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Meromorphic solutions of some nonlinear difference equations of higher order

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1. Introduction.

Here we will consider the nonlinear difference equation

$$\alpha_n y(x+n)+\alpha_{n-1}y(x+n-1)+\cdots+\alpha_1 y(x+1)=R(y(x)), \tag{1.1}$$

where $R(y)$ is a rational function of y:

$$\begin{cases} R(y)=P(y)/Q(y), \\ P(y)=a_p y^p+\cdots+a_0, \\ Q(y)=b_q y^q+\cdots+b_0, \end{cases} \tag{1.2}$$

in which $\alpha_n, \cdots, \alpha_1; a_p, \cdots, a_0; b_q, \cdots, b_0$ are constants, $\alpha_n a_p b_q \neq 0$. We suppose that $P(y)$ and $Q(y)$ are mutually prime. In the sequel, we denote by p and q the degree of the nominator $P(y)$ and of the denominator $Q(y)$, respectively.

We will investigate in this note whether the equation (1.1) admits a meromorphic solution or not. Of course, we mean nontrivial solution, i.e., solution which is not identically equal to a constant.

In [1] and [2], Harris and Sibuya investigated the difference equation

$$\begin{aligned} &\vec{y}(x+1)=\vec{F}(x, \vec{y}(x)), \\ &\vec{F}(x, \vec{y})=(F_j(x, y_1, \cdots, y_n), \quad j=1, \cdots, n), \\ &\vec{F}(\infty, \vec{0})=\vec{0}. \end{aligned} \tag{1.3}$$

When F_j are rational functions of $x, y_1, \cdots, y_n$, then their results imply that the equation (1.3) possesses a meromorphic solution $\vec{y}(x)$ which has an asymptotic expansion

$$\vec{y}(x) \sim \sum_{m=1}^{\infty} \vec{a}_m / x^m \tag{1.4}$$

in an angular domain. This is a very general result. But in the present case (1.1), the solution (1.4) obtained by them has coefficients $\vec{a}_m=\vec{0}$, $m=1, 2, \cdots$. Therefore we need somewhat more detailed study of the equation to get nontrivial solutions.

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Put

(1.5) $$\Lambda=\{\lambda\in \boldsymbol{C}\,;(\alpha_n+\cdots+\alpha_1)\lambda=R(\lambda)\}.$$

Suppose Λ is not void. For a $\lambda\in\Lambda$, we put

(1.6) $$f_\lambda(t)=\alpha_n t^n+\alpha_{n-1}t^{n-1}+\cdots+\alpha_1 t-R'(\lambda)=0\,,$$

and denote the roots of the equation (1.6) as

(1.6′) $$\tau_1(\lambda),\ \cdots,\ \tau_n(\lambda)\,,\qquad |\tau_1(\lambda)|\geqq\cdots\geqq|\tau_n(\lambda)|\geqq 0\,.$$

We proved in [**11**] the following theorems.

THEOREM A. *Suppose Λ is not void and there are a $\lambda\in\Lambda$ and a j, $1\leqq j\leqq n$, such that either*

(1.7) $$|\tau_j(\lambda)|>1\,,\qquad or$$

(1.7′) $$\tau_j(\lambda)=1\,,$$

then the equation (1.1) *admits a nontrivial meromorphic solutions.*

If $p\geqq q+2$, then obviously the set Λ is not void. In this case we have the following theorem [**11**].

THEOREM B. *If $p\geqq q+2$ in* (1.1), *then there are a $\lambda\in\Lambda$ and a j, $1\leqq j\leqq n$, for which either* (1.7) *or* (1.7′) *holds.*

By Theorems A and B, we see the following fact [**11**].

THEOREM C. *If $p\geqq q+2$, then the equation* (1.1) *admits a nontrivial meromorphic solution.*

Thus we will confine ourselves to the case when $p\leqq q+1$. In the sequel, we assume that $R(y)$ is of the form

(1.8) $$R(y)=B_{-1}y+\sum_{k=m}^{\infty}B_k y^{-k},\qquad B_m\neq 0\,,\ m\geqq 0\,,$$

for sufficiently large y.

First we note that, when $p\leqq q+1$, then the set Λ may be void. For example, consider the case when $R(y)=(\alpha_n+\cdots+\alpha_1)y+1/Q(y)$. Further, even if Λ is not void, it may be that neither (1.7) nor (1.7′) holds. For example, consider the equation

$$y(x+3)+y(x+2)+y(x+1)=2y(x)+1/y(x)^2.$$

In this case, $\Lambda=\{1,\ (-1\pm\sqrt{3}i)/2\}$, and $R'(\lambda)=0$ for any $\lambda\in\Lambda$. The equation (1.6) for this case possesses roots $t=0$, $(-1\pm\sqrt{3}i)/2$.

However, we obtain the following results for the case $p\leqq q+1$. Put

(1.9) $$f_\infty(t)=\alpha_n t^n+\alpha_{n-1}t^{n-1}+\cdots+\alpha_1 t-B_{-1}=0\,,$$

where $B_{-1}=R'(\infty)$ is in (1.8). Denote the roots of (1.9) as

$$\tau_1(\infty), \cdots, \tau_n(\infty), \qquad |\tau_1(\infty)| \geqq \cdots \geqq |\tau_n(\infty)| \geqq 0. \tag{1.9'}$$

LEMMA 1. *Suppose that* $p \leqq q+1$ *and* $q \neq 0$. *Further suppose that* $|\alpha_{n-1}|+\cdots+|\alpha_1| \neq 0$. *Then at least one of the following possibilities* (i) *and* (ii) *is valid:*

(i) Λ *is not void and there are a* $\lambda \in \Lambda$ *and a* j, $1 \leqq j \leqq n$, *for which either* (1.7) *or* (1.7′) *holds;*

(ii) *There is a* j, $1 \leqq j \leqq n$, *such that either*

$$0 < |\tau_j(\infty)| < 1, \qquad \textit{or} \tag{1.10}$$

$$\tau_j(\infty) = 1. \tag{1.10'}$$

REMARK. If $p \leqq q+1$ and $q=0$, then the equation (1.1) reduces to a linear (homogeneous or inhomogeneous) equation and we have nothing to consider. Hence we suppose that $q \neq 0$ in Lemma 1.

If $|\alpha_{n-1}|+\cdots+|\alpha_1|=0$, then the equation (1.1) is of the form: $\alpha_n y(x+n) = R(y(x))$ which is essentially an equation of order 1, and has been studied in some detail in [**10**]. Hence we exclude this case in Lemma 1.

THEOREM 2. *Suppose there is a* j, $1 \leqq j \leqq n$, *for which* (1.10) *holds. Write* $\tau_j(\infty)=\tau$. *Put*

$$K = \{k \geqq 0\,;\ \tau^{-k} \text{ is a root of } (1.9)\}. \tag{1.11}$$

(1) *When the set* K *is void, there is a meromorphic solution* $y(x)$ *of* (1.1) *which has an expansion*

$$y(x) = c_{-1}\tau^x + \sum_{k=m}^{\infty} c_k \tau^{-kx} \tag{1.12}$$

in a domain

$$D(\rho) = \{x\,;\ |\tau^{-x}| \leqq \rho\} \tag{1.13}$$

for a sufficiently small $\rho > 0$, *in which* m *is the integer in* (1.8). *The coefficient* c_{-1} *may be arbitrarily prescribed, and* c_k, $k \geqq m$, *are constants determined uniquely if* c_{-1} *is prescribed.*

(2) *When* K *is not void, there is a meromorphic solution* $y(x)$ *of* (1.1) *which has an asymptotic expansion*

$$y(x) \sim c_{-1}\tau^x + \sum_{k=m}^{\infty} c_k(x) \tau^{-kx} \tag{1.14}$$

in a domain

$$D_\varepsilon(\rho) = \left\{x\,;\ |\tau^{-x}| \leqq \rho,\ -\frac{\pi}{2}+\varepsilon \leqq \arg[x \log \tau] \leqq \frac{\pi}{2}-\varepsilon\right\} \tag{1.13'}$$

for a sufficiently small $\rho > 0$ *and an* ε, $0 < \varepsilon < \pi/2$. c_{-1} *is an arbitrarily prescribed* *constant, and* $c_k(x)$, $k \geqq m$, *are polynomials with*

$$\deg[c_k(x)] \leqq C^*(k+1)\,, \tag{1.14'}$$

where C^* *is a constant.* $c_k(x)$ *are indeterminate for* $k \in K$, *but we require*

(1.15) $c_k(x)$, $k \in K$, *are polynomials whose terms are least in number among admissible ones* (see the proof).

With the condition (1.15), $c_k(x)$ *are uniquely determined if* c_{-1} *is fixed.*

THEOREM 3. *Suppose there is a* j, $1 \leqq j \leqq n$, *for which* (1.10′) *holds. Let* m *be the integer in* (1.8), *and* κ *be the integer such that*

$$\kappa = \min\{k \geqq 1\,;\, f_\infty^{(k)}(1) \neq 0\}\,. \tag{1.16}$$

(1) *Suppose that*

(1.17) *either* $m=0$ *or* $\kappa/(m+1)$ *is not an integer.*

Then there is a meromorphic solution $y(x)$ *which has an asymptotic expansion*

$$y(x) \sim \sum_{k=0}^{\infty} p_k(x)\Big(\frac{\log x}{x}\Big)^k, \tag{1.18}$$

$$p_k(x) \sim x^{\kappa/(1+m)} \sum_{j=0}^{\infty} c_{jk} x^{-j/(1+m)} \tag{1.19}$$

in an angular domain

$$D(M, \varepsilon) = \Big\{x\,;\, |\arg(x+M) - \pi| < \frac{\pi}{2} - \varepsilon\Big\}, \tag{1.20}$$

where ε, $0 < \varepsilon < \pi/2$, *is an arbitrarily fixed number, and* M *is a sufficiently large number.* $c_{m+1,0}$ *can be arbitrarily prescribed, and other* c_{jk} *are determined uniquely if* $c_{m+1,0}$ *is prescribed.* M *in* (1.20) *depends on* ε *and* $c_{m+1,0}$.

(2) *Suppose that*

(1.17′) $m \geqq 1$ *and* $\kappa/(m+1)$ *is an integer.*

Then there is a meromorphic solution $y(x)$ *which has an asymptotic expansion*

$$y(x) \sim \sum_{k=0}^{\infty} q_k(x)(\log x)^{(1-k)/(m+1)}, \tag{1.18'}$$

$$q_k(x) \sim x^{\kappa/(1+m)} \sum_{j=0}^{\infty} c_{jk} x^{-j/(1+m)} \tag{1.19'}$$

in an angular domain $D(M, \varepsilon)$ *in* (1.20). $c_{m+1,0}$ *can be arbitrarily prescribed, and other* c_{jk} *are determined uniquely if* $c_{m+1,0}$ *is fixed.* M *in* (1.20) *depends on* ε *and* $c_{m+1,0}$.

In view of Lemma 1, Theorems 2 and 3 (together with Theorem A) assure that the equation (1.1) admits a nontrivial meromorphic solution when $p \leqq q+1$. Therefore, if we note Theorem C, we see that the equation (1.1) possesses always a nontrivial meromorphic solution.

2. Proof of Lemma 1.

We put

$$A(\lambda)=\alpha_n+\cdots+\alpha_1-R'(\lambda) \quad \text{for } \lambda\in \boldsymbol{C}, \tag{2.1}$$

$$A(\infty)=\alpha_n+\cdots+\alpha_1-B_{-1} \quad \text{with } B_{-1} \text{ in (1.8).} \tag{2.2}$$

It is easily seen that the set Λ consists of $(q+1)$ elements (counted according to multiplicities) if and only if $A(\infty)\neq 0$, and that $w=\lambda\in\Lambda$ is a multiple root of $(\alpha_n+\cdots+\alpha_1)w-R(w)=0$ if and only if $A(\lambda)=0$.

$A(\lambda)=0$ if and only if $\tau_j(\lambda)=1$ for some j, $1\leqq j\leqq n$. $A(\infty)=0$ if and only if $\tau_j(\infty)=1$ for some j, $1\leqq j\leqq n$.

Assume that

$$|\tau_j(\lambda)|\leqq 1 \quad \text{and} \quad \tau_j(\lambda)\neq 1 \quad \text{for any } \lambda\in\Lambda \text{ and } j,\ 1\leqq j\leqq n, \tag{2.3}$$

and that

$$\tau_j(\infty)\neq 1 \quad \text{for any } j,\ 1\leqq j\leqq n. \tag{2.3'}$$

If, under the assumptions (2.3) and (2.3′), we could deduce that

$$0<|\tau_j(\infty)|<1 \quad \text{for some } j,\ 1\leqq j\leqq n, \tag{2.3''}$$

then we would be through.

By the assumption (2.3′), $A(\infty)\neq 0$. Hence the set Λ consists of $(q+1)$ elements $\lambda_1, \cdots, \lambda_{q+1}$. By the assumption (2.3), $A(\lambda_h)\neq 0$, $h=1, \cdots, q+1$. Thus λ_h, $1\leqq h\leqq q+1$, are all simple roots of $(\alpha_n+\cdots+\alpha_1)w-R(w)=0$. Therefore we can write

$$\frac{1}{(\alpha_n+\cdots+\alpha_1)w-R(w)}=\sum_{h=1}^{q+1}\frac{1}{A(\lambda_h)}\,\frac{1}{w-\lambda_h}. \tag{2.4}$$

Multiplying by w and letting $w\to\infty$, we obtain

$$\sum_{h=1}^{q+1}\frac{1}{A(\lambda_h)}+\left(-\frac{1}{A(\infty)}\right)=0. \tag{2.5}$$

In (1.6), put $t=(\zeta+1)/\zeta$. Then

$$A(\lambda)\zeta^n+f_\lambda'(1)\zeta^{n-1}+\cdots=0. \tag{2.6}$$

Let $\zeta_j(\lambda)$ be roots of (2.6) corresponding to $\tau_j(\lambda)$, i.e., $\tau_j(\lambda)=[\zeta_j(\lambda)+1]/\zeta_j(\lambda)$.

