(\lambda)$ (cf. [**19**]).

LEMMA 3.3. *Let $\lambda_1, \lambda_2\in\mathfrak{h}_C^*$ be regular. If $\lambda_1-\lambda_2$ is integral for G_C, then $\widetilde{W}_H(\lambda_1)=\widetilde{W}_H(\lambda_2)$ and $W_H(\lambda_1)=W_H(\lambda_2)$.*

PROOF. Take $w\in\widetilde{W}_H(\lambda_2)$. Both $\exp w(\lambda_1-\lambda_2)(X)$ and $\exp w\lambda_2(X)$ are well-defined characters of H_0, we see $\exp w\lambda_1(X)=\exp w\lambda_2(X)\exp w(\lambda_1-\lambda_2)(X)$ is well-defined, i.e., $w\in\widetilde{W}_H(\lambda_1)$. The converse inclusion can be similarly proved. Since $W_H(\lambda_i)$ $(i=1, 2)$ is the largest subgroup which leaves $\widetilde{W}_H(\lambda_i)$ invariant under the right multiplication, we have $W_H(\lambda_1)=W_H(\lambda_2)$. Q.E.D.

LEMMA 3.4. *For regular $\lambda\in\mathfrak{h}_C^*$, put $I(\lambda)=\{w\in W \mid w\lambda-\lambda$ is integral for $G_C\}$. Then, for $t\in\widetilde{W}_H(\lambda)$, $W_H(\lambda)$ contains $tI(\lambda)t^{-1}$. In particular, if the unit e of W is contained in $\widetilde{W}_H(\lambda)$, then $W_H(\lambda)$ contains $I(\lambda)$.*

PROOF. Clearly $I(\lambda)$ is a group. Considering, if necessary, $\widetilde{W}_H(t\lambda)$ instead of $\widetilde{W}_H(\lambda)$, we may assume that $e\in\widetilde{W}_H(\lambda)$ and $t=e$. Take $\sigma\in I(\lambda)$. Since $\sigma\lambda-\lambda$ is integral for G_C, we have $\widetilde{W}_H(\sigma\lambda)=\widetilde{W}_H(\lambda)$ by Lemma 3.3. This means $\exp w\lambda(X)$ $(X\in\mathfrak{h})$ is well-defined on H_0 if and only if $\exp w\sigma\lambda(X)$ $(X\in\mathfrak{h})$ is so. Therefore σ leaves $\widetilde{W}_H(\lambda)$ invariant from the right and we have $\sigma\in W_H(\lambda)$.

Q. E. D.

These two lemmas give us a method to calculate $W_H(\lambda)$ explicitly and show us $W_H(\lambda)$ contains large subgroups of $W(\mathfrak{h}_C)$.

§4. Relation to Zuckerman's representation.

In this section, we describe the relation between our representation τ and Zuckerman's one. So, we put some assumptions on G in addition to those in the former sections, after Zuckerman.

Let G be a connected semisimple linear Lie group. We suppose that there are simply connected complex Lie group G_C with Lie algebra $\mathfrak{g}_C$ and the natural injection $j: G\hookrightarrow G_C$. Let λ be a differential of a character of a Cartan subgroup H_C of G_C. We assume that λ is regular and satisfies the assumption in 1.2. Then by Lemma 3.2, we have $\widetilde{W}_H(\lambda)=W_H(\lambda)=W(\mathfrak{h}_C)$ for any Cartan subgroup H of G under the above assumptions on G. We write W for $W(\mathfrak{h}_C)$ and identify it with $W(\mathfrak{h}'_C)$ for another Cartan subalgebra $\mathfrak{h}'$ by Cayley transforms. Thus we have the representation of W on the virtual character module $V(\lambda)=\sum^{\oplus}V_H(\lambda)$.

Now we define another representation $\mathcal{Z}$ of W on $V(\lambda)$ after G. Zuckerman [**12**]. Let Θ be a virtual character in $V(\lambda)$. Then we can write it on a Cartan subgroup H of G as

$$\Theta(h)=\frac{1}{D(h)}\sum_{s\in W}c(\Theta,\ s\,;\,h)\xi_{s\lambda}(h) \qquad (h\in H'),$$

where $c(\Theta,\ s\,;\,h)$ is a locally constant function on $H'(\boldsymbol{R})$ and $\xi_{s\lambda}$'s are well-defined characters of H (cf. 1.1). Then we define $\mathcal{Z}_\sigma\Theta$ $(\sigma\in W)$ by the equation below.

$$\mathcal{Z}_\sigma\Theta(h)=\frac{1}{D(h)}\sum_{s\in W}c(\Theta,\ s\,;\,h)\xi_{s\sigma^{-1}\lambda}(h) \qquad (h\in H'). \tag{4.1}$$

The system of functions $\mathcal{Z}_\sigma\Theta$ on every H in G determines again a virtual character in $V(\lambda)$. This is proved by Zuckerman [**12**], using tensor products with finite dimensional representations of G. Thus we get a representation $\mathcal{Z}$ of W on $V(\lambda)$.

We want to show the two representations τ and $\mathcal{Z}$ of W are equivalent by giving an intertwining operator explicitly. Before doing this we prepare a technical lemma.

