J. Math. Soc. Japan
Vol. 37, No. 4, 1985

Asymptotic properties of asymptotically homogeneous diffusion processes on a compact manifold

By Osamu TAKEYAMA

(Received July 9, 1984)

§1. Introduction.

The purpose of this paper is to formulate a class of time inhomogeneous diffusion processes and to investigate the asymptotic properties of such processes. Let $\{\xi(t), P_{s,x}\}$ be a time inhomogeneous diffusion process on a manifold M generated by a smooth differential operator

$$L_t=\frac{1}{2}a^{ij}(t, x)\frac{\partial^2}{\partial x^i\partial x^j}+b^i(t, x)\frac{\partial}{\partial x^i}$$

and let $\{\lambda(t), P_x\}$ be a homogeneous diffusion process on M generated by a smooth differential operator

$$L=\frac{1}{2}a^{ij}(x)\frac{\partial^2}{\partial x^i\partial x^j}+b^i(x)\frac{\partial}{\partial x^i}.$$

(Throughout this paper we use the usual summation convention.) Here, $P_{s,x}$ is the probability law governing sample paths $\xi(t)$, $t\geqq s$, starting at x at time s and P_x is that of $\lambda(t)$, $t\geqq 0$, starting at x at time 0. If $L_t\phi\rightarrow L\phi$ uniformly on any compact set on M as $t\rightarrow\infty$ for every smooth function ϕ on M, we shall call $\{\xi(t), P_{s,x}\}$ *asymptotically homogeneous* with the limiting homogeneous process $\{\lambda(t), P_x\}$.

Such a situation was studied by Bhattacharya and Ramasubramanian [2]. They showed that if $\{\lambda(t), P_x\}$ is positively recurrent with the invariant probability measure m, then, under additional assumptions on the process, the law of the shifted process $t\mapsto\xi_s^+(t)=\xi(t+s)$ under $P_{0,x}$ converges to that of $t\mapsto\lambda(t)$ under $P_m=\int_M P_x m(dx)$ as $s\rightarrow\infty$ for every $x\in M$. Conversely, if this convergence holds then $\{\xi(t), P_{s,x}\}$ must be asymptotically homogeneous. Thus we may expect that if an inhomogeneous diffusion $\{\xi(t), P_{s,x}\}$ is asymptotically homogeneous with the limiting homogeneous diffusion process $\{\lambda(t), P_x\}$, the asymptotic properties of the process $\xi(t)$ can be stated in terms of the process $\lambda(t)$. In this paper, we discuss the asymptotics of the occupation distribution (the empirical distribution) for an asymptotically homogeneous diffusion process.

In §2, we obtain some preliminary general results on inhomogeneous diffusion processes. If a general inhomogeneous diffusion operator L_t is smooth, L_t-diffusion process $\{x(t), P_{s,x}\}$ exists uniquely. This can be shown, locally in each coordinate chart, by the method of stochastic differential equations and hence, by a standard argument of piecing out, the global solution of L_t-martingale problem exists uniquely. Here, we further assume that the differential operator L_t is nondegenerate. Then, we can give a more intrinsic construction of an L_t-diffusion process by generalizing the method of stochastic moving frame of Eells and Elworthy (cf. Ikeda and Watanabe [5]) to the time dependent case. This time dependent stochastic moving frame provides us with a time inhomogeneous diffusion process $r(t)$ on the principal frame bundle $GL(M)$ of M, and an L_t-diffusion process $x(t)$ is obtained by the projection of $r(t)$ onto M. Also we obtain the Cameron-Martin-Girsanov theorem for $x(t)$ by making use of the stochastic moving frame $r(t)$.

Assume, for simplicity, that the manifold M is compact. In § 3, we discuss the strong law of large numbers for the asymptotically homogeneous diffusion process $\xi(t)$, i.e., almost sure convergence of the occupation distribution (the empirical distribution) of $\xi(t)$ and the characterization of this limit in terms of the limiting homogeneous diffusion process $\lambda(t)$. The results obtained there apply to many degenerate diffusions. By restricting to nondegenerate diffusions, however, the strong law of large numbers will be refined in two directions. In § 4, we discuss the functional central limit theorem and in § 5, we discuss theorems of large deviations. The variance of the limiting Wiener process is determined by the limiting process $\lambda(t)$ and the rate of the large deviation is measured by the I-functions of $\lambda(t)$. In these two sections the method of the time dependent stochastic moving frame discussed in § 2 will play an important role.

§ 2. Inhomogeneous diffusion processes on a manifold.

The unique existence of a diffusion process generated by a time inhomogeneous diffusion operator L_t is well known. The existence and uniqueness of local solutions in each coordinate chart can be shown by using stochastic differential equations (SDE's) and these can be united to form the unique solution of L_t-martingale problem. When the differential operator L_t is nondegenerate, we can give the following intrinsic construction of the L_t-diffusion. In this case, the inverse matrix of its diffusion matrix defines a time dependent Riemannian metric $g_t(x)$ on M and so, without loss of generality, we may assume that $L_t=\Delta_t/2+b_t$ where Δ_t is the Laplace-Beltrami operator induced by $g_t(x)$, and $b_t(x)$ is a time dependent smooth vector field.

Let us introduce a frame $r=(x, e)$ at $x\in M$ where $e=\{e_1, \cdots, e_d\}$ is a basis of the tangent space T_xM at x. We denote by $GL(M)$ the frame bundle consisting of all frames on M with the usual structure of manifold and by $O_t(M)$, for

$t\geqq 0$, we denote a submanifold of $GL(M)$ defined by $O_t(M)=\{r=(x, e)\in GL(M);$ e is an orthonormal basis of T_xM with respect to the inner product $\langle\ ,\ \rangle_t$ induced by $g_t(x)\}$. In the local coordinate (x^i, e_j^i) of $GL(M)$, $r=(x, e)\in O_t(M)$ if and only if $g_{p,q}(t, x)e_i^p e_j^q=\delta_{ij}$ where $\{g_{ij}(t, x)\}$ is the components of $g_t(x)$. The time dependent Christoffel symbols $\{\Gamma_{j,k}^i(t, x)\}$ are defined by

$$\Gamma_{j,k}^i(t, x)=\frac{1}{2}g^{ih}(t, x)\times\left\{\frac{\partial g_{hk}}{\partial x^j}(t, x)+\frac{\partial g_{hj}}{\partial x^k}(t, x)-\frac{\partial g_{jk}}{\partial x^h}(t, x)\right\}$$

where $\{g^{ij}(t, x)\}$ is the inverse matrix of $\{g_{ij}(t, x)\}$.