In (1.9), put $t=(\zeta-1)/\zeta$. Then

$$A(\infty)\zeta^n-f_\infty'(1)\zeta^{n-1}+\cdots=0. \tag{2.6'}$$

Let $\zeta_j(\infty)$ be roots of (2.6′) corresponding to $\tau_j(\infty)$, $1\leqq j\leqq n$.

From (2.6) and (2.6′), we obtain

(2.7) $$\sum_{j=1}^{n}\zeta_j(\lambda_h)=-f'_{\lambda_h}(1)/A(\lambda_h)\,,\qquad h=1,\cdots,q+1\,,$$

(2.7′) $$\sum_{j=1}^{n}\zeta_j(\infty)=f'_{\infty}(1)/A(\infty)\,.$$

Since $f'_{\lambda_h}(1)=f'_{\infty}(1)=n\alpha_n+(n-1)\alpha_{n-1}+\cdots+\alpha_1$, we obtain by (2.5)

(2.8) $$\sum_{h=1}^{q+1}\sum_{j=1}^{n}\zeta_j(\lambda_h)+\sum_{j=1}^{n}\zeta_j(\infty)=0\,.$$

By the assumption (2.3), we get

(2.9) $$\mathrm{Re}[\zeta_j(\lambda_h)]\leqq -1/2 \qquad \text{for}\quad h=1,\cdots,q+1\quad\text{and}\quad j=1,\cdots,n\,.$$

Hence

$$\sum_{h=1}^{q+1}\sum_{j=1}^{n}\mathrm{Re}[\zeta_j(\lambda_h)]\leqq -\frac{n}{2}(q+1)\leqq -n\,,\qquad \text{since } q\neq 0\,.$$

Therefore by (2.8)

(2.10) $$\sum_{j=1}^{n}\mathrm{Re}[\zeta_j(\infty)]\geqq n\,.$$

Suppose that $|\tau_1(\infty)|\geqq 1$, noting that $|\tau_1(\infty)|\geqq|\tau_j(\infty)|$. Then

$$\mathrm{Re}[\zeta_1(\infty)]\leqq 1/2\,.$$

Hence by (2.10)

$$\sum_{j=2}^{n}\mathrm{Re}[\zeta_j(\infty)]\geqq n-(1/2)\,.$$

Then there must be a j', $2\leqq j'\leqq n$, such that

$$\mathrm{Re}[\zeta_{j'}(\infty)]>1\,.$$

Then obviously we have that $0<|\tau_{j'}(\infty)|<1$ for this j'.

Suppose that $|\tau_j(\infty)|<1$, $j=1,\cdots,n$. If $\tau_j(\infty)=0$, $j=1,\cdots,n$, then $B_{-1}=0$ and $\alpha_{n-1}=\cdots=\alpha_1=0$, which contradicts the assumption. Hence there is a j such that $0<|\tau_j(\infty)|<1$. Q.E.D.

Lemma 1 is a generalization of a lemma of Julia [**4**, p. 158].

3. Proof of Theorem 2. I. Formal solution.

Suppose that $R(y)$ is expanded as in (1.8). Put

(3.1) $$y(x)=c_{-1}\tau^x+\sum_{k=0}^{\infty}c_k(x)\tau^{-kx}=c_{-1}\tau^x\Big(1+\sum_{k=1}^{\infty}c'_k(x)\tau^{-kx}\Big),$$

(3.1′) $$c'_k(x)=c_{k-1}(x)/c_{-1}\,,$$

in which we suppose $c_k(x)$, $k\geqq 1$, to be polynomials which may be constants. Let

(3.2) $$\Big(1+\sum_{k=1}^{\infty}c'_k(x)\tau^{-kx}\Big)^{-1}=1+\sum_{k=1}^{\infty}c''_k(x)\tau^{-kx},$$

then

(3.2′) $$c''_k(x)=-\sum_{l=1}^{k}c'_l(x)c''_{k-l}(x)\,,\qquad c'_0(x)=c''_0(x)=1\,.$$

Further, let

(3.3) $$\left(1+\sum_{k=1}^{\infty}c'_k(x)\tau^{-kx}\right)^{-s}=1+\sum_{k=1}^{\infty}\tilde{c}_k^{(s)}(x)\tau^{-kx}.$$

Then

(3.3′) $$\tilde{c}_k^{(s)}(x)=\sum_{\substack{\nu_1+\cdots+\nu_s=s\\ j_1\nu_1+\cdots+j_s\nu_s=k\\ j_1<j_2<\cdots<j_s}}\frac{s!}{\nu_1!\cdots\nu_s!}c''_{j_1}(x)^{\nu_1}\cdots c''_{j_s}(x)^{\nu_s}\,,$$

$$\tilde{c}_0^{(0)}(x)=1\,,\qquad \tilde{c}_k^{(0)}(x)=0\,,\quad k=1,2,\cdots,$$

$$\tilde{c}_k^{(1)}(x)=c''_k(x)\,,\quad k=1,2,\cdots.$$

Thus

(3.4) $$R(y(x))=B_{-1}\left(c_{-1}\tau^x+\sum_{k=0}^{\infty}c_k(x)\tau^{-kx}\right)+\sum_{s=m}^{\infty}B_s(c_{-1}\tau^x)^{-s}\left(1+\sum_{k=1}^{\infty}c'_k(x)\tau^{-kx}\right)^{-s}$$

$$=B_{-1}c_{-1}\tau^x+\sum_{k=0}^{\infty}B_{-1}c_k(x)\tau^{-kx}+\sum_{s=m}^{\infty}B_sc_{-1}^{-s}\tau^{-sx}\left(1+\sum_{k=1}^{\infty}\tilde{c}_k^{(s)}(x)\tau^{-kx}\right)$$

$$=B_{-1}c_{-1}\tau^x+B_{-1}c_0(x)+\cdots+B_{-1}c_{m-1}(x)\tau^{-(m-1)x}$$

$$+\sum_{k=m}^{\infty}\left[B_{-1}c_k(x)+\sum_{l=m}^{k}B\,c_{-1}^{-l}\tilde{c}_{k-l}^{(l)}(x)\right]\tau^{-kx}$$

and

(3.4′) $$\alpha_ny(x+n)+\cdots+\alpha_1y(x+1)$$

$$=(\alpha_n\tau^n+\cdots+\alpha_1\tau)c_{-1}\tau^x+\sum_{k=0}^{\infty}\left(\sum_{j=1}^{n}\alpha_j\tau^{-jk}c_k(x+j)\right)\tau^{-kx}.$$

Thus

(3.5) $$(\alpha_n\tau^n+\cdots+\alpha_1\tau-B_{-1})c_{-1}=0\,.$$

If $m\geqq 1$,

(3.5′) $$\begin{cases}\alpha_nc_0(x+n)+\cdots+\alpha_1c_0(x+1)=B_{-1}c_0(x)\,,\\ \quad\cdots\cdots\\ \alpha_n\tau^{-(m-1)n}c_{m-1}(x+n)+\cdots+\alpha_1\tau^{-(m-1)}c_{m-1}(x+1)=B_{-1}c_{m-1}(x)\,.\end{cases}$$

For $k\geqq m$,

(3.5″) $$\alpha_n\tau^{-kn}c_k(x+n)+\cdots+\alpha_1\tau^{-k}c_k(x+1)=B_{-1}c_k(x)+S_k(x)\,,$$

where

(3.5‴) $$S_k(x)=\sum_{l=m}^{k}B_lc_{-1}^{-l}\tilde{c}_{k-l}^{(l)}(x)\,.$$

By (3.5), c_{-1} is seen to be arbitrarily prescribed.

(1) When the set K in (1.11) is void, we can suppose that $c_k(x)$ are constants. By (3.5′), $c_k=0$, $0\leqq k\leqq m-1$. Constant coefficients c_k, $k\geqq m$, are determined uniquely if c_{-1} is fixed.

(2) When K is not void. Also in this case, we can take $c_k(x)=0$ for $0\leqq k\leqq m-1$. Suppose $k\geqq m$. If $k\in K$, then (3.5″) can not determine $c_k(x)$ uniquely. But (3.5″) possesses polynomial solutions, and subtracting polynomial solutions of homogeneous equation $\alpha_n\tau^{-kn}u(x+n)+\cdots+\alpha_1\tau^{-k}u(x+1)-B_{-1}u(x)=0$, we obtain $c_k(x)$ so as to satisfy the condition (1.15), which permit us to determine $c_k(x)$ uniquely if c_{-1} is fixed.

If k_0 is sufficiently large, then

$$\alpha_n\tau^{-kn}+\cdots+\alpha_1\tau^{-k}-B_{-1}=f_\infty(\tau^{-k})\neq 0 \qquad \text{for} \quad k\geqq k_0. \tag{3.6}$$

Then obviously we obtain polynomials $c_k(x)$ such that

$$\deg[c_k(x)]=\deg[S_k(x)]. \tag{3.7}$$

Suppose that

$$\deg[c_j(x)]\leqq C^*(j+1) \qquad \text{for} \quad j=0,1,\cdots,k-1, \tag{3.8}$$

where C^* is a suitable constant. Then by (3.1′)

$$\deg[c_j'(x)]\leqq C^*j.$$

By (3.2′), we can easily see that

$$\deg[c_j''(x)]\leqq C^*j.$$

Thus by (3.3′)

$$\deg[\tilde{c}_k^{(s)}(x)]\leqq C^*(j_1\nu_1+\cdots+j_s\nu_s)=C^*k.$$

Therefore by (3.5‴),

$$\deg[S_k(x)]\leqq\deg[\tilde{c}_k^{(s)}(x)]\leqq C^*k.$$

By (3.7), supposing that $k\geqq k_0$,

$$\deg[c_k(x)]=\deg[S_k(x)]\leqq C^*k\leqq C^*(k+1).$$

Thus (3.8) hold for any k, and we obtain a formal solution as stated in Theorem 2.

4. Proof of Theorem 2. II. Existence proof.

(1) When the set K in (1.11) is void. Then we can take a constant $A>0$ such that

$$|f_\infty(\tau^{-k})|=|\alpha_n\tau^{-kn}+\cdots+\alpha_1\tau^{-k}-B_{-1}|\geqq A, \qquad k\geqq 0. \tag{4.1}$$

There are constants $M>0$ and $r>0$ such that

$$|B_k|\leqq M/r^k, \qquad k\geqq 0. \tag{4.2}$$

Let $C_{-1}=|c_{-1}|$. Consider the equation

$$\begin{aligned} C_{-1}|\tau^x|A\,u(x)&=\sum_{k=0}^{\infty} Mr^{-k}(C_{-1}|\tau^x|)^{-k}(1-u(x))^{-k} \\ &=MrC_{-1}|\tau^x|(1-u(x))/[rC_{-1}|\tau^x|(1-u(x))-1], \end{aligned} \tag{4.3}$$

i. e.,

$$A\,u(x)=Mr(1-u(x))/[rC_{-1}|\tau^x|(1-u(x))-1]. \tag{4.3$'$}$$

Then

$$u(x)^2-\Big(1+\frac{Mr-A}{ArC_{-1}|\tau^x|}\Big)u(x)+\frac{Mr}{ArC_{-1}|\tau^x|}=0, \tag{4.3$''$}$$

which obviously possesses a solution in the form

$$u(x)=\sum_{k=1}^{\infty} C_k|\tau^x|^{-k} \qquad \text{in} \quad |\tau^x|^{-1}\leqq\rho \tag{4.4}$$

for sufficiently small $\rho>0$. Obviously we have that, as easily seen from (3.5$''$) and (4.1), (4.2), (4.3),

$$|c_k|\leqq C_k, \qquad k=0,\ 1,\ 2,\ \cdots \tag{4.4$'$}$$

which proves the convergence of (1.12) in $D(\rho)$ of (1.13).

(2) When K is not void. We will prove the theorem by the fixed point theorem. Let k_0 be the integer stated in (3.6). Put for $N\geqq k_0$

$$U_N(x)=c_{-1}\tau^x+\sum_{k=m}^{N-1} c_k(x)\tau^{-kx}. \tag{4.5}$$

Let Υ_N be the family of functions $\Xi(x)$, holomorphic in $D_\varepsilon(\rho_N)$ (see (1.13$'$)) and satisfying the condition

$$|\Xi(x)|\leqq K_N|x|^{C^*(N+1)}|\tau^{-Nx}| \qquad \text{for} \quad x\in D_\varepsilon(\rho_N), \tag{4.6}$$

where C^* is a constant in (3.8), and ρ_N as well as K_N is a constant to be determined later. $\varepsilon>0$ is arbitrarily fixed.