LEMMA 4.1. *Any* $\zeta\in\mathfrak{B}(H;\lambda)$ *can be written as*

$$\zeta(h)=\sum_{w\in W}c_w(h)\xi_{w\lambda}(h)\,, \tag{4.2}$$

for certain locally constant functions c_w $(w\in W)$ *on* H.

PROOF. Indeed, $\zeta_{i,t}$ $(0\leqq i\leqq l,\ t\in W(H_i)\backslash W)$ in Proposition 1.7 can be written as

$$\zeta_{i,t}(wa_i\exp X)=\varepsilon(w,\ a_i)\sum_{s\in W_G(H_i)}\varepsilon(s,\ a_i)\xi^i(s)\exp(t\lambda,\ sX)$$

$$(w\in W_G(H),\ X\in\mathfrak{h}).$$

At first we assume that $w=e$. Then we have

$$\begin{aligned}\zeta_{i,t}(a_i\exp X)&=\sum_{s\in W_G(H_i)}\varepsilon(s,\ a_i)\xi^i(s)\exp(s^{-1}t\lambda,\ X)\\&=\sum_{s\in W_G(H_i)}\varepsilon(s,\ a_i)\xi^i(s)\frac{\exp(s^{-1}t\lambda,\ X)}{\xi_{s^{-1}t\lambda}(a_i\exp X)}\xi_{s^{-1}t\lambda}(a_i\exp X)\\&=\sum_{s\in W_G(H_i)}\varepsilon(s,\ a_i)\frac{1}{\xi_{t\lambda}(a_i)}\frac{\exp(s^{-1}t\lambda,\ X)}{\xi_{s^{-1}t\lambda}(\exp X)}\xi_{s^{-1}t\lambda}(a_i\exp X)\,.\end{aligned}$$

Obviously, we have $\exp(s^{-1}t\lambda,\ X)=\xi_{s^{-1}t\lambda}(\exp X)$. Therefore

$$\zeta_{i,t}(a_i\exp X)=\frac{1}{\xi_{t\lambda}(a_i)}\sum_{s\in W_G(H_i)}\varepsilon(s,\ a_i)\xi_{s^{-1}t\lambda}(a_i\exp X)\,. \tag{4.3}$$

Let $\{w_j\mid 1\leqq j\leqq k\}$ be a complete system of representatives of a left coset space $W(H_i)\backslash W_G(H)$. Then we define $\eta_{i,t}$ as

$$\eta_{i,t}(h)=\sum_{s\in W(H_i)w_j}\varepsilon(s,\ a_i)\xi_{s^{-1}t\lambda}(h)\qquad\text{for}\quad h\in w_j^{-1}H_i\,.$$

If $h=w_j^{-1}a_i\exp X$ $(X\in\mathfrak{h})$, then

$$\begin{aligned}\eta_{i,t}(h)&=\sum_{s\in W(H_i)w_j}\varepsilon(s,\ a_i)\xi_{s^{-1}t\lambda}(w_j^{-1}a_i\exp X)\\&=\sum_{\sigma\in W(H_i)}\varepsilon(\sigma w_j,\ a_i)\xi_{(\sigma w_j)^{-1}t\lambda}(w_j^{-1}a_i\exp X)\\&=\varepsilon(w_j,\ a_i)\sum_{\sigma\in W(H_i)}\varepsilon(\sigma,\ a_i)\xi_{w_j^{-1}\sigma^{-1}t\lambda}(w_j^{-1}a_i\exp X)\\&=\xi_{t\lambda}(a_i)\zeta_{i,t}(w_j^{-1}a_i\exp X)=\xi_{t\lambda}(a_i)\zeta_{i,t}(h)\,.\end{aligned}$$

The fourth equality follows from (4.3) and ε-symmetricity of $\zeta_{i,t}$. Since $\{\zeta_{i,t}\}$ forms a basis of $\mathfrak{B}(H;\lambda)$, $\{\eta_{i,t}\}$ also forms a basis of $\mathfrak{B}(H;\lambda)$. Now it is clear that any element $\zeta\in\mathfrak{B}(H;\lambda)$ can be written as in (4.2). Q.E.D.

DEFINITION 4.2. Let $\zeta\in\mathfrak{B}(H;\lambda)$ and write it as (4.2). Then we define a representation $\mathcal{L}$ of W on $\mathfrak{B}(H;\lambda)$ as follows.

$$(\mathcal{L}_s\zeta)(h)=\sum_{w\in W}c_w(h)\xi_{ws^{-1}\lambda}(h)\,.$$

Let C be a linear map of $\mathfrak{B}(H;\lambda)$ into itself given by

$$C(\zeta_{i,t})=\eta_{i,t}\,.$$

Then clearly $C\mathcal{R}_s=\mathcal{L}_sC$ $(s\in W)$ holds. This means the representations $\mathcal{R}$ and $\mathcal{L}$ are equivalent. Remark that C is a diagonal operator with respect to the basis $\{\zeta_{i,t}\}$. Indeed it is of the form $C=\mathrm{diag}(\xi_{t\lambda}(a_i);\ 0\leqq i\leqq l,\ t\in W(H_i)\backslash W)$.