We define smooth time dependent vector fields on $GL(M)$, in each coordinate chart, by

$$L_\alpha(t, r)=e_\alpha^i\frac{\partial}{\partial x^i}-\Gamma_{k,l}^i(t, x)e_j^k e_\alpha^l\frac{\partial}{\partial e_j^i}, \qquad \alpha=1, 2, \cdots, d,$$

$$\bar{b}(t, r)=b^i(t, x)\frac{\partial}{\partial x^i}-\Gamma_{k,l}^i(t, x)e_j^k b^l(t, x)\frac{\partial}{\partial e_j^i},$$

$$K(t, r)=-\frac{1}{2}g^{ik}(t, x)\frac{\partial g_{kl}}{\partial t}(t, x)e_j^l\frac{\partial}{\partial e_j^i}.$$

The fact that these define vector fields on $GL(M)$ is easy to verify. Consider the following SDE on $GL(M)$

$$dr(t)=L_\alpha(t, r(t))\circ dw^\alpha(t)+\bar{b}(t, r(t))dt+K(t, r(t))dt, \tag{2.1}$$

where $w(t)=(w^\alpha(t))$ is a d-dimensional Wiener process and $\circ$ denotes the Stratnovich stochastic differential (cf. Ikeda and Watanabe [5]). In local coordinates, (2.1) is equivalent to

$$\begin{cases} dx^i(t)=e_\alpha^i(t)\circ dw^\alpha(t)+b^i(t, x(t))dt, \\ de_j^i(t)=-\Gamma_{k,l}^i(t, x(t))e_j^k(t)\circ dx^l(t) \\ \qquad\qquad -\dfrac{1}{2}g^{ik}(t, x(t))\dfrac{\partial g_{kl}}{\partial t}(t, x(t))e_j^l(t)dt. \end{cases} \tag{2.2}$$

Although the following two theorems hold under the condition that diffusion processes do not explode in the general manifold M, we shall assume, for simplicity, M to be compact.

THEOREM 1. *Let $r(t)=(x(t), e(t))$ be the solution of SDE* (2.1) *on $GL(M)$ with initial condition $r(s)=(x, e)\in O_s(M)$ at time s. Then, $r(t)$ lies on $O_t(M)$ for all $t\geqq s$ and its projection $x(t)$ onto M defines an inhomogeneous diffusion process whose generator is $L_t=\Delta_t/2+b_t$.*

PROOF. By using (2.2) and $g_{p,q}(t, x)=g_{q,p}(t, x)$, we have from stochastic differential rules that

$$
\begin{aligned}
&d(g_{p,q}(t, x(t))e_i^p(t)e_j^q(t)) \\
&=\frac{\partial g_{p,q}}{\partial t}(t, x(t))e_i^p(t)e_j^q(t)dt+\frac{\partial g_{p,q}}{\partial x^k}(t, x(t))e_i^p(t)e_j^q(t)\circ dx^k(t) \\
&\quad +g_{p,q}(t, x(t))e_j^q(t)\circ de_i^p(t)+g_{p,q}(t, x(t))e_i^p(t)\circ de_j^q(t) \\
&=\frac{\partial g_{p,q}}{\partial t}(t, x(t))e_i^p(t)e_j^q(t)dt+\frac{\partial g_{p,q}}{\partial x^k}(t, x(t))e_i^p(t)e_j^q(t)\circ dx^k(t) \\
&\quad -g_{p,q}(t, x(t))\Gamma^p_{k,l}(t, x(t))(e_j^q(t)e_i^k(t)+e_i^q(t)e_j^k(t))\circ dx^l(t) \\
&\quad -\frac{1}{2}g_{p,q}(t, x(t))g^{p,k}(t, x(t))\frac{\partial g_{kl}}{\partial t}(t, x(t))e_i^l(t)e_j^q(t)dt \\
&\quad -\frac{1}{2}g_{p,q}(t, x(t))g^{q,k}(t, x(t))\frac{\partial g_{kl}}{\partial t}(t, x(t))e_j^l(t)e_i^p(t)dt \\
&=\frac{\partial g_{p,q}}{\partial x^k}(t, x(t))e_i^p(t)e_j^q(t)\circ dx^k(t) \\
&\quad -(g_{p,q}\Gamma^p_{k,l}+g_{p,k}\Gamma^p_{q,l})(t, x(t))e_i^q(t)e_j^k(t)\circ dx^l(t) \\
&=\frac{\partial g_{p,q}}{\partial x^k}(t, x(t))e_i^p(t)e_j^q(t)\circ dx^k(t)-\frac{\partial g_{q,k}}{\partial x^l}(t, x(t))e_i^q(t)e_j^k(t)\circ dx^l(t) \\
&=0.
\end{aligned}
$$

Here, we used the following relation

$$(g_{p,q}\Gamma^p_{k,l}+g_{p,k}\Gamma^p_{q,l})(t, x)=\frac{\partial g_{q,k}}{\partial x^l}(t, x).$$

Hence, tangent vectors defined by $e_i(t)=e_i^k(t)\partial/\partial x^k$, $i=1, 2, \cdots, d$, on M at $x(t)$ satisfy

$$\langle e_i(t), e_j(t)\rangle_t=\langle e_i(s), e_j(s)\rangle_s=\delta_{ij}, \qquad t\geqq s.$$

This implies that $r(t)=(x(t), e(t))\in O_t(M)$ for $t\geqq s$.