Put for $\Xi(x)\in\Upsilon_N$,

$$\begin{aligned} T[\Xi](x)&=\alpha_n^{-1}[R(U_N(x-n)+\Xi(x-n))-R(U_N(x-n))] \\ &\quad+\alpha_n^{-1}[R(U_N(x-n))-(\alpha_nU_N(x)+\alpha_{n-1}U_N(x-1)+\cdots+\alpha_1U_N(x-n+1))] \\ &\quad+(-\alpha_n^{-1})[\alpha_{n-1}(x-1)+\cdots+\alpha_1(x-n+1)] \\ &=I_1+I_2+I_3. \end{aligned} \tag{4.7}$$

Let $M_1=(1+|R'(\infty)|)/|\alpha_n|$. Then there is a ρ_N such that

$$|I_1|\leqq M_1|\Xi(x-n)|\leqq M_1K_N|x-n|^{C^*(N+1)}|\tau^{-Nx}||\tau|^{nN} \qquad \text{in} \quad D_\varepsilon(\rho_N).$$

Let τ' be a number such that $1<\tau'<1/|\tau|$. Let ρ_N be so small that

$$|(x-n)/x|^{C^*}<\tau' \qquad \text{for} \quad x\in D_\varepsilon(\rho_N).$$

Then

$$|I_1|\leqq|\tau|^{nN}\tau'^{N+1}M_1K_N|x|^{C^*(N+1)}|\tau^{-Nx}| \qquad \text{for} \quad x\in D_\varepsilon(\rho_N). \tag{4.8}$$

Next, since c_{-1} and $c_k(x)$, $k=0, 1, \cdots$, are coefficients of formal solution, I_2 begins with a term $c_N^*(x)\tau^{-Nx}$, where $c_N^*(x)$ is a polynomial of degree less than $C^*(N+1)$. Hence there is a constant M_2 such that

$$|I_2|\leqq M_2|x|^{C^*(N+1)}|\tau^{-Nx}| \qquad \text{for} \quad x\in D_\varepsilon(\rho_N). \tag{4.8$'$}$$

Let $M_3=2(|\alpha_{n-1}|+\cdots+|\alpha_1|+1)/|\alpha_n|$. Then

$$|I_3|\leqq|\tau^N|\tau'^{N+1}M_3K_N|x|^{C^*(N+1)}|\tau^{-Nx}| \qquad \text{for} \quad x\in D_\varepsilon(\rho_N). \tag{4.8$''$}$$

Suppose N is sufficiently large and K_N is so large that

$$\tau'^N|\tau|^{nN}\tau'M_1K_N+M_2+\tau'^N|\tau|^N\tau'M_3K_N<K_N,$$

then T maps Υ_N into Υ_N, and T is obviously continuous in the topology of uniform convergence on compact sets. Thus the fixed point theorem is applied, since Υ_N is convex and a normal family. Let $\Xi_N(x)$ be a fixed point. Then $y_N(x)=U_N(x)+\Xi_N(x)$ is a solution of (1.1) in $D_\varepsilon(\rho_N)$.

Next we will show that the solution $y_N(x)$ is independent of N. Suppose there would be another solution $y_N^*(x)$, holomorphic and satisfying $y_N^*(x)-U_N(x)=O(|x|^{C^*(N+1)}|\tau^{-Nx}|)$ in $D_\varepsilon(\rho_N^*)$ for a ρ_N^*. Put $h(x)=y_N^*(x)-y_N(x)$. If we show that $h(x)\equiv 0$ in $D_\varepsilon(\rho_N)\cap D_\varepsilon(\rho_N^*)$, then it can be easily deduced that $y_N(x)$ is independent of N. Thus it remains to show that: *Let $h(x)$ be holomorphic and satisfy*

$$|h(x)|\leqq K^*|x|^{C^*(N+1)}|\tau^{-Nx}| \qquad \textit{with a constant } K^* \tag{4.9}$$

in $D_\varepsilon(\rho)$ for a ρ, and further satisfy

$$\alpha_n h(x+n)+\cdots+\alpha_1 h(x+1)=R(y_N(x)+h(x))-R(y_N(x)), \tag{4.9$'$}$$

then we will have that $h(x)\equiv 0$.

(i) Suppose $R'(\infty)=B_{-1}\neq 0$. The right hand side of (4.9′) can be written as $R'(\infty)(1+g(x))h(x)$, where $g(x)\to 0$ as $\mathrm{Re}\,x\to-\infty$ in $D_\varepsilon(\rho)$. Put $x=-t$ and $h(-t+n)=u(t)$. Then (4.9′) is written as

$$u(t+n)+\beta_{n-1}(t)u(t+n-1)+\cdots+\beta_0(t)u(t)=0, \tag{4.9$''$}$$

where $\beta_j(t)=-\alpha_{n-j}/[R'(\infty)(1+g(-t))]\to-\alpha_{n-j}/R'(\infty)$ as $\mathrm{Re}\,t\to\infty$, and $\beta_0(t)\neq 0$. Thus (4.9′) is an equation of Poincaré. By a theorem of Perron [**7**, p. 309], [8], [**9**],

$$\limsup_{j\to\infty}|u(t+j)|^{1/j}=\limsup_{j\to\infty}|h(-t+n-j)|^{1/j}$$

$$=\limsup_{j\to\infty}|h(x+n-j)|^{1/j}=1/|\tau^*|,$$

where τ^* is a root of (1.9).

On the other hand, by the assumption (4.9) on $h(x)$,

$$|h(x+n-j)|^{1/j}\leqq(K^*)^{1/j}|x+n-j|^{C^*(N+1)/j}|\tau^{-N(x+n)/j}||\tau|^N$$
$$\to|\tau|^N \qquad \text{as} \quad j\to\infty .$$

This is impossible if N is so large that $|\tau|^N<1/|\tau^*|$ for any root τ^* of (1.9). Hence we must have that $h(x)\equiv 0$.

(ii) Suppose $R'(\infty)=B_{-1}=0$. Put $v=\min\{k\geqq 1;\ \alpha_k\neq 0\}$ and $m'=\min\{k\geqq 1;\ B_k\neq 0\}$. By (4.9′) and (1.8)

(4.10)
$$\alpha_n h(x+n)+\cdots+\alpha_v h(x+v)$$
$$=-m'B_{m'}c_{-1}^{-(m'+1)}\tau^{-(m'+1)x}(1+g_1(x))h(x),$$

where $g_1(x)\to 0$ as $\mathrm{Re}\,x\to-\infty$ in $D_\varepsilon(\rho)$.

Let $t_1, \cdots, t_u$ be roots of the equation

(4.11)
$$\psi(t)=\alpha_n t^{n-v}+\cdots+\alpha_v=0,$$

with multiplicities $s_1, \cdots, s_u$ $(s_1+\cdots+s_u=n-v)$, respectively. Let $\sigma_j=\tau^N t_j$, $j=1, \cdots, u$. We take N so large that

$$|\sigma_j|<1, \qquad j=1, \cdots, u .$$

Put $h(x)=\tau^{-Nx}H(x)$ in (4.10). Then

$$\alpha_n\tau^{-nN}H(x+n)+\cdots+\alpha_v\tau^{-vN}H(x+v)$$
$$=-m'B_{m'}\tau^{-(m'+1)x}c_{-1}^{-(m'+1)}(1+g_1(x))H(x)=\phi(x)$$

and

(4.12)
$$|\phi(x)|\leqq B'|\tau^{-(m'+1)x}||H(x)|$$

with a constant $B'\leqq 2m'|B_{m'}||c_{-1}^{-(m'+1)}|$.

For simplicity, we suppose that $s_1=\cdots=s_u=1$. Then by [7, p. 396]

(4.13)
$$H(x)=\sum_{j=1}^{u}\pi_j(x)\sigma_j^x+\sum_{j=1}^{u}\frac{\sigma_j^{x-1}}{\psi'(\sigma_j)}\mathop{\mathrm{S}}_{-\infty}^{x}\phi(z)\sigma_j^{-z}\Delta z,$$

where $\pi_j(x)$ are periodic functions with period 1, and S denotes the summation [7, p. 43]:

(4.13′)
$$\mathop{\mathrm{S}}_{-\infty}^{x}f(z)\Delta z=\sum_{k=1}^{\infty}f(x-k).$$

By the definition, $H(x)$ satisfies

(4.13″) $\quad |H(x)|\leqq K^*|x|^{C^*(N+1)}$ with a constant K^*.

Hence we must have that $\pi_j(x)\equiv 0$, $j=1, \cdots, u$ in (4.13), as seen by letting

$\operatorname{Re} x \to -\infty$.

Since $|H(x-k)| \leqq K^*|x-k|^{C^*(N+1)} \leqq K^*K'|x|^{C^*(N+1)} k^{C^*(N+1)}$ with a constant K', we have by (4.12)

$$(4.14) \qquad \sum_{k=1}^{\infty} |\phi(x-k)|\,|\sigma_j^{k-1}| \leqq B'|\tau^{-(m'+1)x}| \sum_{k=1}^{\infty} |\tau^{(m'+1)k}\sigma_j^{k-1}|\,|H(x-k)|$$

$$\leqq \Big(B' \sum_{k=1}^{\infty} |\tau^{(m'+1)k}\sigma_j^{k-1}|\, k^{C^*(N+1)}\Big) K'K^*|\tau^{-(m'+1)x}|\,|x|^{C^*(N+1)}.$$

Therefore, if we put

$$K^{**} = \sum_{j=1}^{u} \frac{1}{|\psi'(\sigma_j)|}\Big(B' \sum_{j=1}^{\infty} |\tau^{(m'+1)k}\sigma_j^{k-1}|\, k^{C^*(N+1)}\Big)K',$$

then

$$(4.15) \qquad |H(x)| \leqq K^{**}K^*|\tau^{-(m'+1)x}|\,|x|^{C^*(N+1)}$$

by (4.13). Again by (4.12), using (4.15), we get

$$\sum_{k=1}^{\infty} |\phi(x-k)|\,|\sigma_j^{k-1}| \leqq B'|\tau^{-(m'+1)x}| \sum_{k=1}^{\infty} |\tau^{(m'+1)k}\sigma_j^{k-1}|$$

$$\times K^{**}K^*|\tau^{-(m'+1)x}|\,|x-k|^{C^*(N+1)}|\tau^{(m'+1)k}|$$

$$\leqq K^{**}K^*|\tau^{-(m'+1)x}|^2\Big(B' \sum_{k=1}^{\infty} |\tau^{(m'+1)k}\sigma_j^{k-1}|\, k^{C^*(N+1)}\Big)K'|x|^{C^*(N+1)}$$

$$\leqq (K^{**})^2K^*|\tau^{-(m'+1)x}|^2|x|^{C^*(N+1)}.$$

Repeating this procedure, we obtain

$$(4.15') \qquad |H(x)| \leqq (K^{**})^j|\tau^{-(m'+1)x}|^j K^*|x|^{C^*(N+1)}.$$

If $|\operatorname{Re} x|$ is so large ($\operatorname{Re} x<0$) that

$$K^{**}|\tau^{-(m'+1)x}|<1,$$

then we have

$$H(x)=0 \qquad \text{if} \quad \operatorname{Re} x<(\log K^{**})/[(m'+1)\log\tau]$$

by letting $j\to\infty$ in (4.15′). Hence $H(x)\equiv 0$, and we obtain $h(x)\equiv 0$.

5. Proof of Theorem 3(1). I. Determination of formal solution.

LEMMA 5.1. *We have*

$$(5.1) \qquad \alpha_n u(x+n)+\cdots+\alpha_1 u(x+1)-B_{-1}u(x)$$

$$=\beta_n\Delta^n u(x)+\cdots+\beta_1\Delta u(x)+\beta_0 u(x),$$

where Δ^k denotes the k-th difference, and

$$(5.1') \qquad \beta_k=f_\infty^{(k)}(1)/k \qquad (\beta_0=f_\infty(1)),$$

in which $f_\infty(t)=\alpha_n t^n+\cdots+\alpha_1 t-B_{-1}$ (see (1.9)).

PROOF. Let $u(x)=t^x$ in (5.1). Then we obtain easily

$$f_\infty(t)=\beta_n(t-1)^n+\cdots+\beta_1(t-1)+\beta_0,$$

from which we get (5.1′). Q.E.D.