In order to show that the representation τ and $\mathcal{Z}$ are equivalent, it is sufficient to show the next theorem.

THEOREM 4.3. *Hirai's method* $\boldsymbol{T}$ *intertwines* $\mathcal{Z}$ *and* $\mathcal{L}$, *i.e., for any* $s\in W$ *and* $\zeta\in\mathfrak{B}(H;\lambda)$,

$$\boldsymbol{T}(\mathcal{L}_s\zeta)=\mathcal{Z}_s(\boldsymbol{T}\zeta)\,.$$

By this theorem, the representations $\mathcal{L}$ and $\mathcal{Z}$ are equivalent. So we have on $V_H(\lambda)$, $\tau\cong\mathcal{R}\cong\mathcal{L}\cong\mathcal{Z}$. The first equivalence follows from the definition of τ.

$$\begin{array}{ccccccc}
V_H(\lambda) & \xleftarrow[\boldsymbol{T}]{\sim} & \mathfrak{B}(H;\lambda) & \xleftarrow[C]{\sim} & \mathfrak{B}(H;\lambda) & \xrightarrow[\boldsymbol{T}]{\sim} & V_H(\lambda)\\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \mathcal{R}} & & \downarrow{\scriptstyle \mathcal{L}} & & \downarrow{\scriptstyle \mathcal{Z}}\\
V_H(\lambda) & \xleftarrow[\boldsymbol{T}]{\sim} & \mathfrak{B}(H;\lambda) & \xleftarrow[C]{\sim} & \mathfrak{B}(H;\lambda) & \xrightarrow[\boldsymbol{T}]{\sim} & V_H(\lambda)
\end{array}$$

COROLLARY. *Representations* τ *and* $\mathcal{Z}$ *of* W *on* $V(\lambda)$ *are equivalent. An intertwining operator is given by* $\boldsymbol{T}\circ C\circ\boldsymbol{T}^{-1}$.

REMARK. We can also treat $\lambda\in\mathfrak{h}_{\mathbf{C}}^*$ which satisfies the following condition (*) instead of that in 1.2. In this case, we define an action τ' of $W_H(\lambda)$ slightly different from τ.

(*) Each $\xi_{t\lambda}$ $(t\in\widetilde{W}_H(\lambda))$ on H_0 can be extended to the whole H in such a way that

$$\xi_{st\lambda}(sh)=\xi_{t\lambda}(h)\qquad\text{for}\quad s\in W_G(H)\,.$$

Here we naturally define τ' by

$$\tau'_s(\boldsymbol{T}\zeta)=\boldsymbol{T}(\mathcal{L}_s\zeta)\qquad(s\in W_H(\lambda),\ \zeta\in\mathfrak{B}(H;\lambda))\,,$$

with the same formulas for ζ and $\mathcal{L}_s\zeta$ as in Definition 4.2.

If λ is integral for G_C, the above assumption (*) is satisfied. Moreover, the representation τ' is also equivalent to $\mathcal{Z}$.

PROOF OF THE THEOREM. We use the notations in §2. For an IED $\Theta\in V(\lambda)$ and any Cartan subgroup J of G, we can write

$$C_J(\Theta)(h)=\sum_{w\in W} c_w(h)\xi_{w\lambda}(h) \qquad (h\in J'(\boldsymbol{R}))\,,$$

where $c_w(h)$ are locally constant functions on $J'(\boldsymbol{R})$. Then we define $\mathcal{L}'_s C_J(\Theta)$ $(s\in W)$ by

$$\mathcal{L}'_s C_J(\Theta)(h)=\sum_{w\in W} c_w(h)\xi_{ws^{-1}\lambda}(h) \qquad (h\in J'(\boldsymbol{R}))\,.$$

We show that for any Cartan subgroup J of G,

$$C_J(\boldsymbol{T}(\mathcal{L}_s\zeta))=\mathcal{L}'_s C_J(\boldsymbol{T}\zeta) \qquad (s\in W)\,, \tag{4.4}$$

by induction with respect to the order on $\mathrm{Car}(G)$. If we can establish (4.4), then by the definition of $\mathcal{Z}$, we have for any $[J]\in\mathrm{Car}(G)$,

$$C_J(\boldsymbol{T}(\mathcal{L}_s\zeta))=\mathcal{L}'_s C_J(\boldsymbol{T}\zeta)=C_J(\mathcal{Z}_s(\boldsymbol{T}\zeta)) \qquad (s\in W)\,,$$

and this proves the theorem. So let us prove (4.4).