To prove that $x(t)$ is an inhomogeneous diffusion process on M with generator L_t, we first remark that the law of $x(t)$, $t\geqq s$ with $r(s)=(x, e)$, is uniquely determined by x and is independent of the choice of orthonormal frame e. This can be shown by the same argument as in [5] p. 269. Hence, $x(t)$ is an inhomogeneous diffusion process on M. To see that its generator is L_t it is sufficient to show that, for any smooth function f on M,

$$df(x(t))=e_\alpha^i(t)\frac{\partial f}{\partial x^i}(x(t))\cdot dw^\alpha(t)+L_t f(x(t))dt,$$

where $\cdot$ denotes the Ito stochastic differential. Ito's formula says that

$$
\begin{aligned}
df(x(t))&=\frac{\partial f}{\partial x^i}(x(t))\circ dx^i(t) \\
&=\frac{\partial f}{\partial x^i}(x(t))e_\alpha^i(t)\circ dw^\alpha(t)+\frac{\partial f}{\partial x^i}(x(t))b^i(t, x(t))dt
\end{aligned}
$$

$$=\frac{\partial f}{\partial x^i}(x(t))e^i_\alpha(t)\cdot dw^\alpha(t)+\frac{1}{2}d\Big(\frac{\partial f}{\partial x^i}(x(t))e^i_\alpha(t)\Big)\cdot dw^\alpha(t)$$
$$+\frac{\partial f}{\partial x^i}(x(t))b^i(t,\ x(t))dt$$

By stochastic differential rules, for example $dw^\alpha(t)\cdot dw^\beta(t)=\delta_{\alpha\beta}dt$ and $dw^\alpha(t)\cdot dt=0$, we see from (2.2) that $dx^j(t)\cdot dw^\alpha(t)=e^j_\alpha(t)dt$. Using this,

$$d\Big(\frac{\partial f}{\partial x^i}(x(t))e^i_\alpha(t)\Big)\cdot dw^\alpha(t)$$
$$=\frac{\partial^2 f}{\partial x^j\partial x^i}(x(t))e^i_\alpha(t)\circ dx^j(t)\cdot dw^\alpha(t)+\frac{\partial f}{\partial x^i}(x(t))\circ de^i_\alpha(t)\cdot dw^\alpha(t)$$
$$=\sum_{\alpha=1}^d \frac{\partial^2 f}{\partial x^j\partial x^i}(x(t))e^i_\alpha(t)e^j_\alpha(t)dt$$
$$+\frac{\partial f}{\partial x^i}(x(t))\Big\{-\Gamma^i_{kl}(t,\ x(t))e^k_\alpha(t)\circ dx^l(t)$$
$$-\frac{1}{2}g^{ik}(t,\ x(t))\frac{\partial g_{kl}}{\partial t}(t,\ x(t))e^l_\alpha(t)dt\Big\}\cdot dw^\alpha(t)$$
$$=\sum_{\alpha=1}^d \frac{\partial^2 f}{\partial x^j\partial x^i}(x(t))e^i_\alpha(t)e^j_\alpha(t)dt-\sum_{\alpha=1}^d\frac{\partial f}{\partial x^i}(x(t))\Gamma^i_{k,l}(t,\ x(t))e^k_\alpha(t)e^l_\alpha(t)dt$$
$$=\sum_{\alpha=1}^d e^k_\alpha(t)e^l_\alpha(t)\Big\{\frac{\partial^2 f}{\partial x^k\partial x^l}(x(t))-\Gamma^i_{k,l}(t,\ x(t))\frac{\partial f}{\partial x^i}(x(t))\Big\}dt.$$

It is easy to see from $r(t)\in O_t(M)$ or $g_{k,l}(t,\ x(t))e^k_\alpha(t)e^l_\beta(t)=\delta_{\alpha\beta}$ that $\sum_{\alpha=1}^d e^k_\alpha(t)e^l_\alpha(t)=g^{kl}(t,\ x(t))$. Hence,

$$d\Big(\frac{\partial f}{\partial x^i}(x(t))e^i_\alpha(t)\Big)\cdot dw^\alpha(t)$$
$$=g^{kl}(t,\ x(t))\Big\{\frac{\partial^2 f}{\partial x^k\partial x^l}(x(t))-\Gamma^i_{k,l}(t,\ x(t))\frac{\partial f}{\partial x^i}(x(t))\Big\}dt$$
$$=\Delta_t f(x(t))dt.$$

This completes the proof.

Now we shall present the Cameron-Martin-Girsanov formula for inhomogeneous diffusion processes on a manifold. Our reason for doing so at this point is that it will play an important role in our study of large deviations in §5.

THEOREM 2. *Let $P_{s,x}$ and $\tilde{P}_{s,x}$ be probability laws starting at x at time s of inhomogeneous diffusion processes on M generated by $L_t=\Delta_t/2+b_t$ and $\tilde{L}_t=\Delta_t/2+b_t+c_t$ respectively, where c_t is another time dependent smooth vector field. Then, $P_{s,x}$ and $\tilde{P}_{s,x}$ are mutually absolutely continuous with the following density on $\mathcal{F}^t_s=\sigma(x(\tau)\,;\ s\leqq\tau\leqq t)$*

$$\left.\frac{d\tilde{P}_{s,x}}{dP_{s,x}}\right|_{\mathcal{F}_s^t}=\exp\left\{\int_s^t\langle c(\tau,\ x(\tau)),\ e_\alpha(\tau)\rangle_\tau\cdot dw^\alpha(\tau)\right.$$
$$\left.-\frac{1}{2}\int_s^t\|c(\tau,\ x(\tau))\|_\tau^2 d\tau\right\}$$

where $r(t)=(x(t),\ e(t))$, *constructed in Theorem* 1, *is the inhomogeneous diffusion process on the principal frame bundle* $GL(M)$ *generated by* $A(t, r)=(1/2)\sum_{\alpha=1}^d L_\alpha(t, r)^2+\bar{b}(t, r)+K(t, r)$, *and* $\|\ \|_\tau$ *denotes a Riemannian norm induced by* $g_\tau(x)$.