Suppose that $R(y)$ is expanded as in (1.8). We consider here the case that $\beta_0=0$ in (5.1). Thus

$$\alpha_n y(x+n)+\cdots+\alpha_1 y(x+1)-R(y(x)) \tag{5.2}$$
$$=\beta_n\Delta^n y(x)+\cdots+\beta_1\Delta y(x)-F(y(x))=0,$$

where

$$F(y)=B_m y^{-m}+B_{m+1}y^{-m-1}+\cdots \qquad \text{(see (1.8))}. \tag{5.2′}$$

Let κ be the number such that

$$\kappa=\min\{k\geqq 1\,;\,\beta_k\neq 0\}. \tag{5.2″}$$

We assume a formal solution of (5.2) in the form (1.18):

$$y(x)=\sum_{k=0}^{\infty}p_k(x)\Big(\frac{\log x}{x}\Big)^k, \tag{5.3}$$

where

$$p_k(x)=x^{\kappa/(m+1)}\Big[c_{0k}+\sum_{j=1}^{\infty}c_{jk}x^{-j/(m+1)}\Big]. \tag{5.3′}$$

Then

$$\Delta y(x)=\sum_{k=0}^{\infty}p_k(x+1)\Big[\Big(\frac{\log(x+1)}{x+1}\Big)^k-\Big(\frac{\log x}{x}\Big)^k\Big]$$
$$+\sum_{k=0}^{\infty}[p_k(x+1)-p_k(x)]\Big(\frac{\log x}{x}\Big)^k,$$

in which

$$\Big(\frac{\log(x+1)}{x+1}\Big)^j-\Big(\frac{\log x}{x}\Big)^j$$
$$=\Big(\frac{\log x}{x}+\frac{1}{x}\log\Big(1+\frac{1}{x}\Big)\Big)^j\Big(1+\frac{1}{x}\Big)^{-j}-\Big(\frac{\log x}{x}\Big)^j$$
$$=\Big[\Big(1+\frac{1}{x}\Big)^{-j}-1\Big]\Big(\frac{\log x}{x}\Big)^j$$
$$+\Big(1+\frac{1}{x}\Big)^{-j}\sum_{h=1}^{j}\binom{j}{h}\Big(\frac{1}{x}\log\Big(1+\frac{1}{x}\Big)\Big)^h\Big(\frac{\log x}{x}\Big)^{j-h}.$$

Thus, if we write

$$\Delta y(x)=\sum_{k=0}^{\infty}p_k^{(1)}(x)\Big(\frac{\log x}{x}\Big)^k,$$

then

$$(5.4)\qquad p_k^{(1)}(x)=p_k(x+1)-p_k(x)+p_k(x+1)\Big[\Big(1+\frac{1}{x}\Big)^{-k}-1\Big]$$
$$+\sum_{h=1}^{\infty}\Big(1+\frac{1}{x}\Big)^{-k-h}\binom{k+h}{h}\Big(\frac{1}{x}\log\Big(1+\frac{1}{x}\Big)\Big)^h p_{k+h}(x+1).$$

In general, if we write

$$\Delta^l y(x)=\sum_{k=0}^{\infty}p_k^{(l)}(x)\Big(\frac{\log x}{x}\Big)^k,$$

then

$$(5.5)\qquad p_k^{(l)}(x)=p_k^{(l-1)}(x+1)-p_k^{(l-1)}(x)+p_k^{(l-1)}(x+1)\Big[\Big(1+\frac{1}{x}\Big)^{-k}-1\Big]$$
$$+\sum_{h=1}^{\infty}\binom{k+h}{h}\Big(1+\frac{1}{x}\Big)^{-k-h}\Big(\frac{1}{x}\log\Big(1+\frac{1}{x}\Big)\Big)^h p_{k+h}^{(l-1)}(x+1),$$
$$l=1,\ 2,\ \cdots.$$

Then

$$(5.6)\qquad \alpha_n y(x+n)+\cdots+\alpha_1 y(x+1)-R'(\infty)y(x)$$
$$=\sum_{k=0}^{\infty}[\beta_n p_k^{(n)}(x)+\cdots+\beta_\kappa p_k^{(\kappa)}(x)]\Big(\frac{\log x}{x}\Big)^k=F(y(x)).$$

On the other hand, write

$$(5.7)\qquad y(x)=p_0(x)\Big(1+\sum_{k=1}^{\infty}p_k'(x)\Big(\frac{\log x}{x}\Big)^k\Big),$$

where

$$(5.7')\qquad p_k'(x)=p_k(x)/p_0(x)\qquad (p_0'(x)=1)$$

and

$$(5.8)\qquad \Big(1+\sum_{k=1}^{\infty}p_k'(x)\Big(\frac{\log x}{x}\Big)^k\Big)^{-1}=1+\sum_{k=1}^{\infty}p_k''(x)\Big(\frac{\log x}{x}\Big)^k\qquad (p_0''(x)=1),$$

where

$$(5.8')\qquad p_k''(x)=-\sum_{l=1}^{k}p_l'(x)p_{k-l}''(x).$$

Further, let

$$(5.9)\qquad \Big(1+\sum_{k=1}^{\infty}p_k'(x)\Big(\frac{\log x}{x}\Big)^k\Big)^{-s}=\Big(1+\sum_{k=1}^{\infty}p_k''(x)\Big(\frac{\log x}{x}\Big)^k\Big)^s$$
$$=1+\sum_{k=1}^{\infty}\tilde{p}_{*k}^{(s)}(x)\Big(\frac{\log x}{x}\Big)^k\qquad (\tilde{p}_{*0}^{(s)}(x)=1).$$

Then

$$\tilde{p}_{*k}^{(s)}(x)=\sum_{\substack{\nu_1+\cdots+\nu_s=s\\ j_1\nu_1+\cdots+j_s\nu_s=k\\ j_1<\cdots<j_s}}\frac{s!}{\nu_1!\cdots\nu_s!}p_{j_1}''(x)^{\nu_1}\cdots p_{j_s}''(x)^{\nu_s}.$$

Thus

$$(5.10)\qquad F(y)=\sum_{s=m}^{\infty}B_s p_0(x)^{-s}\Big(1+\sum_{k=1}^{\infty}p_{*k}^{(s)}(x)\Big(\frac{\log x}{x}\Big)^k\Big)$$

$$=\sum_{s=m}^{\infty}B_s p_0(x)^{-s}+\sum_{k=1}^{\infty}\Big(\sum_{s=m}^{\infty}B_s p_0(x)^{-s}\tilde{p}_{*k}^{(s)}(x)\Big)\Big(\frac{\log x}{x}\Big)^k .$$

From (5.6) and (5.10), we have

(5.11) $$\beta_n p_k^{(n)}(x)+\cdots+\beta_\kappa p_k^{(\kappa)}(x)=\sum_{s=m}^{\infty}B_s p_0(x)^{-s}\tilde{p}_{*k}^{(s)}(x) .$$

By these formulas, we will determine coefficients c_{jk}.

Put

(5.12) $$p_0(x)^{-1}=(x^{-\kappa/(m+1)}/c_{00})\Big(1+\sum_{j=1}^{\infty}c_{j0}^{(-1)}x^{-j/(m+1)}\Big)$$

and

(5.12′) $$p_0(x)^{-s}=(x^{-s\kappa/(m+1)}/c_{00}^{s})\Big(1+\sum_{j=1}^{\infty}c_{j0}^{(-s)}x^{-j/(m+1)}\Big).$$

Then

(5.13) $$c_{j0}^{(-1)}=-\sum_{l=1}^{j}(c_{l0}/c_{00})c_{j-l,0}^{(-1)} , \qquad c_{00}^{(-1)}=1 ,$$

and

(5.13′) $$c_{j0}^{(-s)}=\sum_{\substack{\nu_1+\cdots+\nu_s=s\\ k_1\nu_1+\cdots+k_s\nu_s=j\\ k_1<\cdots<k_s}}\frac{s!}{\nu_1!\cdots\nu_s!}(c_{k_10}^{(-1)})^{\nu_1}\cdots(c_{k_s0}^{(-1)})^{\nu_s} .$$

Further, put

$$p_k'(x)=(1/c_{00})\Big(c_{0k}+\sum_{j=1}^{\infty}c_{jk}x^{-j/(m+1)}\Big)\Big(1+\sum_{j=1}^{\infty}c_{j0}^{(-1)}x^{-j/(m+1)}\Big)=\sum_{j=0}^{\infty}c_{jk}'x^{-j/(m+1)} ,$$

then

(5.14) $$c_{0k}'=c_{0k}/c_{00} , \qquad c_{jk}'=\sum_{l=0}^{j}(c_{lk}/c_{00})c_{(j-l),0}^{(-1)} \quad (k\geqq 1) .$$

Moreover

$$p_k''(x)=-\sum_{l=1}^{k}p_l'(x)p_{k-l}''(x)=\sum_{j=0}^{\infty}c_{jk}''x^{-j/(m+1)},$$

then

(5.15) $$c_{jk}''=-\sum_{l=1}^{k}\Big(\sum_{t=0}^{j}c_{tl}'c_{(j-t)(k-l)}''\Big) \qquad (c_{j'0}''=0 \text{ if } j'\geqq 1) .$$

Thus, if we put

(5.16) $$\tilde{p}_{*k}^{(s)}(x)=\sum_{j=0}^{\infty}\tilde{c}_{*jk}^{(s)}x^{-j/(m+1)}=sp_k''(x)+\cdots ,$$

then

(5.16′) $\tilde{c}_{*jk}^{(s)}$=(a polynomial of $c_{j'k'}''$, $j'=0, \cdots, j$; $k'=0, \cdots, k-1$)$+sc_{jk}''$,

hence

(5.16″) $\tilde{c}_{*jk}^{(s)}$=(a polynomial of (c_{l0}/c_{00}), $1\leqq l\leqq j$, and of $c_{j'k'}$, $j'=0, \cdots, j$; $k'=1, \cdots, k-1$)$+(-s/c_{00})c_{jk}$.

Thus, if we put

$$p_0(x)^{-s}\tilde{p}^{(s)}_{*k}(x)=(x^{-s\kappa/(m+1)}/c^s_{00})\sum_{j=0}^{\infty} b^{(s)}_{jk}x^{-j/(m+1)}, \tag{5.17}$$

then

$$b^{(s)}_{jk}=\sum_{l=0}^{j} c^{(-s)}_{l0}\tilde{c}^{(s)}_{*(j-l)k}=c^{(s)}_{*jk}+B^{(s)}_{jk}(c_{l0}/c_{00},\ c_{j'k'}), \tag{5.17'}$$
$$1\leqq l\leqq j,\ 0\leqq j'\leqq j,\ 1\leqq k'\leqq k-1,$$

where $B^{(s)}_{jk}(\cdots)$ is a polynomial of the variables displayed there.

Write

$$\tilde{b}^{(s)}_{j,k}=b^{(s)}_{j-(s+1)\kappa,k}\quad \text{for}\quad j\geqq(s+1)\kappa, \tag{5.18}$$
$$=0\quad \text{for}\quad j<(s+1)\kappa.$$

As seen from (5.5), $p^{(l)}_k(x)$ begins with the term $x^{\kappa/(m+1)-l}$. Therefore we can write

$$p^{(l)}_k(x)=x^{\kappa/(m+1)}\sum_{j=(m+1)l}^{\infty} c^{(l)}_{jk}x^{-j/(m+1)},\qquad c^{(l)}_{jk}=0\quad \text{if } j<(m+1)l. \tag{5.19}$$

Then by (5.11)

$$\beta_n c^{(n)}_{jk}+\beta_{n-1}c^{(n-1)}_{jk}+\cdots+\beta_\kappa c^{(\kappa)}_{jk}=\sum_{s=m}^{\infty} B_s\tilde{b}^{(s)}_{jk}/c^s_{00}. \tag{5.20}$$

By (5.19)

$$p^{(l-1)}_k(x+1)-p^{(l-1)}_k(x)=x^{\kappa/(m+1)}\sum_j c^{(l-1)}_{jk}\left[\left(1+\frac{1}{x}\right)^{(\kappa-j)/(m+1)}-1\right]x^{-j/(m+1)}, \tag{5.21}$$

$$p^{(l-1)}_k(x+1)\left[\left(1+\frac{1}{x}\right)^{-k}-1\right] \tag{5.21'}$$
$$=x^{\kappa/(m+1)}\sum_j^{\infty} c^{(l-1)}_{jk}\left[\left(1+\frac{1}{x}\right)^{(\kappa-j)/(m+1)-k}-\left(1+\frac{1}{x}\right)^{(\kappa-j)/(m+1)}\right]x^{-j/(m+1)}.$$

If we write

$$\left(1+\frac{1}{x}\right)^{(\kappa-j)/(m+1)}=1+\sum_{t=1}^{\infty}\gamma_{tj}x^{-t},\qquad \gamma_{tj}=\binom{(\kappa-j)/(m+1)}{t},$$

then

$$p^{(l-1)}_k(x+1)-p^{(l-1)}_k(x)=x^{\kappa/(m+1)}\sum_j^{\infty}\Bigg(\sum_{\substack{j'+(m+1)t=j\\ t\geqq 1}} c^{(l-1)}_{j'k}\gamma_{tj'}\Bigg)x^{-j/(m+1)} \tag{5.22}$$

and

$$p^{(l-1)}_k(x+1)\left[\left(1+\frac{1}{x}\right)^{-k}-1\right] \tag{5.22'}$$
$$=x^{\kappa/(m+1)}\sum_j^{\infty}\Bigg(\sum_{\substack{j'+(m+1)t=j\\ t\geqq 1}} c^{(l-1)}_{j'k}(\gamma_{t,j'+k(m+1)}-\gamma_{t,j'})\Bigg)x^{-j/(m+1)}.$$

Further write

$$\left(1+\frac{1}{x}\right)^{-k-h}\left[\frac{1}{x}\log\left(1+\frac{1}{x}\right)\right]^h=x^{-2h}\sum_{t=0}^{\infty}\delta_{thk}x^{-t}\qquad (\delta_{0hk}=1)$$

and

$$\Big(1+\sum_{t=1}^{\infty}\delta_{thk}x^{-t}\Big)\Big(1+\sum_{t=1}^{\infty}\gamma_{tj}x^{-t}\Big)=1+\sum_{t=1}^{\infty}\Gamma_{thkj}x^{-t},$$

$$\Gamma_{thkj}=\sum_{t'+t''=t}\delta_{t'hk}\gamma_{t''j},$$

then

$$(5.22'')\qquad \binom{k+h}{h}\Big(1+\frac{1}{x}\Big)^{-k-h}\Big(\frac{1}{x}\log\Big(1+\frac{1}{x}\Big)\Big)^{h}p_{k+h}^{(l-1)}(x+1)$$

$$=\sum_{j}\Big(\binom{k+h}{h}\Big(\sum_{\substack{j'+t(m+1)+2h(m+1)=j\\ j'\geqq(m+1)(l-1),\ h\geqq 1}}\Gamma_{thkj'}c_{j',k+h}^{(-1)}\Big)\Big)x^{-j/(m+1)}.$$

Thus, by (5.5), (5.22), (5.22′), (5.22″), we have

$$(5.23)\qquad c_{jk}^{(l)}=\Big(\frac{\kappa-j}{m+1}+1-k\Big)c_{j-(m+1),k}^{(l-1)}$$

$$+F_{jk}^{(l)}(c_{j-2(m+1),k}^{(l-1)},\ \cdots,\ c_{j-(m+1)[j/(m+1)],k}^{(l-1)})$$

$$+G_{jk}^{(l)}(c_{j-2(m+1),k+1}^{(l-1)},\ \cdots,\ c_{j-2(m+1)[j/2(m+1)],k+[j/2(m+1)]}^{(l-1)}),$$

where $F_{jk}^{(l)}$ and $G_{jk}^{(l)}$ are linear functions of the variables displayed there. [ ] denotes the Gauss symbol, i.e., $[a]$, $a>0$, is the largest integer which does not exceed a.