For $[J]\not\leqq[H]$, (4.4) is trivially valid. For $J=H$, the unique height of $\boldsymbol{T}\zeta$ $(\zeta\in\mathfrak{B}(H;\lambda))$, we have

$$C_H(\boldsymbol{T}\mathcal{L}_s\zeta)=\mathcal{L}'_s C_H(\boldsymbol{T}\zeta) \qquad (s\in W)\,, \tag{4.5}$$

by the definition of $\boldsymbol{T}$. The equation (4.5) shows (4.4) is valid for $J=H$. So we assume that (4.4) is valid for $[J]>[A]$ and prove it for $[A]$. Let $\{B^m \mid 1\leqq m\leqq r\}$ be Cartan subgroups given in connection with A in § 2. By the induction hypothesis we have

$$C_{B^m}(\boldsymbol{T}(\mathcal{L}_s\zeta))=\mathcal{L}'_s C_{B^m}(\boldsymbol{T}\zeta) \qquad (s\in W)\,.$$

Recall that $\boldsymbol{T}$ has main two steps $\boldsymbol{R}$ and $\boldsymbol{S}$. By the definition of $\boldsymbol{T}$,

$$C_A(\boldsymbol{T}(\mathcal{L}_s\zeta))=\boldsymbol{S}\circ\boldsymbol{R}(C_{B^m}(\boldsymbol{T}(\mathcal{L}_s\zeta)) \mid 1\leqq m\leqq r)\,.$$

Therefore it is sufficient to show that

$$\boldsymbol{S}\circ\boldsymbol{R}(\mathcal{L}'_s C_{B^m}(\boldsymbol{T}\zeta) \mid 1\leqq m\leqq r)=\mathcal{L}'_s(\boldsymbol{S}\circ\boldsymbol{R}(C_{B^m}(\boldsymbol{T}\zeta) \mid 1\leqq m\leqq r)) \tag{4.6}$$
$$(=\mathcal{L}'_s C_A(\boldsymbol{T}\zeta)) \qquad (s\in W)\,.$$

We write $C_m(\boldsymbol{T}\zeta)$ instead of $C_{B^m}(\boldsymbol{T}\zeta)$ for brevity.

(I) *Step* $\boldsymbol{R}$. We express $C_m(\boldsymbol{T}\zeta)$ as

$$C_m(\boldsymbol{T}\zeta)(h)=\sum_{w\in W} c_w^m(h)\xi_{w\lambda}(h) \qquad (h\in(B^m)'(\boldsymbol{R}))\,,$$

where $c_w^m(h)$ is a locally constant function on $(B^m)'(\boldsymbol{R})$. Since by the definition of $\mathcal{L}'_s$,

$$\mathcal{L}'_s C_m(\boldsymbol{T}\zeta)(h)=\sum_{w\in W} c_w^m(h)\xi_{ws^{-1}\lambda}(h) \qquad (h\in(B^m)'(\boldsymbol{R}))\,,$$

we have for $h=a\exp X$ $(a\in\Sigma'_m$ and $X\in\mathfrak{a}$ sufficiently small),

$$\begin{aligned} \boldsymbol{R}_{\alpha_m}(\mathcal{L}_s' C_m(\boldsymbol{T}\zeta))(h) &= \sum_{w\in W} c_w^m(a)\xi_{ws^{-1}\lambda}(a\exp\nu_m X) \\ &= \sum_{w\in W} c_w^m(a)\xi_{t_{\nu_m}(ws^{-1}\lambda)}(a\exp X) \\ &= \sum_{w\in W} c_w^m(a)\xi_{ws^{-1}\lambda}(h). \end{aligned}$$

Here we identify $ws^{-1}\lambda\in(\mathfrak{h}_C^m)^*$ with $ws^{-1}\lambda\in(\mathfrak{a}_C)^*$ by Cayley transform ν_m. Thus we have proved

$$(4.7)\qquad \boldsymbol{R}_{\alpha_m}(\mathcal{L}_s' C_m(\boldsymbol{T}\zeta)) = \mathcal{L}_s' \boldsymbol{R}_{\alpha_m}(C_m(\boldsymbol{T}\zeta)) \qquad (s\in W).$$

(II) *Step S.* By (4.7) it holds

$$\begin{aligned} \boldsymbol{S}\circ\boldsymbol{R}(\mathcal{L}_s' C_m(\boldsymbol{T}\zeta) \mid 1\leqq m\leqq r) &= \boldsymbol{S}(\boldsymbol{R}_{\alpha_m}(\mathcal{L}_s' C_m(\boldsymbol{T}\zeta)) \mid 1\leqq m\leqq r) \\ &= \boldsymbol{S}(\mathcal{L}_s' \boldsymbol{R}_{\alpha_m}(C_m(\boldsymbol{T}\zeta)) \mid 1\leqq m\leqq r). \end{aligned}$$

Put

$$\begin{aligned} P_{s_m} &= (1-s_m)\boldsymbol{R}_{\alpha_m}(C_m(\boldsymbol{T}\zeta)) \qquad (1\leqq m\leqq r) \\ P'_{s_m} &= (1-s_m)(\mathcal{L}_s' \boldsymbol{R}_{\alpha_m}(C_m(\boldsymbol{T}\zeta))) \qquad (1\leqq m\leqq r). \end{aligned}$$