PROOF.

$$N_s(t)=\sum_{\alpha=1}^d\int_s^t\langle c(\tau,\ x(\tau)),\ e_\alpha(\tau)\rangle_\tau\cdot dw^\alpha(\tau)$$

is a continuous square integrable $P_{s,x}$-martingale and its quadratic variational process is, from the fact that $r(t)=(x(t),\ e(t))\in O_t(M)$ for $t\geqq s$,

$$\langle N_s\rangle(t)=\int_s^t\sum_{\alpha=1}^d\langle c(\tau,\ x(\tau)),\ e_\alpha(\tau)\rangle_\tau^2 du=\int_s^t\|c(\tau,\ x(\tau))\|_\tau^2 d\tau .$$

Set $M_s(t)=\exp\{N_s(t)-(1/2)\langle N_s\rangle(t)\}$. By the general theory of exponential martingales [5] p. 140, we easily see that $M_s(t)$ is $P_{s,x}$-martingale. Let $\hat{P}$ be a probability measure defined by $\hat{P}(F)=\int_F M_s(t)dP_{s,x}$ for $F\in\mathcal{F}_s^t$. We shall show that $\hat{P}=\tilde{P}_{s,x}$.

Transform the d-dimensional Wiener process $w(t)=(w^\alpha(t))$ with respect to $P_{s,x}$ into

$$\tilde{w}^\alpha(t)=w^\alpha(t)-\langle w^\alpha,\ N_s\rangle(t)=w^\alpha(t)-\int_s^t\langle c(\tau,\ x(\tau)),\ e_\alpha(\tau)\rangle_\tau d\tau .$$

Since $\tilde{w}^\alpha(t)$, $\alpha=1, 2, \cdots, d$, are $\hat{P}$-martingales and their quadratic variational processes satisfy $\langle\tilde{w}^\alpha,\ \tilde{w}^\beta\rangle(t)=\langle w^\alpha,\ w^\beta\rangle(t)=\delta_{\alpha\beta}(t-s)$, $\tilde{w}(t)=(\tilde{w}^\alpha(t))$ is a d-dimensional Wiener process with respect to $\hat{P}$. From (2.1),

$$\begin{aligned}dr(t)=&\sum_{\alpha=1}^d L_\alpha(t,\ r(t))\circ\{dw^\alpha(t)-\langle c(t,\ x(t)),\ e_\alpha(t)\rangle_t dt\}\\&+\sum_{\alpha=1}^a L_\alpha(t,\ r(t))\langle c(t,\ x(t)),\ e_\alpha(t)\rangle_t dt\\&+\bar{b}(t,\ r(t))dt+K(t,\ r(t))dt .\end{aligned}$$

Let $\bar{c}(t, r)$ be a time dependent smooth vector field on $GL(M)$ defined by

$$\bar{c}(t,\ r)=c^i(t,\ x)\frac{\partial}{\partial x^i}-\Gamma_{k,l}^i(t,\ x)e_j^k c^l(t,\ x)\frac{\partial}{\partial e_j^i}$$

in local coordinates. Since $r(t)=(x(t),\ e(t))\in O_t(M)$, it is easy to see that

$$\sum_{\alpha=1}^d L_\alpha(t,\ r(t))\langle c(t,\ x(t)),\ e_\alpha(t)\rangle_t=\bar{c}(t,\ r(t)) .$$

Then,

$$dr(t)=L_\alpha(t,\ r(t))\circ d\tilde{w}^\alpha(t)+\bar{b}(t,\ r(t))dt+\bar{c}(t,\ r(t))dt+K(t,\ r(t))dt .$$

The path $r(t)$ is, therefore, an inhomogeneous diffusion process on $GL(M)$ generated by

$$\tilde{A}(t, r)=\frac{1}{2}\sum_{\alpha=1}^{d} L_{\alpha}(t, r)^2+\bar{b}(t, r)+\bar{c}(t, r)+K(t, r)$$

with respect to $\hat{P}$ and hence its projection $x(t)$ onto M is an inhomogeneous diffusion process generated by $\tilde{L}_t$ with respect to $\hat{P}$.

§ 3. Strong law of large numbers.

Let M be a smooth manifold which we assume, for simplicity, to be compact, although our results hold in more general cases under necessary modifications. Let

$$L_t=\frac{1}{2}a^{ij}(t, x)\frac{\partial^2}{\partial x^i \partial x^j}+b^i(t, x)\frac{\partial}{\partial x^i}$$

and

$$L=\frac{1}{2}a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j}+b^i(x)\frac{\partial}{\partial x^i}$$

be diffusion operators ($a^{ij}(t, x)$ and $a^{ij}(x)$ may degenerate) with smooth coefficients. Consider a time inhomogeneous diffusion process $\{\xi(t), P_{s,x}\}$ generated by L_t and a homogeneous diffusion process $\{\lambda(t), P_x\}$ generated by L.

DEFINITION. We say that $\{\xi(t), P_{s,x}\}$ is *asymptotically homogeneous* with the limiting homogeneous diffusion process $\{\lambda(t), P_x\}$ if, for every smooth function ϕ on M, $L_t\phi(x)\to L\phi(x)$ uniformly in $x\in M$ as $t\to\infty$.

Let $l_{s,t}$ be, for each sample path $\xi(\tau)$, its occupation distribution defined by $\frac{1}{t-s}\int_s^t \delta_{\xi(\tau)}d\tau$ where δ_x is the unit mass at $x\in M$. Thus $l_{s,t}$ is a process taking values in $\mathcal{M}(M)$=the totality of probability measures on M. We endow $\mathcal{M}(M)$ with the topology of the weak convergence. $m\in\mathcal{M}(M)$ is called an invariant probability measure of the homogeneous diffusion process $\{\lambda(t), P_x\}$ if

$$\int_M E_x(f(\lambda(t)))m(dx)=\int_M f(x)m(dx)$$

for every continuous function f on M where E_x is the expectation with respect to P_x.