We write (5.11) as (5.11_k). When $k=0$, we have

$$(5.11_0)\qquad \beta_n p_0^{(n)}(x)+\cdots+\beta_\kappa p_0^{(\kappa)}(x)=B_m p_0(x)^{-m}\tilde{p}_{*0}^{(m)}(x)+\cdots$$

or, writing (5.20) as (5.20_{jk}),

$$(5.20_{j0})\qquad \beta_n c_{j0}^{(n)}+\cdots+\beta_\kappa c_{j0}^{(\kappa)}=c_{00}^{-m}B_m b_{j-(m+1)\kappa,0}^{(m)}+c_{00}^{-m-1}B_{m+1}b_{j-(m+2)\kappa,0}^{(m+1)}+\cdots.$$

Since $c_{jk}^{(l)}=0$ for $j<(m+1)l$, we obtain by (5.20_{j0}), noting (5.18),

$$(5.24)\qquad \beta_\kappa c_{(m+1)\kappa,0}^{(\kappa)}=c_{00}^{-m}B_m b_{00}^{(m)}=c_{00}^{-m}B_m.$$

By (5.23)

$$(5.24')\qquad c_{(m+1)\kappa,0}^{(\kappa)}=\Big(\frac{\kappa}{m+1}-\kappa+1\Big)c_{(m+1)(\kappa-1),0}^{(\kappa-1)}$$

$$=\cdots=\Big(\frac{\kappa}{m+1}-\kappa+1\Big)\Big(\frac{\kappa}{m+1}-\kappa+2\Big)\cdots\Big(\frac{\kappa}{m+1}\Big)c_{00}.$$

From (5.24) and (5.24′), we obtain by our assumption (1.17)

$$(5.25)\qquad c_{00}^{m+1}=(B_m/\beta_\kappa)\Big(\Big(\frac{\kappa}{m+1}-\kappa+1\Big)\Big(\frac{\kappa}{m+1}-\kappa+2\Big)\cdots\Big(\frac{\kappa}{m+1}\Big)\Big)^{-1}\neq 0.$$

For j, $(m+1)\kappa<j<(m+1)(\kappa+1)$, we have $c_{j0}^{(\kappa+1)}=0$. Hence (5.20_{j0}) determines $c_{j0}^{(\kappa)}$, and the right hand side of (5.23) for $c_{j0}^{(\kappa)}$, $(m+1)\kappa<j<(m+1)(\kappa+1)$, contains only $c_{j-(m+1)\kappa,0}$. Therefore $c_{10},\ \cdots,\ c_{m0}$ are determined by (5.20_{j0}). In fact, we note the following relations from (5.14), (5.15), (5.16″), and (5.17′):

(5.26) $c_{j0}^{(-1)}=(-1/c_{00})c_{j0}+$(a polynomial of $c_{j'0},\ j'\leqq j-1$),

(5.26′) $c_{j0}^{(-s)}=(-s/c_{00})c_{j0}+$(a polynomial of $c_{j'0},\ j'\leqq j-1$),

(5.27) $c'_{jk}=(1/c_{00})c_{jk}+$(a polynomial of $c_{j'k},\ j'\leqq j-1$),

(5.27′) $c''_{jk}=(-1/c_{00})c_{jk}+$(a polynomial of $c_{j'k'},\ j'\leqq j,\ k'\leqq k-1$),

(5.28) $\tilde{c}_{*jk}^{(s)}=(-s/c_{00})c_{jk}+$(a polynomial of $c_{j'k'},\ j'\leqq j,\ k'\leqq k-1$),

(5.29) $b_{jk}^{(s)}=\tilde{c}_{*jk}^{(s)}+\cdots=(-s/c_{00})c_{jk}+$(a polynomial of $c_{j'k'},\ j'\leqq j,\ k'\leqq k-1$).

Put

$$C_{jk}=\Big(\frac{\kappa-j}{m+1}-k+1\Big)\cdots\Big(\frac{\kappa-j}{m+1}-k+\kappa\Big), \tag{5.30}$$

$$C'_{jk}=-m\Big(\frac{\kappa}{m+1}\Big)\cdots\Big(\frac{\kappa}{m+1}-\kappa+1\Big). \tag{5.30′}$$

(We note that C'_{jk} does not depend on $j,\ k$.) Then

$$\begin{cases} c_{jk}^{(\kappa)}=C_{jk}c_{j-(m+1)\kappa,\,k}+\cdots \\ \beta_\kappa^{-1}c_{00}^{-m}B_m b_{j-(m+1)\kappa,\,k}^{(m)}=C'_{jk}c_{j-(m+1)\kappa,\,k}+\cdots. \end{cases} \tag{5.31}$$

Since, for $j\geqq(m+1)$,

$$\begin{cases} C_{jk}=C'_{jk} \quad \text{if and only if} \\ \quad \text{either } j=(m+1)(\kappa+1),\ k=0 \quad \text{or} \quad j=(m+1)\kappa,\ k=1. \end{cases} \tag{5.32}$$

Thus, by (5.31) and (5.32), we see that $c_{10},\cdots,c_{m0}$ are determined. Further by (5.32), we see that $c_{m+1,0}$ can be arbitrarily prescribed. In fact, by (5.20_{j0}) for $j=(m+1)(\kappa+1)$,

$$\beta_{\kappa+1}c_{j0}^{(\kappa+1)}+\beta_\kappa c_{j0}^{(\kappa)}=c_{00}^{-m}B_m b_{m+1,0}^{(m)}+c_{00}^{-m-1}B_{m+1}b_{m+1-\kappa,0}^{(m+1)}+\cdots \tag{5.33}$$

in which $c_{j0}^{(\kappa)}$ contains c_{01}. By (5.32), the coefficients of $c_{m+1,0}$ on the both sides of (5.33) are equal, hence $c_{m+1,0}$ can be arbitrary. Further, (5.33) determines c_{01}. By (5.32), we see that this is consistent with other formulas.

Thus we obtain a formal solution in the form stated in the theorem.

6. Proof of Theorem 3(1). II. Existence of solution.

We will show the existence of solution by an application of Laplace transform, following the method of Harris and Sibuya [2].

6.1. As easily seen, there exists a function $U(x)$ such that

(6.1.1) $U(x)$ is holomorphic in $S_0=\Big\{x\,;\ |\arg(x+a)-\pi|<\dfrac{\pi}{2}+\varepsilon_0\Big\}$

and

(6.1.1′) $$U(x) \sim x^{\kappa/(m+1)}\left[c_{00}+\sum_{j+k\geq 1} c_{jk}x^{-j/(m+1)}\left(\frac{\log x}{x}\right)^{k}\right]$$

as x tends to ∞ in the sector S_0, where a $(a>0)$, ε_0 $(0<\varepsilon_0<\pi/2)$ are constants.

We fix a, ε_0 and such a function $U(x)$. Put

(6.1.2) $$y(x)=U(x)+z(x).$$

Then the difference equation (1.1) becomes

(6.1.3) $$\alpha_n z(x+n)+\cdots+\alpha_1 z(x+1)-B_{-1}z(x)=g(x, z(x)),$$

where

(6.1.4) $$g(x, z)=\sum_{\mu=m}^{\infty}\frac{B_\mu}{(U(x)+z)^\mu}-[\alpha_n U(x+n)+\cdots+\alpha_1 U(x+1)-B_{-1}U(x)].$$

$g(x, z)$ is holomorphic in

(6.1.5) $$|z|<\delta_0,\ |\arg(x+b)-\pi|<\frac{\pi}{2}+\varepsilon_0,$$

if δ_0 is sufficiently small and $b>0$ is sufficiently large. Further

(6.1.6) $$g(x, z)=x^{-2}h_0(x)+g_1(x, z),$$

where

(6.1.7) $$x^{-2}h_0(x)=\sum_{\mu=m}^{\infty}\frac{B_\mu}{U(x)^\mu}-[\alpha_n U(x+n)+\cdots+\alpha_1 U(x+1)-B_{-1}U(x)]$$

and

(6.1.8) $$g_1(x, z)=\sum_{l=1}^{\infty}\left[\sum_{\mu=m}^{\infty}C_{l\mu}\frac{B_\mu}{U(x)^{l+\mu}}\right]z^l,$$

in which $C_{l\mu}$ are coefficients of

(6.1.8′) $$(1+x)^{-\mu}=1+\sum_{l=1}^{\infty}C_{l\mu}x^l.$$

We write

(6.1.9) $$g_1(x, z)=B_1'x^{-\kappa}z+B_1''(x)z+\sum_{l=2}^{\infty}B_l(x)z^l,$$

where

(6.1.10) $$B_1'=C_{1m}B_m/c_{00}^{m+1},$$

(6.1.10′) $$B_1''(x)=\sum_{\mu=m+1}^{\infty}C_{1\mu}B_\mu/U(x)^{\mu+1}+C_{1m}B_m[U(x)^{-m-1}-c_{00}^{-m-1}x^{-\kappa}]$$

and

(6.1.10″) $$B_l(x)=\sum_{\mu=m}^{\infty}C_{l\mu}B_\mu/U(x)^{l+\mu},\qquad l\geqq 2.$$

6.2. Since the solution (1.18) of the equation (1.1) corresponds to a solution $z=\phi(x)$ of the equation (6.1.3) such that

(6.2.1) $$\phi(x) \sim 0, \quad \text{i.e.,} \quad \phi(x) \sim 0+0/x+0/x^2+\cdots$$

as x tends to ∞ in a sector, we consider the following problem.

We can write in (6.1.10′) and (6.1.10″)

(6.2.2) $$B_1''(x)=h_1(x)B_1^*(x), \qquad B_l(x)=h_1(x)B_l^*(x),$$

where

(6.2.3) $$h_1(x)=x^{-\kappa-1/(m+1)} \quad \text{if} \quad m\geqq 1;$$
$$=x^{-\kappa}\left(\frac{\log x}{x}\right) \quad \text{if} \quad m=0.$$

$h_0(x)$, $B_1''(x)$, and $B_l(x)$ are holomorphic in

(6.2.4) $$S_1=\left\{x\,;\,|\arg(x+b)-\pi|<\frac{\pi}{2}+\varepsilon_0\right\}$$

and

(6.2.5) $$h_0(x) \sim 0, \quad \text{i.e.,} \quad h_0(x) \sim 0+0/x+0/x^2+\cdots$$

as x tends to ∞ in the sector S_1.

Let $w(t)$, $k_0(t)$, $K(t)$, and $k_l(t)$ be inverse Laplace transforms of $z(x)$, $x^{-2}h_0(x)$, $h_1(x)B_1^*(x)$, and $h_1(x)B_l^*(x)$, respectively. Then the equation (6.1.3) corresponds to the following integral equation

(6.2.6) $$(\alpha_n e^{-nt}+\cdots+\alpha_1 e^{-t}-B_{-1})w(t)$$
$$=k_0(t)+B_1^*\int_0^t (t-s)^{\kappa-1}w(s)ds+\int_0^t K(t-s)w(s)ds$$
$$+\sum_{l=2}^{\infty}\int_0^t k_l(t-s)[w(s)]^l ds,$$

where $B_1^*=B_1'/[(\kappa-1)!]$, and $[w(t)]^l$ denotes an iterated convolution which is the inverse Laplace transform of $z(x)^l$.

Let $T_0'=\{t\,;\,|\arg t+\pi|<\varepsilon_0'\}$, and T_0 be

(6.2.7) $$T_0=\{t\,;\,|\arg t+\theta_0|<\varepsilon_0''\} \quad \text{for some } \theta_0 \text{ and } \varepsilon_0'',$$

which is a subdomain of T_0' such that $\alpha_n e^{-nt}+\cdots+\alpha_1 e^{-t}-B_{-1}\neq 0$ for $t\in T_0$. We shall prove the existence of a solution $w(t)$ which is

(i) holomorphic in T_0 of (6.2.7),

(ii) of exponential order as t tends to ∞ in T_0,

(iii) asymptotically equal to 0 as t tends to 0 in T_0.

Further, the Laplace transform of this solution $w(t)$ will be the solution satisfying (6.2.1), which corresponds to the desired solution (1.18).

6.3. First, we need some estimates of $k_0(t)$, $K(t)$, $k_l(t)$.

Let S_0 be the sector in (6.1.1) with sufficiently large $a>0$, and $S_0'=\{x\,;\,|\arg(x+a')-\pi|<\pi/2+\varepsilon_0\}$ with $0<a'<a-2$. Suppose $f(x)$ be a function holomorphic and bounded in S_0':

$$|f(x)|\leqq M \qquad \text{for} \quad x\in S_0'.$$

Further, let $h(x)$ be holomorphic in S_0' and satisfying

$$|h(x)|\leqq M'|x|^{-\alpha} \qquad \text{with } \alpha>1, \quad \text{for } x\in S_0'. \tag{6.3.0}$$

Let t be a number in T_0 of (6.2.7) and Γ_t be the path of integration in the x-plane defined by

$$\Gamma_t: \quad x=-a+se^{i\theta}, \quad -\infty<s<\infty, \tag{6.3.1}$$

where $\theta=\pi/2-\arg t$. Put

$$F(t)=\int_{\Gamma_t} h(\xi)f(\xi)e^{\xi t}d\xi. \tag{6.3.2}$$

LEMMA 6.3.1. *Let* $0<\varepsilon_0'<\varepsilon_0$. *Then*

$$|F(t)e^{at}|\leqq MM'|a\sin\theta|^{-\alpha+1}\int_{-\infty}^{\infty}|\mu-i|^{-\alpha}d\mu \tag{6.3.3}$$

for $t\in T_0$. *Further*

$$|F(t)e^{at}|\leqq MM'K(\alpha)|t|^{\alpha-1} \qquad \text{as} \quad t\to 0, \tag{6.3.4}$$

where $K(\alpha)$ *is a constant depending only on* α.