By the definition of $\boldsymbol{S}$, $\boldsymbol{S}(P_{s_1}, \cdots, P_{s_r})$ can be written as

$$\begin{aligned} \boldsymbol{S}(P_{s_1}, \cdots, P_{s_r}) &= \frac{1}{|S|}\sum_{\sigma\in S} P_\sigma \\ &= \sum_{1\leq m\leq r}\sum_{\sigma\in S} q_{m,\sigma}\sigma(\boldsymbol{R}_{\alpha_m}(C_m(\boldsymbol{T}\zeta))), \end{aligned}$$

where P_σ is defined as in § 2, and $\{q_{m,\sigma}\}$ are some rational numbers. Since

$$\begin{aligned} \sigma(\boldsymbol{R}_{\alpha_m} C_m(\boldsymbol{T}\zeta))(h) &= \boldsymbol{R}_{\alpha_m} C_m(\boldsymbol{T}\zeta)(\sigma^{-1}h) \\ &= \sum_{w\in W} c_w^m(\sigma^{-1}h)\xi_{w\lambda}(\sigma^{-1}h) \\ &= \sum_{w\in W} c_w^m(\sigma^{-1}h)\xi_{\sigma w\lambda}(h) \qquad (h\in A'(\boldsymbol{R})), \end{aligned}$$

we get

$$\begin{aligned} (4.8)\qquad & \mathcal{L}_s'(\boldsymbol{S}\circ\boldsymbol{R}(C_m(\boldsymbol{T}\zeta) \mid 1\leqq m\leqq r))(h) \\ &= \mathcal{L}_s'\Big(\sum_{1\leq m\leq r}\sum_{\sigma\in S} q_{m,\sigma}\sum_{w\in W} c_w^m(\sigma^{-1}h)\xi_{\sigma w\lambda}(h)\Big) \\ &= \sum_{1\leq m\leq r}\sum_{\sigma\in S} q_{m,\sigma}\sum_{w\in W} c_w^m(\sigma^{-1}h)\xi_{\sigma w s^{-1}\lambda}(h) \qquad (h\in A'(\boldsymbol{R})). \end{aligned}$$

On the other hand, by similar calculations, we have

$$\begin{aligned} (4.9)\qquad \boldsymbol{S}(P'_{s_1}, \cdots, P'_{s_r})(h) &= \frac{1}{|S|}\sum_{\sigma\in S} P'_\sigma(h) \\ &= \sum_{1\leq m\leq r}\sum_{\sigma\in S} q_{m,\sigma}\sigma(\mathcal{L}_s' \boldsymbol{R}_{\alpha_m} C_m(\boldsymbol{T}\zeta))(h) \end{aligned}$$

$$=\sum_{1\leq m\leq r}\sum_{\sigma\in S}q_{m,\sigma}\sigma(\sum_{w\in W}c_w^m(h)\xi_{ws^{-1}\lambda}(h))$$
$$=\sum_{1\leq m\leq r}\sum_{\sigma\in S}q_{m,\sigma}\sum_{w\in W}c_w^m(\sigma^{-1}h)\xi_{ws^{-1}\lambda}(\sigma^{-1}h)$$
$$=\sum_{1\leq m\leq r}\sum_{\sigma\in S}q_{m,\sigma}\sum_{w\in W}c_w^m(\sigma^{-1}h)\xi_{\sigma ws^{-1}\lambda}(h)\,.$$

Therefore we get

$$\boldsymbol{S}\circ\boldsymbol{R}(\mathcal{L}'_s C_m(\boldsymbol{T}\zeta)\mid 1\leqq m\leqq r)=\boldsymbol{S}(P'_{s_1},\cdots,P'_{s_r})$$
$$=\mathcal{L}'_s(\boldsymbol{S}\circ\boldsymbol{R}(C_m(\boldsymbol{T}\zeta)\mid 1\leqq m\leqq r))\,,$$

combining (4.8) and (4.9). This proves (4.6) and thus the proof is completed.

Q. E. D.

§ 5. Decompositions of the representations on $V(\lambda)$.

In this section, we assume again that G is a connected semisimple Lie group which is acceptable and has finite centre. Let λ be a regular infinitesimal character. For any Cartan subgroup H of G, we constructed the representation τ of $W_H(\lambda)$ on $V_H(\lambda)$ in § 3. Here we will give a canonical decomposition of τ which clarifies the structure of τ. We keep to the notations in §§ 1 and 3.

Let H be a Cartan subgroup of G and $\{H_i \mid 0\leqq i\leqq l\}$ be a complete system of representatives of connected components of H under the action of $W_G(H)$. Let H_0 be the identity component of H. We denote the kernel of the map $\exp:\mathfrak{h}\to H_0$ by L. Then we defined $\widetilde{W}_H(\lambda)$ as $\widetilde{W}_H(\lambda)=\{w\in W(\mathfrak{h}_C)\mid \langle w\lambda, L\rangle \subset 2\pi\sqrt{-1}\boldsymbol{Z}\}$, where we consider λ as a dominant element of $\mathfrak{h}_C^*$. By Proposition 1.5, $\widetilde{W}_H(\lambda)$ is invariant by $W(H_i)$ under the left multiplication and is also invariant by the integral Weyl group $W_H(\lambda)$ under the right multiplication. Therefore we can consider a double coset space $W(H_i)\backslash\widetilde{W}_H(\lambda)/W_H(\lambda)$. Let $\Gamma\subset\widetilde{W}_H(\lambda)$ be a complete system of representatives of a coset space $W(H_i)\backslash\widetilde{W}_H(\lambda)/W_H(\lambda)$, and put

$$W^{(i,\gamma)}=W_H(\lambda)\cap\gamma^{-1}W(H_i)\gamma \qquad (\gamma\in\Gamma)\,,$$
$$\varepsilon^{(i,\gamma)}(w)=\varepsilon(\gamma w\gamma^{-1}, a_i)\xi^i(\gamma w\gamma^{-1}) \qquad (a_i\in H_i,\ w\in W^{(i,\gamma)})\,,$$

where $\varepsilon(w, h)$ $(w\in W_G(H),\ h\in H)$ is defined as (1.1). Then $\varepsilon^{(i,\gamma)}$ is a character of the group $W^{(i,\gamma)}$.