THEOREM 3. *Let $\{\xi(t), P_{s,x}\}$ be an inhomogeneous diffusion process which is asymptotically homogeneous with the limiting homogeneous diffusion process $\{\lambda(t), P_x\}$. Let $l_{s,t}$ be the occupation distribution of the process $\xi(t)$. Then, with $P_{s,x}$-probability one for every $s\geqq 0$ and $x\in M$,*

$$\operatorname*{limit\,point}_{t\to\infty}\{l_{s,t}\}\subset\{\textit{invariant probability measures of } \lambda(t)\}.$$

COROLLARY. *If $\lambda(t)$ is ergodic in the sense that it possesses the unique invariant* *measure m, then, with $P_{s,x}$-probability one for every $s \geqq 0$ and $x \in M$,*

$$\lim_{t\to\infty} l_{s,x} = m$$

or equivalently

$$\lim_{t\to\infty} \frac{1}{t-s}\int_s^t f(\xi(\tau))d\tau = \int_M f(x)m(dx)$$

for every continuous function f on M with $P_{s,x}$-probability one and this limit also coincides with $\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\lambda(\tau))d\tau$ with P_x-probability one.

PROOF. Let $p(s, x, t, dy)$ and $p(t, x, dy)$ be the transition probability functions of $\xi(t)$ and $\lambda(t)$. The associated transition operators are defined by $T_{s,t}f(x) = \int_M p(s, x, t, dy)f(y)$ and $T_t f(x) = \int_M p(t, x, dy)f(y)$ respectively. We shall prove, for simplicity, in the case where $s=0$. It is sufficient to show that if μ is a typical limit point of $l_{s,t}$ as $t\to\infty$ then $\int_M T_u f(x)\mu(dx) = \int_M f(x)\mu(dx)$ for every given $u>0$ and given continuous function f on M, because the usual argument should save some trouble about the exceptional set with $P_{0,x}$-probability one. Since we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\xi(\tau))d\tau = \lim_{t\to\infty}\int_M f(x)l_{0,t}(dx) = \int_M f(x)\mu(dx)$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t T_u f(\xi(\tau))d\tau = \lim_{t\to\infty}\int_M T_u f(x)l_{0,t}(dx) = \int_M T_u f(x)\mu(dx),$$

we shall show that, with $P_{0,x}$-probability one,

$$\frac{1}{t}\int_0^t \{T_u f(\xi(\tau)) - f(\xi(\tau))\}\, d\tau \to 0 \qquad \text{as} \quad t\to\infty.$$

Divide this into the following three parts:

$$\frac{1}{t}\int_0^t \{T_u f(\xi(\tau)) - T_{\tau,\tau+u} f(\xi(\tau))\}\, d\tau + \frac{1}{t}\int_0^t \{T_{\tau,\tau+u} f(\xi(\tau)) - f(\xi(\tau+u))\}\, d\tau$$
$$+\frac{1}{t}\int_0^t \{f(\xi(\tau+u)) - f(\xi(\tau))\}\, d\tau.$$

Notice that $T_{\tau,\tau+u}f(x) \to T_u f(x)$ uniformly in $x\in M$ as $\tau\to\infty$ (see Strook and Varadhan [7] p. 272). Hence, the first part is negligible. We can easily see that the absolute value of the third part is dominated by $(2u/t)\cdot\sup_{x\in M}|f(x)|$.

To prove that the second part tends to 0, set

$$M_t = \int_1^t \frac{1}{\tau}\{T_{\tau,\tau+u} f(\xi(\tau)) - f(\xi(\tau+u))\}\, d\tau$$

and show that it converges with $P_{0,x}$-probability one as $t\to\infty$. Then Kronecker's lemma can be applied to complete the proof. We denote by $E(\ )$ and $E(\ |\mathcal{F}_t)$ the expectation and the conditional expectation with respect to $P_{0,x}$ and

$\mathcal{F}_t=\sigma(\xi(\tau)\,;\,0\leqq\tau\leqq t)$. We have by the Markov property that, for $t_2>t_1\geqq 1$,

$$E(M_{t_2}-M_{t_1}|\mathcal{F}_{t_1})=\int_{t_1}^{t_2}\frac{1}{\tau}E(E(T_{\tau,\tau+u}f(\xi(\tau))-f(\xi(\tau+u)))|\mathcal{F}_\tau)|\mathcal{F}_{t_1})d\tau=0,$$

$$\begin{aligned}E(M_t^2)&=E\Big(\Big(\int_1^t\frac{1}{\tau}\{T_{\tau,\tau+u}f(\xi(\tau))-f(\xi(\tau+u))\}\,d\tau\Big)^2\Big)\\&=2\int_1^t ds\int_s^t d\tau\frac{1}{s\tau}E(\{T_{\tau,\tau+u}f(\xi(\tau))-f(\xi(\tau+u))\}\\&\qquad\times\{T_{s,s+u}f(\xi(s))-f(\xi(s+u))\})\\&=2\int_1^t ds\int_s^{s+u} d\tau\frac{1}{s\tau}E(\{T_{\tau,\tau+u}f(\xi(\tau))-f(\xi(\tau+u))\}\\&\qquad\times\{T_{s,s+u}f(\xi(s))-f(\xi(s+u))\})\\&\quad+2\int_1^t ds\int_{s+u}^{t} d\tau\frac{1}{s\tau}E(\{T_{\tau,\tau+u}f(\xi(\tau))-f(\xi(\tau+u))\}\\&\qquad\times\{T_{s,s+u}f(\xi(s))-f(\xi(s+u))\})\\&\leqq 8\sup_{x\in M}|f(x)|\int_1^t ds\frac{u}{s^2}<\infty.\end{aligned}$$

It is easy to see from these facts that $\tilde{M}_t=E(M_t|\mathcal{F}_t)$ is L^2-bounded $(\mathcal{F}_t)$-martingale and hence we can apply the martingale convergence theorem to show that $\tilde{M}_t$ converges with $P_{0,x}$-probability one. The convergence of M_t with $P_{0,x}$-probability one as $t\to\infty$ is, therefore, obtained from the fact that

$$|\tilde{M}_t-M_t|=\Big|\int_{t-u}^t\frac{1}{\tau}\{f(\xi(\tau+u))-E(f(\xi(\tau+u))|\mathcal{F}_t)\}\,d\tau\Big|\leqq\frac{2u}{t-u}\sup_{x\in M}|f(x)|.$$

§ 4. Central limit theorem.