PROOF. We note that, for $t\in T_0$,

$$0<\frac{\pi}{2}-\varepsilon_0'<\theta-\pi<\frac{\pi}{2}+\varepsilon_0'.$$

On the other hand, since we have for $x\in\Gamma_t$

$$\begin{aligned}\arg(x+a)&=\theta && (s>0)\\ &=\theta-\pi && (s<0),\end{aligned}$$

we also have

$$|\arg(x+a)-\pi|<\frac{\pi}{2}+\varepsilon_0'<\frac{\pi}{2}+\varepsilon_0.$$

Thus

$$\begin{aligned}F(t)&=\int_{-\infty}^{\infty}h(-a+se^{i\theta})f(-a+se^{i\theta})e^{(-a+se^{i\theta})t}e^{i\theta}ds\\ &=e^{-at}e^{i\theta}\int_{-\infty}^{\infty}h(-a+se^{i\theta})f(-a+se^{i\theta})e^{is|t|}ds.\end{aligned}$$

Hence

$$|F(t)e^{at}|\leqq M\int_{-\infty}^{\infty}|h(-a+se^{i\theta})|ds\leqq MM'\int_{-\infty}^{\infty}|-a+se^{i\theta}|^{-\alpha}ds.$$

Put $s=\sigma+a\cos\theta$. Then

$$\int_{-\infty}^{\infty}|-a+se^{i\theta}|^{-\alpha}ds=\int_{-\infty}^{\infty}|\sigma-ia\sin\theta|^{-\alpha}d\sigma$$

$$=|a \sin\theta|^{-\alpha}\int_{-\infty}^{\infty}|(\sigma/a\sin\theta)-i|^{-\alpha}d\sigma$$

$$=|a\sin\theta|^{-\alpha+1}\int_{-\infty}^{\infty}|\mu-i|^{-\alpha}d\mu\,, \qquad \mu=\sigma/a\sin\theta\,,$$

which proves (6.3.3).

Further, if $\xi=-a+\zeta\in\Gamma_t$, then

$$F(t)=\int_{\Gamma_\theta}h(-a+\zeta)f(-a+\zeta)e^{-at+t\zeta}d\zeta\,,$$

where $\Gamma_\theta=\{se^{i\theta}\,;\,-\infty<s<\infty\}$. We write $t\zeta=\eta$, then

$$F(t)e^{at}=t^{-1}\int_{\Gamma^*}h\Big(-a+\frac{\eta}{t}\Big)f\Big(-a+\frac{\eta}{t}\Big)e^{\eta}d\eta\,,$$

where Γ^* is the imaginary axis. Write

$$h(x)=x^{-\alpha}h'(x)\,, \qquad |h'(x)|\leqq M' \quad \text{for } x\in S_0'\,.$$

Then

$$h\Big(-a+\frac{\eta}{t}\Big)=t^{\alpha}(-at+\eta)^{-\alpha}h'\Big(-a+\frac{\eta}{t}\Big),$$

and we can write

$$F(t)e^{at}=t^{\alpha-1}\int_{\Gamma^*}(-at+\eta)^{-\alpha}h'\Big(-a+\frac{\eta}{t}\Big)f\Big(-a+\frac{\eta}{t}\Big)e^{\eta}d\eta\,.$$

We change the path Γ^* of integration to Γ_η:

$$\Gamma_\eta=\{\eta=i\gamma\,;\,|\gamma|\geqq 1\}\cup\{\eta=e^{i\phi}\,;\,-\pi/2\leqq\phi\leqq\pi/2\}\,,$$

then $|-at+\eta|\geqq\delta$ on Γ_η for a $\delta>0$. Thus we obtain

$$|F(t)e^{at}|\leqq|t|^{\alpha-1}MM'\int_{\Gamma_\eta}|-at+\eta|^{-\alpha}d\eta\leqq MM'K(\alpha)|t|^{\alpha-1}\,.$$

LEMMA 6.3.2. *Let ε_1 be a constant such that*

$$\varepsilon_0'<\varepsilon_1<\varepsilon_0\,, \tag{6.3.5}$$

and let the path Γ_0 of integration be defined by

$$\Gamma_0\,:\; x=\begin{cases} -a+s\exp\Big[i\Big(\dfrac{3\pi}{2}+\varepsilon_1\Big)\Big] & s\geqq 0\,,\\ -a+s\exp\Big[i\Big(\dfrac{3\pi}{2}-\varepsilon_1\Big)\Big] & s<0\,.\end{cases} \tag{6.3.6}$$

Then for $t\in T_0$

$$F(t)=\int_{\Gamma_0}h(\xi)f(\xi)e^{\xi t}d\xi\,. \tag{6.3.7}$$

Therefore, $F(t)$ is holomorphic in T_0.

PROOF. Note that, on Γ_0,

$$|\arg(x+a)-\pi|<\frac{\pi}{2}+\varepsilon_1<\frac{\pi}{2}+\varepsilon_0 .$$

Put $\omega=\arg t$. Consider the relation

$$\begin{aligned}&\int_{\Gamma_0} h(\xi)f(\xi)e^{\xi t}d\xi\\ &=\int_0^{\infty} h(\xi)f(\xi)e^{-at+i(3\pi/2+\varepsilon_1)}\exp[s|t|e^{i(3\pi/2+\varepsilon_1+\omega)}]ds\\ &\quad+\int_{-\infty}^{0} h(\xi)f(\xi)e^{-at+i(3\pi/2-\varepsilon_1)}\exp[s|t|e^{i(3\pi/2-\varepsilon_1+\omega)}]ds .\end{aligned}$$

Since

$$\pi/2<\pi/2+\varepsilon_1-\varepsilon_0'\leqq\pi/2+\varepsilon_1+\omega+\pi\leqq\pi/2+\varepsilon_1+\varepsilon_0'<3\pi/2 ,$$

$$-\pi/2<\pi/2-\varepsilon_1-\varepsilon_0'\leqq\pi/2-\varepsilon_1+\omega+\pi\leqq\pi/2-\varepsilon_1+\varepsilon_0'<\pi/2 ,$$

the integral is well defined. To prove the equality (6.3.7), it is sufficient to prove that the integrals of $h(\xi)f(\xi)e^{\xi t}$ on the arcs

$$|x+a|=R , \quad \theta\leqq\arg(x+a)\leqq3\pi/2+\varepsilon_1 , \qquad \text{and}$$

$$|x+a|=R , \quad \pi/2-\varepsilon_1\leqq\arg(x+a)\leqq\theta-\pi$$

tend to 0 as $R\to\infty$. It is easily seen that on these arcs we have

$$\pi/2=\theta+\omega\leqq\arg(x+a)+\omega\leqq3\pi/2+\varepsilon_1+\omega<3\pi/2 ,$$

$$-3\pi/2<\pi/2-\varepsilon_1+\omega\leqq\arg(x+a)+\omega\leqq\theta+\omega-\pi=-\pi/2 .$$

This implies that these integrals tend to 0 as $R\to\infty$. Thus the proof of Lemma 6.3.2 is completed.

LEMMA 6.3.3. *Let C_ω be the path of integration in the t-plane defined by*

(6.3.8) $$C_\omega:\ t=\tau e^{i\omega},\quad 0\leqq\tau<\infty \qquad (\omega=\arg t) .$$

Then we have

(6.3.9) $$h(x)f(x)=\frac{1}{2\pi i}\int_{C_\omega} F(t)e^{-xt}dt$$

for x in

(6.3.10) $$S_t=\{x\,;\,|\arg(x+a)+\omega|<\pi/2\} .$$

PROOF. Note that $F(t)e^{at}$ is bounded and that

$$F(t)e^{-xt}=F(t)e^{at}e^{-(x+a)t}.$$

Hence the right member of (6.3.9) is well defined and holomorphic for $x\in S_t$. If $|\arg(x+a)-\pi|<\pi/2-\varepsilon_0$, then $x\in S_t$. Therefore

$$\frac{1}{2\pi i}\int_{C_\omega}F(t)e^{-xt}dt=\frac{1}{2\pi i}\int_{C_\omega}\left[\int_{\Gamma_0}h(\xi)f(\xi)e^{\xi t}dt\right]e^{-xt}dt$$
$$=\frac{1}{2\pi i}\int_{\Gamma_0}h(\xi)f(\xi)d\xi\int_{C_\omega}e^{(\xi-x)t}dt=-\frac{1}{2\pi i}\int_{\Gamma_0}\frac{h(\xi)f(\xi)}{\xi-x}d\xi$$
$$=h(x)f(x).$$

Since the both sides of (6.3.9) are holomorphic in S_t, we have the equality (6.3.9) for $x\in S_t$.

LEMMA 6.3.4. *When $h(x)\sim 0$ as x tends to ∞ in the sector S_0 of (6.1.1), then $F(t)\sim 0$ as t tends to 0 in the sector T_0 of (6.2.7).*

The proof is easily obtained by Lemma 6.3.1.

LEMMA 6.3.5. *Assume that $g(t)$ is holomorphic in T_0 and*

$$|g(t)|\leqq M_1\exp[\sigma|t|] \qquad (\sigma>0).$$

Further, assume that $g(t)\sim 0$ as t tends to 0 in T_0. Put

$$f(x)=\int_{C_\omega}g(t)e^{-at}e^{-xt}dt.$$

Then $f(x)$ is holomorphic in

$$|\arg(x+a)+\omega|\leqq\pi/2-\gamma, \qquad |x+a|>\sigma/\sin\gamma$$

and

$$f(x)\sim 0$$

as x tends to 0 in this sector, where $\gamma>0$ is sufficiently small.

The proof is easy and may be omitted, see [**2**, p. 128].

6.4. Put

$$(6.4.1)\qquad \begin{cases} k_0(t)=\int_{\Gamma_t}\xi^{-2}h_0(\xi)e^{\xi t}d\xi, \\ K(t)=\int_{\Gamma_t}h_1(\xi)B_1^*(\xi)e^{\xi t}d\xi, \\ k_l(t)=\int_{\Gamma_t}h_1(\xi)B_l^*(\xi)e^{\xi t}d\xi, \end{cases} \qquad \begin{pmatrix}\Gamma_t \text{ is the path of integration}\\ \text{in the sector } S_1 \text{ of (6.2.4)}\end{pmatrix},$$

where $t\in T_0$ and $\Gamma_t=\{x;\ x=-b+se^{i\theta},\ -\infty<s<\infty,\ \theta=\pi/2-\arg t\}$. We note that $h_1(x)$ satisfies (6.3.0) with an $\alpha>1$, as seen from (6.2.3).

Let $C(t)$ be the path of integration in the t-plane defined by

$$(6.4.2)\qquad C(t):\ s=\tau e^{i\omega},\quad 0\leqq\tau\leqq|t|,$$

where $\omega=\arg t$. Consider the equation

(6.4.3)
$$(\alpha_n e^{-nt}+\cdots+\alpha_1 e^{-t}-B_{-1})w(t)$$
$$=k_0(t)+B_1^*\int_{C(t)}(t-s)^{\kappa-1}w(s)ds+\int_{C(t)}K(t-s)w(s)ds$$
$$+\sum_{l=2}^{\infty}\int_{C(t)}k_l(t-s)[w(s)]^l ds\,,$$

where $[w(s)]^l$ is an iterated convolution defined as

$$[w(t)]^k=\int_{C(t)}w(t-s)[w(s)]^{k-1}ds\,.$$

Put

(6.4.4)
$$w(t)=e^{-bt}u(t)\,,$$
$$k_0(t)=e^{-bt}\hat{k}_0(t)\,,$$
$$K(t)=e^{-bt}\hat{K}(t)\,,$$
$$k_l(t)=e^{-bt}\hat{k}_l(t)\,,$$

with b in (6.2.4). Since

$$[w(t)]^l=e^{-bt}[u(t)]^l,$$

the equation (6.4.3) becomes

(6.4.5)
$$h_3(t)u(t)=\hat{k}_0(t)+B_1^*\int_{C(t)}e^{-b(s-t)}(t-s)^{\kappa-1}u(s)ds$$
$$+\int_{C(t)}\hat{K}(t-s)u(s)ds+\sum_{l=2}^{\infty}\int_{C(t)}\hat{k}_l(t-s)[u(s)]^l ds\,,$$

where

(6.4.6)
$$h_3(t)=\alpha_n e^{-nt}+\cdots+\alpha_1 e^{-t}-B_{-1}$$
$$=\beta_n(e^{-t}-1)^n+\cdots+\beta_\kappa(e^{-t}-1)^\kappa.$$

6.5. It is easy to see that

(6.5.1)
$$|h_3(t)|\geqq|t|^\kappa/L \qquad \text{for} \quad t\in T_0$$

with a constant $L>0$.

By the assumption (6.2.5) and Lemma 6.3.4, we have

(6.5.2)
$$\hat{k}_0(t)\sim 0, \qquad \text{hence} \qquad \hat{k}_0(t)/h_3(t)\sim 0$$

as t tends to 0 in T_0. Hence for every positive integer μ, there exists a positive constant L_μ such that

(6.5.3)
$$|\hat{k}_0(t)/h_3(t)|\leqq L_\mu L|t|^\mu.$$

We can assume that

(6.5.4) $$|B_1^*(x)|\leqq M_1, \qquad |B_l^*(x)|\leqq M_2/\delta_1^l$$

for $x\in S_1$ in (6.2.4), where M_1, M_2, δ_1 are positive constants.