THEOREM 5.1. *The representation τ of $W_H(\lambda)$ on $V_H(\lambda)$ given in Definition 3.1 is decomposed into a direct sum of subrepresentations as follows:*

$$V_H(\lambda)\cong\sum_{i=0}^{l}{}^{\oplus}\sum_{\gamma\in\Gamma}{}^{\oplus}\operatorname{Ind}(\varepsilon^{(i,\gamma)}\,;W^{(i,\gamma)}\uparrow W_H(\lambda))\,,$$

where $\operatorname{Ind}(\varepsilon\,;A\uparrow B)=\operatorname{Ind}_A^B\varepsilon$.

PROOF. Since τ and $\mathcal{R}$ are equivalent by definition, we decompose the representation $\mathcal{R}$ of $W_H(\lambda)$ on $\mathfrak{V}(H;\lambda)$. For $0\leqq i\leqq l$ and $t\in\widetilde{W}_H(\lambda)$ we put

$$\zeta_{i,t}(wa_i\exp X)=\varepsilon(w, a_i)\sum_{s\in W_G(H_i)}\varepsilon(s, a_i)\xi^i(s)\exp(t\lambda, sX)$$
$$(w\in W_G(H),\ X\in\mathfrak{h}).$$

We denote by $\mathfrak{V}^i(H;\lambda)$ a subspace of $\mathfrak{V}(H;\lambda)$ spanned by the elements $\{\zeta_{i,t}\mid t\in\widetilde{W}_H(\lambda)\}$ for a fixed i $(0\leqq i\leqq l)$. Then by the definition of $\mathcal{R}$, $\mathfrak{V}^i(H;\lambda)$ is clearly an invariant subspace of $\mathfrak{V}(H;\lambda)$ under $W_H(\lambda)$. Moreover as $W_H(\lambda)$-modules,

$$\mathfrak{V}(H;\lambda)=\sum_{i=0}^{l}{}^{\oplus}\,\mathfrak{V}^i(H;\lambda). \tag{5.1}$$

So it is sufficient to see that how a $W_H(\lambda)$-module $\mathfrak{V}^i(H;\lambda)$ is decomposed into a direct sum of submodules. For a fixed i $(0\leqq i\leqq l)$, we write ζ_t $(t\in\widetilde{W}_H(\lambda))$ instead of $\zeta_{i,t}$. We denote by $\mathfrak{V}^{(i,\gamma)}(H;\lambda)$ $(\gamma\in\Gamma)$ a subspace of $\mathfrak{V}^i(H;\lambda)$ spanned by the elements $\{\zeta_t\mid t\in W(H_i)\gamma W_H(\lambda)\}$. Then clearly

$$\mathfrak{V}^i(H;\lambda)=\sum_{\gamma\in\Gamma}{}^{\oplus}\,\mathfrak{V}^{(i,\gamma)}(H;\lambda) \tag{5.2}$$

gives a decomposition of the $W_H(\lambda)$-module $\mathfrak{V}^i(H;\lambda)$. We show that

$$\mathfrak{V}^{(i,\gamma)}(H;\lambda)\cong\mathrm{Ind}(\varepsilon^{(i,\gamma)}\,;W^{(i,\gamma)}\uparrow W_H(\lambda)). \tag{5.3}$$

We realize the representation $\mathrm{Ind}\,\varepsilon^{(i,\gamma)}$ as follows. The representation space E is given by

$$E=\{f: W_H(\lambda)\to \boldsymbol{C}\mid f(wv)=\varepsilon^{(i,\gamma)}(w^{-1})f(v),\ w\in W^{(i,\gamma)},\ v\in W_H(\lambda)\},$$

with the action of $W_H(\lambda)$ being the right translation. We define a linear map $\mathcal{Y}$ from $\mathfrak{V}^{(i,\gamma)}(H;\lambda)$ to E by

$$(\mathcal{Y}\zeta_{\sigma\gamma u})(v)=\begin{cases}\varepsilon(\sigma, a_i)\xi^i(\sigma)\varepsilon^{(i,\gamma)}(uv^{-1}) & \text{if } uv^{-1}\in W^{(i,\gamma)},\\ 0 & \text{otherwise,}\end{cases}$$

where $\sigma\in W(H_i)$, $\gamma\in\Gamma$ and $u, v\in W_H(\lambda)$. This linear map $\mathcal{Y}$ gives equivalence of $\mathfrak{V}^{(i,\gamma)}(H;\lambda)$ and E. Indeed, for $w\in W^{(i,\gamma)}$,