Consider an asymptotically homogeneous nondegenerate diffusion process $\xi(t)$ with the limiting process $\lambda(t)$ on a manifold M which we assume to be compact, connected and without boundary although the following result can be carried over to the case of diffusion processes with reflection on a compact manifold with boundary. In this case, without loss of generality, we may assume that $\xi(t)$ and $\lambda(t)$ are generated by $L_t=\Delta_t/2+b_t$ and $L=\Delta/2+b$ where Δ_t and Δ are Laplace-Beltrami operators induced by a time dependent Riemannian metric $g_t(x)$ and a time independent Riemannian metric $g(x)$, b_t and b are a time dependent smooth vector field and a time independent smooth vector field on M respectively. The condition that $L_t\phi(x)\to L\phi(x)$ uniformly in $x\in M$ for any smooth function ϕ on M as $t\to\infty$ is assumed.

We will denote by $\langle\ ,\ \rangle_t$ and $\langle\ ,\ \rangle$ the inner products of vector fields ($\|v\|_t=\sqrt{\langle v,v\rangle_t}$ and $\|v\|=\sqrt{\langle v,v\rangle}$), by ∇_t and ∇ the Riemannian gradients, with respect to the Riemannian metrics $g_t(x)$ and $g(x)$ respectively. If we assume the convergence rate of $L_t\phi$, we can formulate a refinement of Theorem 3 in § 3.

THEOREM 4. *Suppose that there is some* $r<-1/2$ *such that* $\sup_{x\in M}|L_t\phi(x)-L\phi(x)|=O(t^r)$ *for every smooth function* ϕ *on* M. *Then, for any continuously differentiable function* f *on* M, *the family of stochastic processes* $\left\{\frac{1}{\sqrt{\lambda}}\int_0^{\lambda t}(f(\xi(s))-m(f))ds\,;\ t\geqq 0\right\}$ *converges in distribution as* $\lambda\to\infty$ *to a Wiener process* $W_{v(f)}$ *with zero drift and variance parameter* $v(f)=\int_M\|\nabla\chi\|^2dm$, *where* m *is the unique invariant measure of* $\lambda(t)$ *on* M, *i.e.,* $L^*m=0$, $m(f)=\int_M fdm$ *and* χ *is a solution of the equation* $L\chi=-\{f-m(f)\}$.

PROOF. By Fredholm alternative theorem, we can solve the following equation on M:

$$L\chi=-\{f-m(f)\}. \tag{4.1}$$

Applying Ito's formula to $\chi(x)$,

$$\chi(\xi(t))-\chi(\xi(0))-\int_0^t L_s\chi(\xi(s))ds=\int_0^t\langle\nabla_s\chi(\xi(s)),\ e_\alpha(s)\rangle_s\cdot dw^\alpha(s) \tag{4.2}$$

where $r(s)=(\xi(s),\ e(s))$ is an ihomogeneous diffusion process on the principal frame bundle $GL(M)$ over M, constructed in Theorem 1, satisfying $r(s)\in O_s(M)=\{r=(x,\ e)\in GL(M)\,;\ \langle e_i,\ e_j\rangle_s=\delta_{ij}\}$ for each $s\geqq 0$. Set

$$M(t)=\int_0^t\langle\nabla_s\chi(\xi(s)),\ e_\alpha(s)\rangle_s\cdot dw^\alpha(s).$$

From (4.1) and (4.2),

$$\frac{1}{\sqrt{\lambda}}\left\{\int_0^{\lambda t}(f(\xi(s))-m(f))ds\right\}=\frac{M(\lambda t)}{\sqrt{\lambda}}+\frac{\chi(\xi(0))-\chi(\xi(\lambda t))}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}\int_0^{\lambda t}(L_s\chi-L\chi)(\xi(s))ds.$$

For any $T>0$,

$$\sup_{0\leq t\leq T}\left|\frac{\chi(\xi(0))-\chi(\xi(\lambda t))}{\sqrt{\lambda}}\right|\to 0\qquad \text{a.s. as}\quad \lambda\to\infty$$

and also, by hypothesis,

$$\sup_{0\leq t\leq T}\left|\frac{1}{\sqrt{\lambda}}\int_1^{\lambda t}(L_s\chi-L\chi)(\xi(s))ds\right|\leqq\sup_{0\leq t\leq T}\left|\frac{1}{\sqrt{\lambda}}\int_1^{\lambda t}cs^rds\right|\to 0\qquad \text{as}\quad \lambda\to\infty.$$

For the family of the martingales $M_\lambda(t)=M(\lambda t)/\sqrt{\lambda}$, we have from Corollary of Theorem 3 in § 3 that

$$\langle M_\lambda\rangle(t)=\frac{1}{\lambda}\int_0^{\lambda t}\sum_{\alpha=1}^d\langle\nabla_s\chi(\xi(s)),\ e_\alpha(s)\rangle_s^2ds$$

$$=\frac{1}{\lambda}\int_0^{\lambda t}\|\nabla_s\chi\|_s^2(\xi(s))ds \quad\rightarrow\quad t\int_M\|\nabla\chi\|^2 dm.$$

We can now complete the proof by applying the theorem in Liptser and Shiryayev [**6**].

§ 5. Large deviations.

Consider the same situation as in § 3, and assume that an inhomogeneous diffusion process $\{\xi(t), P_{s,x}\}$ generated by L_t is asymptotically homogeneous with the limiting homogeneous diffusion process $\{\lambda(t), P_x\}$ generated by L. For each $s<t$, let $l_{s,t}$ be the occupation distribution of $\xi(t)$; $l_{s,t}=\frac{1}{t-s}\int_s^t \delta_{\xi(\tau)}d\tau$. Let $Q_{s,x}^t$ be the probability measure on $\mathcal{M}(M)$ induced by $l_{s,t}$ from $P_{s,x}$. We obtain asymptotic estimates of $Q_{s,x}^t(B)$ for $B\subset\mathcal{M}(M)$ from two directions. For this we introduce I-function of $\lambda(t)$ as in [**4**].

$$I(\mu)=-\inf_{\substack{u>0\\ u\in C^\infty(M)}}\int\frac{Lu}{u}d\mu, \qquad \mu\in\mathcal{M}(M).$$

First, we shall state the upper estimate.