By Lemma 6.3.1,

(6.5.5) $$|\hat{K}(t)|\leqq M_1K(\alpha)|t|^{\alpha-1},$$
$$|\hat{k}_l(t)|\leqq(M_2/\delta_1^l)K(\alpha)|t|^{\alpha-1},$$

where

(6.5.6) $$\begin{aligned}\alpha&=\kappa+(m+1)^{-1} && \text{if } m\geqq 1,\\ &=\kappa+\alpha' \quad \text{for any } \alpha',\ 0<\alpha'<1, && \text{if } m=0.\end{aligned}$$

6.6. For convenience in constructing a solution of the integral equation (6.4.5), we introduce a parameter ε into (6.4.5) and consider the equation

(6.6.1) $$\begin{aligned}h_3(t)u(t,\varepsilon)=\hat{k}_0(t)+B_1^*\int_{C(t)}e^{-\delta(s-t)}(t-s)^{\kappa-1}[\varepsilon u(s,\varepsilon)]ds\\ +\int_{C(t)}\hat{K}(t-s)[\varepsilon u(s,\varepsilon)]ds+\sum_{l=2}^{\infty}\int_{C(t)}\hat{k}_l(t-s)[\varepsilon u(s,\varepsilon)]^l ds.\end{aligned}$$

We can construct a formal solution of (6.6.1) in the form

(6.6.2) $$u(t,\varepsilon)=\sum_{\nu=0}^{\infty}\varepsilon^{\nu}u_{\nu}(t),$$

by solving the sequence of equations

(6.6.3) $$\begin{aligned}&h_3(t)u_0(t)=\hat{k}_0(t),\\ &h_3(t)u_{\nu}(t)=\Upsilon_{\nu}(t), \qquad \nu=1,2,\cdots\end{aligned}$$

where $\Upsilon_{\nu}(t)$ depends only on $u_0(t),\cdots,u_{\nu-1}(t)$. It is easily seen that $u_{\nu}(t)$ are holomorphic in T_0, if ε_0' is sufficiently small. If the series (6.6.2) converges uniformly for $|\varepsilon|\leqq 1$ and for t in any compact set of T_0, then

(6.6.4) $$u(t)=u(t,1)$$

is a solution of (6.4.5).

6.7. We shall prove the convergence of (6.6.2) for $|\varepsilon|\leqq 1$ by the method of majorants. Let τ be a real nonnegative variable. Consider the following integral equation (writing $|B_1^*|$ as B_1'):

(6.7.1) $$\begin{aligned}L^{-1}\tau^{\kappa}v(\tau,\varepsilon)=L_{\mu}\tau^{\mu+\kappa}+B_1'\tau^{\kappa-1}\int_0^{\tau}\varepsilon v(s,\varepsilon)ds\\ +M_1K(\alpha)\tau^{\kappa-1}\int_0^{\tau}\varepsilon v(s,\varepsilon)ds\\ +\sum_{l=2}^{\infty}(M_2/\delta_1^l)K(\alpha)\tau^{\kappa-1}\int_0^{\tau}[\varepsilon v(s,\varepsilon)]^l ds,\end{aligned}$$

i.e.,

(6.7.2) $$L^{-1}\tau v(\tau,\ \varepsilon)=L_{\mu}\tau^{\mu+1}+B_1'\int_0^{\tau}\varepsilon v(s,\ \varepsilon)ds+M_1K(\alpha)\int_0^{\tau}\varepsilon v(s,\ \varepsilon)ds + \sum_{l=2}^{\infty}(M_2/\delta_1^l)K(\alpha)\int_0^{\tau}[\varepsilon v(s,\ \varepsilon)]^l ds\,,$$

with $K(\alpha)$ in Lemma 6.3.1. We can construct a formal solution of (6.7.1) in the form

(6.7.3) $$v(\tau,\ \varepsilon)=\sum_{\nu=0}^{\infty}\varepsilon^{\nu}v_{\nu}(\tau)$$

by solving the sequence of equations

(6.7.4) $$L^{-1}v_0(\tau)=L_{\mu}\tau^{\mu},$$
$$L^{-1}v_{\nu}(\tau)=\mathfrak{Q}_{\nu}(\tau)\,, \qquad \nu=1,\ 2,\ \cdots$$

where $\mathfrak{Q}_{\nu}(\tau)$ depends only on $v_0(\tau),\ \cdots,\ v_{\nu-1}(\tau)$.

It is easily seen that $v_{\nu}(\tau)$ are nonnegative for $\tau\geqq 0$ and that

(6.7.5) $$|u_{\nu}(t)|\leqq v_{\nu}(|t|)$$

for $t\in T_0$ in (6.2.7). Hence, if the series (6.7.3) converges uniformly for $|\varepsilon|\leqq 1$ and for τ in any bounded interval in $0\leqq\tau<\infty$, the series (6.6.2) also converges uniformly for $|\varepsilon|\leqq 1$ and for t in any compact set of T_0.

6.8. Consider the following differential equation:

(6.8.1) $$-L^{-1}\frac{d}{dx}p(x,\ \varepsilon)=(\mu+1)\cdot L_{\mu}x^{-\mu-2}+B_1'x^{-1}\varepsilon p(x,\ \varepsilon) +M_1K(\alpha)x^{-1}\varepsilon p(x,\ \varepsilon)+x^{-1}\sum_{l=2}^{\infty}(M_2/\delta_1^l)K(\alpha)\varepsilon^l p(x,\ \varepsilon)^l.$$

Put $x=1/\zeta$. Then (6.8.1) becomes

(6.8.2) $$L^{-1}\zeta\frac{d}{d\zeta}\tilde{p}(\zeta,\ \varepsilon)=(\mu+1)\cdot L_{\mu}\zeta^{\mu+1}+B_1'\varepsilon\tilde{p}(\zeta,\ \varepsilon) +M_1K(\alpha)\varepsilon\tilde{p}(\zeta,\ \varepsilon)+\sum_{l=2}^{\infty}(M_2/\delta_1^l)K(\alpha)\varepsilon^l\tilde{p}(\zeta,\ \varepsilon)^l,$$

where we write $p(1/\zeta,\ \varepsilon)$ as $\tilde{p}(\zeta,\ \varepsilon)$. (6.8.2) is an equation of Briot-Bouquet type, and admits a unique solution which is holomorphic at $\zeta=0$ and $\tilde{p}(0,\ \varepsilon)=0$ [**3**, p. 403]. Therefore (6.8.1) possesses a solution $p(x,\ \varepsilon)$ such that

(6.8.3) $$p(x,\ \varepsilon)=\sum_{\beta=1}^{\infty}x^{-\beta}p_{\beta}(\varepsilon)\,.$$

The coefficients $p_{\beta}(\varepsilon)$ can be determined by inserting this series into (6.8.1) and equating the coefficients of $x^{-\beta}$. Then $p_{\beta}(\varepsilon)=0$ for $\beta=1,\ \cdots,\ \mu$. If μ is so large that

(6.8.4) $$-\mu+\varepsilon L(B_1'+M_1K(\alpha))\neq 0 \qquad \text{for}\ \ |\varepsilon|\leqq 2\,,$$

then $p_\beta(\varepsilon)$ are holomorphic in $|\varepsilon|\leqq 2$. Thus

(6.8.5) $$p(x, \varepsilon)=\sum_{\beta=\mu+1}^{\infty} x^{-\beta} p_\beta(\varepsilon).$$

Since (6.8.5) is convergent, we have the estimates

(6.8.6) $$|p_\beta(\varepsilon)|\leqq M(\rho_0)/\xi_0^\beta \quad \text{for } |\varepsilon|\leqq\rho_0<2,$$

where $M(\rho_0)$ is a positive constant. Put

(6.8.7) $$\tilde{v}(\tau, \varepsilon)=\sum_{\beta=\mu+1}^{\infty}(\tau^{\beta-1}/(\beta-1)!)p_\beta(\varepsilon),$$

(6.8.8) $$\sum_{\beta=\mu+1}^{\infty}(\tau^{\beta-1}/(\beta-1)!)|p_\beta(\varepsilon)|\leqq M(\rho_0)\xi_0^{-1}\sum_{\beta=\mu+1}^{\infty}(\tau/\xi_0)^{\beta-1}/(\beta-1)! \leqq M(\rho_0)\xi_0^{-1}e^{\tau/\xi_0}$$

for $|\varepsilon|\leqq\rho_0<2$ and arbitrary τ, then the function $\tilde{v}(\tau, \varepsilon)$ is an entire function of τ and is holomorphic for ε, $|\varepsilon|<2$. Hence we may write

(6.8.9) $$\tilde{v}(\tau, \varepsilon)=\sum_{\nu=0}^{\infty}\varepsilon^\nu v_\nu(\tau),$$

where this series converges uniformly on any compact set of $\{|\tau|<\infty\}\times\{|\varepsilon|<2\}$.

We shall show that, as formal series in ε, we have

(6.8.10) $$v(\tau, \varepsilon)=\tilde{v}(\tau, \varepsilon),$$

where $v(\tau, \varepsilon)$ is the formal solution (6.7.3) of the integral equation (6.7.1). To demonstrate this, it is sufficient to show that $\tilde{v}(\tau, \varepsilon)$ is a solution of (6.7.1).

Note that the identity

$$x^{-\beta}=\int_0^\infty \frac{\tau^{\beta-1}}{(\beta-1)!}e^{-\tau x}d\tau$$

and (6.8.7) yield the representation

(6.8.11) $$p(x, \varepsilon)=\int_0^\infty \tilde{v}(\tau, \varepsilon)e^{-\tau x}d\tau$$

for $|\varepsilon|\leqq\rho_0<2$ and $\mathrm{Re}[x]>\xi_0^{-1}$. By substituting $\tilde{v}(\tau, \varepsilon)$ into both sides of the equation (6.7.1) we obtain two functions which are holomorphic in (τ, ε), $|\tau|<\infty$, $|\varepsilon|<2$. The Laplace transforms of these two functions are equal since $p(x, \varepsilon)$ is the unique solution of (6.8.1). Therefore these two functions are the same, and $\tilde{v}(\tau, \varepsilon)$ is a solution of (6.7.1).

Since $\tilde{v}=v$, $v(\tau, \varepsilon)$ and hence $u(t, \varepsilon)$ also converge on any compact subset of the region T_0 for $|\varepsilon|<2$.

6.9. Inequalities (6.7.5) and (6.8.8) imply

(6.9.1) $$|u(t, \varepsilon)|\leqq M(\rho_0)\xi_0^{-1}e^{|t|/\xi_0}$$

for $|\varepsilon|\leqq\rho_0<2$ and arbitrary values of t in T_0. On the other hand, since $v(\tau, \varepsilon)=O(\tau^{\mu+1})$ as $\tau\to 0$, we have

(6.9.2) $$u(t, \varepsilon)=O(|t|^{\mu+1})$$

as t tends to 0 in T_0. Since μ is arbitrary, we have

(6.9.3) $$u(t, \varepsilon) \sim 0$$

as t tends to 0 in T_0.

If we define $u(t)$ by $u(t)=u(t, 1)$, we get a solution $u(t)$ of the equation (6.4.5) which satisfies the following conditions:

(i) $u(t)$ is holomorphic in T_0,

(ii) $u(t)$ is of exponential order as t tends to ∞ in T_0,

(iii) $u(t)\sim 0$ as t tends to 0 in T_0.

6.10. Put

(6.10.1) $$w(t)=e^{-bt}u(t)\,,$$

where $u(t)$ is the function determined in § 6.9.

Since $u(t)$ is a solution of (6.4.5), $w(t)$ is a solution of (6.4.3) which satisfies the following conditions:

(i) $w(t)$ is holomorphic in T_0,

(ii) $w(t)$ is of exponential order as t tends to ∞ in T_0,

(iii) $w(t)\sim 0$ as t tends to 0 in T_0,

which proves our theorem.