$$\begin{aligned}(\mathcal{Y}\zeta_{\sigma\gamma u})(wv)&=\varepsilon(\sigma, a_i)\xi^i(\sigma)\varepsilon^{(i,\gamma)}(uv^{-1}w^{-1})\\&=\varepsilon^{(i,\gamma)}(w^{-1})(\varepsilon(\sigma, a_i)\xi^i(\sigma)\varepsilon^{(i,\gamma)}(uv^{-1}))\\&=\varepsilon^{(i,\gamma)}(w^{-1})(\mathcal{Y}\zeta_{\sigma\gamma u})(v),\end{aligned}$$

if $uv^{-1}\in W^{(i,\gamma)}$. This proves $\mathcal{Y}\zeta_{\sigma\gamma u}$ belongs to E. Recall that, for $s\in W_H(\lambda)$, we defined $\mathcal{R}_s$ by $\mathcal{R}_s\zeta_t=\zeta_{ts^{-1}}$. Therefore we have

$$
\begin{aligned}
(\mathcal{Y}\mathcal{R}_s\zeta_{\sigma\gamma u})(v) &= (\mathcal{Y}\zeta_{\sigma\gamma us^{-1}})(v) \\
&= \varepsilon(\sigma, a_i)\xi^i(\sigma)\varepsilon^{(i,\gamma)}(us^{-1}v^{-1}) \\
&= \varepsilon(\sigma, a_i)\xi^i(\sigma)\varepsilon^{(i,\gamma)}(u(vs)^{-1}) = (\mathcal{Y}\zeta_{\sigma\gamma u})(vs),
\end{aligned}
$$

if $us^{-1}v^{-1}$ belongs to $W^{(i,\gamma)}$. This proves $\mathcal{Y}$ intertwines $\mathcal{R}$ and the right translation. Since it is easy to see that $\mathcal{Y}$ is an isomorphism, we have proved (5.3). Formulas (5.1), (5.2) and (5.3) prove the theorem. Q. E. D.

Let λ be integral for G_C, i.e., λ is a differential of a character of H_C. Then by Lemma 3.2 it holds that $\widetilde{W}_H(\lambda)=W_H(\lambda)=W(\mathfrak{h}_C)$. As in § 4, we identify all the $W(\mathfrak{h}_C)$'s by Cayley transforms and write it W. In this situation we get the representation τ of W on $V(\lambda)=\sum^{\oplus} V_H(\lambda)$.

THEOREM 5.2. *If λ is integral for G_C, W-module $V(\lambda)$ is decomposed as follows:*

$$
V(\lambda) \cong \sum_{[H]\in \mathrm{Car}(G)}^{\oplus} \sum_{i=0}^{l} {}^{\oplus} \mathrm{Ind}(\varepsilon_i; W(H_i)\uparrow W).
$$

Here $\{H_i \mid 0\leqq i\leqq l\}$ is a complete system of representatives of connected components of H under the conjugation of $W_G(H)$, and ε_i is a character of $W(H_i)$ defined by $\varepsilon_i(w)=\varepsilon(w, a_i)\xi^i(w)$ $(w\in W(H_i),\ a_i\in H_i)$.

PROOF. Since $\widetilde{W}_H(\lambda)=W_H(\lambda)\cong W$, the coset space $W(H_i)\backslash\widetilde{W}_H(\lambda)/W_H(\lambda)$ consists of one element. So we can take $\Gamma=\{e\}$. Now, applying Theorem 5.1, we get Theorem 5.2. Q. E. D.

Theorem 5.2 is a generalization of a result of D. Barbasch and D. Vogan [**1**, Prop. 2.4].

§ 6. Examples.

6.1. Let $G=U(p, 1)$ $(p\geqq 2)$. For classification of irreducible representations and their characters of $U(p, 1)$, see [**9**]. G has two conjugacy classes of Cartan subgroups, namely a class of a compact Cartan subgroup H and that of a maximal split one J. In this case, both H and J are connected. We give H and $\mathfrak{h}$ as

$$
H=\{\mathrm{diag}(a_1, \cdots, a_{p+1}) \mid a_i\in \boldsymbol{C},\ |a_i|=1\},
$$

$$
\mathfrak{h}=\{\mathrm{diag}(\sqrt{-1}\phi_1, \cdots, \sqrt{-1}\phi_{p+1}) \mid \phi_i\in \boldsymbol{R}\},
$$

where $\mathrm{diag}(a_1, \cdots, a_{p+1})$ denotes a diagonal matrix with diagonal elements $a_1, \cdots, a_{p+1}$. We consider $\lambda=(\lambda_1, \cdots, \lambda_{p+1})\in \boldsymbol{C}^{p+1}$ as an element of $\mathfrak{h}_C^*$ by

$$
\lambda(\mathrm{diag}(\sqrt{-1}\phi_1, \cdots, \sqrt{-1}\phi_{p+1}))=\sum_{i=1}^{p+1}\sqrt{-1}\phi_i\lambda_i.
$$

Fix $\lambda\in \boldsymbol{Z}^{p+1}$. Then we have $\widetilde{W}_H(\lambda)=W_H(\lambda)\cong W$ (the full Weyl group).

LEMMA 6.1. (1) *$W_G(H)=\{$permutations of $(\phi_1, \cdots, \phi_p)\} \cong \mathfrak{S}_p$.* (2) *$W_G(J)=\{$permutations of $(\phi_1, \cdots, \phi_{p-1})\} \times \{$permutations of $(\phi_p+\phi_{p+1}, \phi_p-\phi_{p+1})\} \cong \mathfrak{S}_{p-1}\times\mathfrak{S}_2$.*

PROOF. This is given by direct calculations.