THEOREM 5. *For any closed set* $C\subset\mathcal{M}(M)$,

$$\varlimsup_{t\to\infty}\frac{1}{t}\log Q_{s,x}^t(C)\leqq-\inf_{\mu\in C}I(\mu).$$

The proof of this theorem, which is omitted, is analogous to that in Donsker and Varadhan [**3**] by using Feynman-Kac formula.

To obtain the lower estimate, we need to assume that both $\xi(t)$ and $\lambda(t)$ are nondegenerate. Consider the same situation as in § 4 that L_t and L are expressed in the intrinsic forms $L_t=\Delta_t/2+b_t$ and $L=\Delta/2+b$ respectively. We can show that the I-function of $\lambda(t)$ is computed in explicit form for this case.

LEMMA. *For* $\mu\in\mathcal{M}(M)$ *which is absolutely continuous with respect to the Riemannian volume* dx *induced by* $g(x)$ *and whose density* p *is a strictly positive smooth function,*

$$I(\mu)=\frac{1}{2}\int_M\|\nabla\phi\|^2 d\mu$$

where ϕ *is the unique solution up to a constant term of the equation* $\Delta\phi+(1/p)\langle\nabla p, \nabla\phi\rangle=L^*p/p$ *and* L^* *is the formal adjoint to* L *with respect to* dx.

PROOF. Since μ is the unique invariant probability measure of the diffusion operator $A=\Delta+(1/p)\langle\nabla p, \nabla\cdot\rangle$ (i.e. $A^*p=0$), and the integral by μ of L^*p/p is 0, there is a solution ϕ unique up to a constant term by the Fredholm alternative theorem

Set $h=\log u$ for any strictly positive smooth function u on M. Then, we have

$$\int_M \frac{Lu}{u} p dx = \int_M Lhpdx + \frac{1}{2}\int_M \|\nabla h\|^2 pdx$$
$$= \int_M hL^*pdx + \frac{1}{2}\int_M \|\nabla h\|^2 pdx .$$

We can substitute $p\Delta\phi+\langle\nabla p, \nabla\phi\rangle$ for L^*p from the equation of ϕ. Elementary calculations making use of integration by parts yield the following:

$$\int_M \frac{Lu}{u} pdx = -\frac{1}{2}\int_M \|\nabla\phi\|^2 pdx + \frac{1}{2}\int_M \|\nabla h - \nabla\phi\|^2 pdx .$$

The infimum is, therefore, attained at $h=\phi$ or $u=e^{\phi}$.

In order to show the lower estimate of large deviation we essentially have to consider the absolutely continuous transformation of the given diffusion measure, to get some ergodic diffusion measure and to apply the ergodic theorem. Donsker and Varadhan in [4] used the method of the discrete approximation by Markov chain and dealt with diffusion processes on a general state space because there is no concrete representation of the absolutely continuous transformation density in this case. On the other hand, the lower large deviation estimate of one dimensional Brownian motion was established by using the Cameron-Martin-Girsanov formula in [3].

Then, we shall show the lower large deviation estimate of asymptotically homogeneous diffusion processes on a compact manifold by means of Theorem 2 (the Cameron-Martin-Girsanov formula for inhomogeneous diffusion processes on a manifold) and Theorem 3 (the strong law of large number for asymptotically homogeneous diffusion processes).

THEOREM 6. *Suppose that $g_t(x)\to g(x)$ in $C^3(M)$ and $b_t(x)\to b(x)$ in $C^2(M)$ as $t\to\infty$, that is, converge uniformly in $x\in M$ up to the third derivatives and the second derivatives respectively. Then, for any open set $G\subset\mathcal{M}(M)$,*

$$\underline{\lim}_{t\to\infty}\frac{1}{t}\log Q^t_{s,x}(G) \geqq -\inf_{\mu\in G} I(\mu) .$$

PROOF. It suffices to show that $\underline{\lim}_{t\to\infty}(1/t)\log Q^t_{s,x}(G)\geqq -I(\mu)$ for every $\mu\in G$ which has the properties stated in Lemma.

Using the function ϕ in Lemma or the solution ϕ to the equation $A\phi=f$ where $A=\Delta+(1/p)\langle\nabla p, \nabla\cdot\rangle$ and $f=L^*p/p$, we shall consider the following transformation of drift: $\tilde{L}=L+\langle\nabla\phi, \nabla\cdot\rangle$. Then μ is an invariant probability measure of the diffusion operator $\tilde{L}$, since $\tilde{L}^*p=L^*p-\langle\nabla\phi, \nabla p\rangle-\Delta\phi\cdot p=0$. This is the reason why the above drift is selected. The measure μ is also absolutely continuous with respect to the Riemannian volume $d_t x$ induced by $g_t(x)$ with the density $p_t(x)$. Consider, in the same manner as above, the following transformation of drift: $\tilde{L}_t = L_t + \langle \nabla_t \phi_t, \nabla_t \cdot \rangle_t$, using the solution ϕ_t to the equation $A_t \phi_t = f_t$ where $A_t = \Delta_t + (1/p_t)\langle \nabla_t p_t, \nabla_t \cdot \rangle_t$, $f_t = L_t^* p_t / p_t$ and L_t^* is the formal adjoint to L_t with respect to $d_t x$.

First, we shall show that an inhomogeneous diffusion process generated by $\tilde{L}_t$ is asymptotically homogeneous with the limiting homogeneous diffusion process generated by $\tilde{L}$. Regard t as a parameter. Denote by $q_t(\tau, x, dy)$ the transition probability of a homogeneous diffusion process whose generator is A_t and denote by $q(\tau, x, dy)$ that of a homogeneous diffusion process whose generator is A. It is well known that ϕ_t and ϕ are expressed in the following form:

$$\phi_t(x) = -\int_0^\infty d\tau \int_M q_t(\tau, x, dy) f_t(y) \qquad \text{and}$$

$$\phi(x) = -\int_0^\infty d\tau \int_M q(\tau, x, dy) f(y).$$

The coefficients of A_t, by hypotheses, converge uniformly on M to the corresponding coefficients of A as $t \to \infty$. Hence, from [7] p. 272,

$$\lim_{t\to\infty} \sup_{x \in M} \left| \int_M q_t(\tau, x, dy) f_t(y) - \int_M q(\tau, x, dy) f(y) \right| = 0.$$

Further, from [1] p. 373,

$$\sup_{x\in M} \left| \int_M q_t(\tau, x, dy) f_t(y) \right| = \sup_{x\in M} \left| \int_M \{q_t(\tau, x, dy) - \mu(dy)\} f_t(y) \right| \leqq Ce^{-\rho\tau},$$

where the positive constants C and ρ are independent of t. Then, the Lebesgue dominated convergence theorem implies that $\phi_t(x) \to \phi(x)$ uniformly in $x \in M$ as $t \to \infty$. Moreover, $A_t(\phi_t - \phi) = f_t - f + (A - A_t)\phi$. The right hand side, by hypotheses, converges to 0 in $C^1(M)$ as $t \to \infty$. We can, therefore, show from the Schauder estimates for A_t that $\phi_t(x) \to \phi(x)$ in $C^3(M)$ as $t \to \infty$. This sufficiently implies that, for every smooth function ϕ on M, $\tilde{L}_t \phi(x) \to \tilde{L}\phi(x)$ uniformly in $x \in M$ as $t \to \infty$.

Let $\tilde{P}_{s,x}$ be the probability law of the inhomogeneous diffusion process generated by $\tilde{L}_t$ starting at x at time s. By Theorem 2 in §2,

$$\left.\frac{dP_{s,x}}{d\tilde{P}_{s,x}}\right|_{\mathcal{F}_s^t} = \exp\left\{-\int_s^t \langle \nabla_\tau \phi_\tau(x(\tau)), e_\alpha(\tau)\rangle_\tau \cdot dw^\alpha(\tau) \right.$$
$$\left. -\frac{1}{2}\int_s^t \|\nabla_\tau \phi_\tau\|_\tau^2(x(\tau))d\tau \right\}$$

where $r(\tau) = (x(\tau), e_\alpha(\tau))$ is an inhomogeneous diffusion process on the principal frame bundle $GL(M)$ over M associated with the $\tilde{L}_t$-diffusion process on M. Define $E_t = \{\omega; l_{s,t}(\omega) \in G\}$. Then, from Jensen's inequality,

$$Q_{s,x}^t(G) = P_{s,x}(E_t) = \int_{E_t} \left.\frac{dP_{s,x}}{d\tilde{P}_{s,x}}\right|_{\mathcal{F}_s^t} d\tilde{P}_{s,x}$$
$$= \tilde{P}_{s,x}(E_t) \frac{1}{\tilde{P}_{s,x}(E_t)} \int_{E_t} \exp\left\{-\int_s^t \langle \nabla_\tau \phi_\tau(x(\tau)), e_\alpha(\tau)\rangle_\tau \cdot dw^\alpha(\tau)\right.$$

$$-\frac{1}{2}\int_s^t \|\nabla_\tau\phi_\tau\|_\tau^2(x(\tau))d\tau\Big\}d\tilde{P}_{s,x}$$

$$\geqq \tilde{P}_{s,x}(E_t)\exp\Big(\frac{1}{\tilde{P}_{s,x}(E_t)}\int_{E_t}\Big\{-\int_s^t\langle\nabla_\tau\phi_\tau(x(\tau)),\ e_\alpha(\tau)\rangle_\tau\cdot dw^\alpha(\tau)$$
$$-\frac{1}{2}\int_s^t \|\nabla_\tau\phi_\tau\|_\tau^2(x(\tau))d\tau\Big\}d\tilde{P}_{s,x}\Big).$$

By Theorem 3 in § 3,

$$\lim_{t\to\infty}\tilde{P}_{s,x}(E_t)=1 \qquad \text{and} \qquad \lim_{t\to\infty}\frac{1}{t}\int_s^t \|\nabla_\tau\phi_\tau\|_\tau^2(x(\tau))d\tau=\int_M\|\nabla\phi\|^2d\mu.$$

We can easily see that there is some constant C such that

$$\Big|\int_{E_t}\Big\{-\frac{1}{t}\int_s^t\langle\nabla_\tau\phi_\tau(x(\tau)),\ e_\alpha(\tau)\rangle_\tau\cdot dw^\alpha(\tau)\Big\}d\tilde{P}_{s,x}\Big|\leqq\frac{C}{\sqrt{t}}.$$

From these facts, we obtain that

$$\varliminf_{t\to\infty}\frac{1}{t}\log Q_{s,x}^t(G)\geqq-\frac{1}{2}\int_M\|\nabla\phi\|^2d\mu.$$

The desired result now follows from Lemma.

References

[1] A. Bensoussan, J. L. Lions and G. Papanicolaou, Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.

[2] R. N. Bhattacharya and S. Ramasubramanian, Recurrence and ergodicity of diffusions, J. Multivariate Anal., **12** (1982), 95-122.

[3] M. D. Donsker and S. R. S. Varadhan, Asymptotic evaluation of certain Wiener integrals for large time, Proceeding of International Conference on Function Space Integration, Oxford, 1974, 15-33.

[4] M. D. Donsker and S. R. S. Varadhan, Asymptotic evaluation of certain Markov process expectation for large time. I, Comm. Pure Appl. Math., **28** (1975), 1-47; III, Comm. Pure Appl. Math., **29** (1976), 389-461.

[5] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, Kodansha, 1981.

[6] R. S. Liptser and A. N Shiryayev, A functional limit theorem for semimartingales, Theor. Probability Appl., **25** (1980), 667-688.

[7] D. W. Strook and S. R. S. Varadhan, Multidimensional Diffusion Processes, Springer-Verlag, 1979.

Osamu TAKEYAMA
Department of Mathematics
Kyoto University
Kyoto 606, Japan