LEMMA 6.2. *Let $\lambda\in \boldsymbol{Z}^{p+1}$ be regular, i.e., λ is not fixed by any permutation of coordinates. Then the Weyl group W is isomorphic to $\mathfrak{S}_{p+1}$, and as W-module we have*

$$V_H(\lambda)\cong \mathrm{Ind}(\det_p ; \mathfrak{S}_p \uparrow \mathfrak{S}_{p+1}) \cong [1^{p+1}]\oplus[2\cdot 1^{p-1}],$$

$$V_J(\lambda)\cong \mathrm{Ind}(\det_{p-1}\otimes(\mathrm{trivial}) ; \mathfrak{S}_{p-1}\times\mathfrak{S}_2 \uparrow \mathfrak{S}_{p+1})$$
$$\cong[2\cdot 1^{p-1}]\oplus[3\cdot 1^{p-2}],$$

where $\det_p$ is a one dimensional representation of $\mathfrak{S}_p$ which sends $\sigma\in\mathfrak{S}_p$ to determinant of σ.

For notations, see [14]. The irreducible representations corresponding to Young tableaux $[1^{p+1}]$, $[2\cdot 1^{p-1}]$ and $[3\cdot 1^{p-2}]$ are of dimension 1, p and $p(p-1)/2$ respectively.

PROOF. Use Theorem 5.2 and we have the first equivalence for each H and J. The second equivalences are given by direct calculations. Q. E. D.

It can be easily seen that the vector space $V_J(\lambda)$ has a basis consisting of characters of all the principal series representations with infinitesimal character λ. However it's not trivial to find out a basis of $V_H(\lambda)$. We only show the results here without calculations. For notations, see [9].

LEMMA 6.3. (1) *$V_H(\lambda)$ has a basis $\{B^{i,i+1}(1\leqq i\leqq p), D^{0,p+1}\}$, where*

$$B^{i,i+1}=\frac{(-1)^i}{2}\{(D^{0,i}+D^{0,i+1})+(-1)^p(D^{i,p+1}+D^{i+1,p+1})\}-\frac{(-1)^p}{p+1}D^{0,p+1}.$$

Moreover $\{B^{i,i+1}(1\leqq i\leqq p)\}$ generates the p-dimensional invariant space $[2\cdot 1^{p-1}]$ of $V_H(\lambda)$ and $D^{0,p+1}$ generates the one-dimensional invariant space $[1^{p+1}]$ of $V_H(\lambda)$.

(2) *$V_J(\lambda)$ has a basis consisting of the characters of principal series representations and $\dim V_J(\lambda)=p(p+1)/2$.*

6.2. Let $G=SL(2, \boldsymbol{R})$. For classification of irreducible representations and their characters of $SL(2, \boldsymbol{R})$, see [2], [4], [7, p. 50]. G has two conjugacy classes of Cartan subgroups. Put

$$K=\left\{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \;\middle|\; \theta\in\boldsymbol{R}\right\},$$

$$J=A_+\cup A_- \quad \text{with} \quad A_\pm=\left\{\pm\begin{pmatrix}e^t & 0\\ 0 & e^{-t}\end{pmatrix} \;\middle|\; t\in\boldsymbol{R}\right\}.$$

Then K is a compact Cartan subgroup and J is a maximal split one.

LEMMA 6.4. (1) *K is connected and $W_G(K)=\{e\}$.* (2) *J has two connected component A_+, A_- and $W_G(A_+)=W_G(A_-)\cong\mathfrak{S}_2$.*

Let $\lambda\in\mathfrak{k}_C^*$ be a differential of a non-trivial character of K. Then we have

LEMMA 6.5. *The Weyl group W is isomorphic to $\mathfrak{S}_2$ and as W-module, we have*

$$V_K(\lambda)\cong \mathrm{Ind}(\mathrm{trivial}\,;\,\{e\}\uparrow\mathfrak{S}_2)\cong(\mathrm{sgn})\oplus(\mathrm{trivial}),$$

$$V_J(\lambda)\cong \mathrm{Ind}(\mathrm{trivial}\,;\,\mathfrak{S}_2\uparrow\mathfrak{S}_2)\oplus\mathrm{Ind}(\mathrm{trivial}\,;\,\mathfrak{S}_2\uparrow\mathfrak{S}_2)$$
$$\equiv 2(\mathrm{trivial}).$$

PROOF. Theorem 5.2 and direct calculations will show the results. Q.E.D.

As in the case of $U(p, 1)$, $V_J(\lambda)$ has a basis consisting of characters of principal series representations with infinitesimal character λ. The invariant space $V_K(\lambda)$ has a basis $\{D^+-D^-, F\}$, where D^+ (respectively D^-) is the holomorphic (resp. anti-holomorphic) discrete series representation and F is the finite dimensional representation.

We can write down the similar results for the groups $SO_0(2n, 1)$ $(n\geqq 1)$. However, it needs new notations to state them. Here, we only refer the readers to [**16**].

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