7. Proof of Theorem 3(2). I. Determination of formal solution.

As in (5.2), we obtain

(7.1) $$\beta_n \Delta^n y(x)+\cdots+\beta_\kappa \Delta^\kappa y(x)=F(y(x))\,,$$

where

(7.1′) $$F(y)=B_m y^{-m}+B_{m+1}y^{-m-1}+\cdots \qquad \text{(see (1.8))}.$$

We assume a formal solution of (7.1) in the form (1.18′):

(7.2) $$y(x)=\sum_{k=0}^{\infty} q_k(x)(\log x)^{(1-k)/(m+1)},$$

where

(7.2′) $$q_k(x)=x^{\kappa/(m+1)}\Big[c_{0k}+\sum_{j=1}^{\infty} c_{jk}x^{-j/(m+1)}\Big].$$

Then

$$\varDelta y(x)=\sum_{k=0}^{\infty}q_k(x+1)[(\log(x+1))^{(1-k)/(m+1)}-(\log x)^{(1-k)/(m+1)}]$$

$$+\sum_{k=0}^{\infty}[q_k(x+1)-q_k(x)](\log x)^{(1-k)/(m+1)},$$

in which

$$(\log(x+1))^{(1-k')/(m+1)}=(\log x)^{(1-k')/(m+1)}\left[1+\log\left(1+\frac{1}{x}\right)\Big/\log x\right]^{(1-k')/(m+1)}$$

$$=(\log x)^{(1-k')/(m+1)}\left\{1+\sum_{h=1}^{\infty}\binom{(1-k')/(m+1)}{h}\left(\log\left(1+\frac{1}{x}\right)\right)^h(\log x)^{-h}\right\}.$$

Thus, if we write

$$\varDelta y(x)=\sum_{k=0}^{\infty}q_k^{(1)}(x)(\log x)^{(1-k)/(m+1)},$$

then

$$q_k^{(1)}(x)=q_k(x+1)-q_k(x)+\sum_{\substack{k'+h(m+1)=k\\ h\geqq 1}}\binom{(1-k')/(m+1)}{h}\left(\log\left(1+\frac{1}{x}\right)\right)^h q_{k'}(x+1).$$

In general, if we write

$$\varDelta^l y(x)=\sum_{k=0}^{\infty}q_k^{(l)}(x)(\log x)^{(1-k)/(m+1)}, \tag{7.3}$$

then

$$q_k^{(l)}(x)=\sum_{\substack{k'+h(m+1)=k\\ h\geqq 1}}\binom{(1-k')/(m+1)}{h}\left(\log\left(1+\frac{1}{x}\right)\right)^h q_{k'}^{(l-1)}(x+1) \tag{7.4}$$

$$+q_k^{(l-1)}(x+1)-q_k^{(l-1)}(x).$$

Let

$$q_k(x)=x^{\kappa/(m+1)}\sum_{j=0}^{\infty}c_{jk}x^{-j/(m+1)}.$$

Then

$$q_k(x+1)-q_k(x)=x^{\kappa/(m+1)}\sum_{j=0}^{\infty}c_{jk}x^{-j/(m+1)}\left[\left(1+\frac{1}{x}\right)^{(\kappa-j)/(m+1)}-1\right] \tag{7.5}$$

$$=x^{\kappa/(m+1)}\sum_{j=0}^{\infty}\Big(\sum_{\substack{j'+t(m+1)=j\\ t\geqq 1}}c_{j'k}D_{tj'}\Big)x^{-j/(m+1)}$$

where $D_{tj'}$ are the coefficients of the expansion

$$(1+1/x)^{(\kappa-j')/(m+1)}=1+\sum_{t=1}^{\infty}D_{tj'}x^{-t}. \tag{7.5'}$$

Further

$$q_{k'}(x+1)\left(\log\left(1+\frac{1}{x}\right)\right)^h=x^{\kappa/(m+1)}\sum_{j=0}^{\infty}\Big(\sum_{\substack{j'+t(m+1)=j\\ t\geqq h}}c_{j'k'}E_{thj'}\Big)x^{-j/(m+1)}, \tag{7.6}$$

where $E_{thj'}$ are the coefficients of the expansion

$$\left(1+\frac{1}{x}\right)^{(\kappa-j')/(m+1)}\left(\log\left(1+\frac{1}{x}\right)\right)^h=\sum_{t=h}^{\infty}E_{thj'}x^{-t}. \tag{7.6'}$$

Thus, if we write

$$q_k^{(l)}(x)=x^{\kappa/(m+1)}\sum_{j=0}^{\infty}c_{jk}^{(l)}x^{-j/(m+1)}, \tag{7.7}$$

then

$$c_{jk}^{(l)}=\sum_{\substack{j'+t(m+1)=j\\ t\geq 1}}c_{j'k}^{(l-1)}D_{tj'} + \sum_{\substack{k'+h(m+1)=k\\ h\geq 1}}\left\{\binom{(1-k')/(m+1)}{h}\sum_{\substack{j'+t(m+1)=j\\ t\geq h}}c_{j'k'}^{(l-1)}E_{thj'}\right\}. \tag{7.8}$$

Obviously

$$c_{jk}^{(l)}=0 \qquad \text{if} \quad j<(m+1)l. \tag{7.8'}$$

By assumption, $\kappa/(m+1)$ is an integer. We put

$$\Gamma=\frac{\Gamma(z)}{\Gamma(z-\kappa)}\left.\frac{1/(m+1)}{(z-\kappa/(m+1))}\right|_{z=\kappa/(m+1)}. \tag{7.9}$$

Then we can easily obtain that

$$c_{(m+1)\kappa,\,m+1}^{(\kappa)}=\Gamma c_{00}. \tag{7.10}$$

Write

$$\begin{aligned} y(x)&=\sum_{k=0}^{\infty}q_k(x)(\log x)^{(1-k)/(m+1)}\\ &=q_0(x)(\log x)^{1/(m+1)}\left(1+\sum_{k=1}^{\infty}q_k'(x)(\log x)^{-k/(m+1)}\right), \end{aligned} \tag{7.11}$$

then

$$q_k'(x)=q_k(x)/q_0(x), \qquad q_0'(x)=1. \tag{7.11'}$$

Further write

$$1/y(x)=(q_0(x)(\log x)^{1/(m+1)})^{-1}\left(1+\sum_{k=1}^{\infty}q_k''(x)(\log x)^{-k/(m+1)}\right), \tag{7.12}$$

then

$$q_k''(x)=-\sum_{l=1}^{k}q_l'(x)q_{k-l}''(x), \qquad q_0''(x)=1. \tag{7.13}$$

Moreover

$$\begin{aligned} y(x)^{-s}&=q_0(x)^{-s}(\log x)^{-s/(m+1)}\left(1+\sum_{k=1}^{\infty}\tilde{q}_{*k}^{(s)}(x)(\log x)^{-k/(m+1)}\right)\\ &=q_0(x)^{-s}\left[(\log x)^{-s/(m+1)}+\sum_{k=1}^{\infty}\tilde{q}_{*k}^{(s)}(x)(\log x)^{(-k-s)/(m+1)}\right]\\ &=q_0(x)^{-s}\sum_{k=s+1}^{\infty}\bar{q}_k^{(s)}(x)(\log x)^{(1-k)/(m+1)}, \end{aligned} \tag{7.14}$$

in which

$$\begin{aligned} &\bar{q}_{s+1}^{(s)}(x)=1, \qquad \bar{q}_k^{(s)}(x)=0 \quad \text{if} \quad k\leqq s,\\ &\bar{q}_k^{(s)}(x)=\tilde{q}_{*(k-s-1)}^{(s)}(x) \qquad \text{if} \quad k\geqq s+1, \end{aligned} \tag{7.14'}$$

and

(7.15) $$\bar{q}_{*k}^{(s)}(x)=\sum_{\substack{\nu_1+\cdots+\nu_s=s\\ j_1\nu_1+\cdots+j_s\nu_s=k\\ j_1<\cdots<j_s}}\frac{s!}{\nu_1!\cdots\nu_s!}q''_{j_1}(x)^{\nu_1}\cdots q''_{j_s}(x)^{\nu_s}.$$

Then

(7.16) $$\sum_{s=m}^{\infty}B_s/y(x)^s=\sum_{s=m}^{\infty}B_sq_0(x)^{-s}\Big(\sum_{k=s+1}^{\infty}\bar{q}_k^{(s)}(x)(\log x)^{(1-k)/(m+1)}\Big)$$
$$=\sum_{k=m+1}^{\infty}\Big(\sum_{s=m}^{k-1}B_sq_0(x)^{-s}\bar{q}_k^{(s)}(x)\Big)(\log x)^{(1-k)/(m+1)}.$$

Therefore

(7.17) $$\beta_nq_k^{(n)}(x)+\cdots+\beta_\kappa q_k^{(\kappa)}(x)=0 \qquad \text{if} \quad k\leqq m,$$

(7.17′) $$\beta_nq_{m+1}^{(n)}(x)+\cdots+\beta_\kappa q_{m+1}^{(\kappa)}(x)=B_m/q_0(x)^m.$$

In general,

(7.18) $$\beta_nq_k^{(n)}(x)+\cdots+\beta_\kappa q_k^{(\kappa)}(x)=\sum_{s=m}^{k-1}(B_s/q_0(x)^s)\bar{q}_k^{(s)}(x), \qquad \text{if} \quad k\geqq m+1.$$

By these formulas, we determine coefficients c_{jk}.

Write

(7.19) $$q_0(x)=x^{\kappa/(m+1)}\sum_{j=0}^{\infty}c_{j0}x^{-j/(m+1)}=c_{00}x^{\kappa/(m+1)}\Big[1+\sum_{j=1}^{\infty}c'_{j0}x^{-j/(m+1)}\Big],$$

(7.19′) $$c'_{j0}=c_{j0}/c_{00}, \qquad c'_{00}=1.$$

Further, write

(7.20) $$q_0(x)^{-1}=\frac{x^{-\kappa/(m+1)}}{c_{00}}\Big[1+\sum_{j=1}^{\infty}c_{j0}^{(-1)}x^{-j/(m+1)}\Big] \qquad (c_{00}^{(-1)}=1)$$

and

(7.21) $$q_0(x)^{-s}=(x^{-s\kappa/(m+1)}/c_{00}^s)\Big[1+\sum_{j=1}^{\infty}c_{j0}^{(-s)}x^{-j/(m+1)}\Big]$$
$$=(x^{\kappa/(m+1)}/c_{00}^s)\Big[x^{-(s+1)\kappa/(m+1)}+\sum_{j=1}^{\infty}c_{j0}^{(-s)}x^{-(j+(s+1)\kappa)/(m+1)}\Big]$$
$$=(x^{\kappa/(m+1)}/c_{00}^s)\sum_{j=(s+1)\kappa}^{\infty}\bar{\bar{c}}_{j0}^{(-s)}x^{-j/(m+1)},$$

where

(7.21′) $$\bar{\bar{c}}_{j0}^{(-s)}=0 \qquad \text{if} \quad j<(s+1)\kappa,$$
$$\bar{\bar{c}}_{(s+1)\kappa,0}^{(-s)}=1,$$

in which

(7.22) $$c_{j0}^{(-s)}=\sum_{\substack{\nu_1+\cdots+\nu_s=s\\ k_1\nu_1+\cdots+k_s\nu_s=j\\ k_1<\cdots<k_s}}\frac{s!}{\nu_1!\cdots\nu_s!}(c_{k_10}^{(-1)})^{\nu_1}\cdots(c_{k_s0}^{(-1)})^{\nu_s}$$

and

(7.22′) $$\bar{\bar{c}}_{j0}^{(-s)}=c_{j-(s+1)\kappa,0}^{(-s)}.$$

Further, if we write

(7.23) $$q'_k(x)=q_k(x)/q_0(x)=c_{00}^{-1}\Big[c_{0k}+\sum_{j=1}^{\infty}c_{jk}x^{-j/(m+1)}\Big]\Big[1+\sum_{j=1}^{\infty}c_{j0}^{(-1)}x^{-j/(m+1)}\Big]$$
$$=\sum_{j=0}^{\infty}c'_{jk}x^{-j/(m+1)},$$

then

$$c'_{jk}=c_{00}^{-1}\sum_{l=0}^{j}c_{lk}c_{j-l,0}^{(-1)}.$$

Moreover, write

(7.24) $$q''_k(x)=-\sum_{l=1}^{k}q'_l(x)q''_{k-l}(x)=\sum_{j=0}^{\infty}c''_{jk}x^{-j/(m+1)},$$

then

(7.24′) $$c''_{jk}=-\sum_{l=1}^{k}\Big(\sum_{t=0}^{j}c'_{tl}c''_{(j-t)(k-l)}\Big),\qquad c''_{j'0}=0\quad\text{if } j'\geqq 1.$$

Thus, if we put

(7.25) $$\tilde{q}_{*k}^{(s)}(x)=\sum_{j=0}^{\infty}\tilde{c}_{*jk}^{(s)}x^{-j/(m+1)}=sq''_k(x)+\cdots$$

and

(7.25′) $$\bar{q}_k^{(s)}(x)=\sum_{j=0}^{\infty}\bar{c}_{jk}^{(s)}x^{-j/(m+1)}=\tilde{q}_{*(k-s-1)}^{(s)}(x),$$

then

(7.26) $$\tilde{c}_{*jk}^{(s)}=sc''_{jk}+(\text{a polynomial of } c''_{j'k'},\ 0\leqq j'\leqq j,\ 0\leqq k'\leqq k-1),$$

hence

(7.26′) $$\bar{c}_{jk}^{(s)}=(-s/c_{00})c_{j(k-s-1)}+(\text{a polynomial of } (c_{l0}/c_{00}) \text{ and of } c_{j'k'},$$
$$1\leqq l\leqq j,\ 0\leqq j'\leqq j,\ 1\leqq k'\leqq k-s-2).$$

Since

(7.27) $$q_0(x)^{-s}\bar{q}_k^{(s)}(x)=(x^{\kappa/(m+1)}/c_{00}^s)\sum_{j=(s+1)\kappa}^{\infty}\bar{\bar{c}}_{j0}^{(-s)}x^{-j/(m+1)}\sum_{j=0}^{\infty}\bar{c}_{jk}^{(s)}x^{-j/(m+1)}$$
$$=(x^{\kappa/(m+1)}/c_{00}^s)\sum_{j=(s+1)\kappa}\Big(\sum_{\substack{j'+j''=j\\ j'\geqq(s+1)\kappa}}\bar{\bar{c}}_{j'0}^{(-s)}\bar{c}_{j''k}^{(s)}\Big)x^{-j/(m+1)},$$

we obtain by (7.18) and (7.27)

(7.28) $$\beta_n c_{jk}^{(n)}+\cdots+\beta_\kappa c_{jk}^{(\kappa)}=\sum_{s=m}^{k-1}\frac{B_s}{c_{00}^s}\Big(\sum_{\substack{j'+j''=j\\ j'\geqq(s+1)\kappa}}\bar{\bar{c}}_{j'0}^{(-s)}\bar{c}_{j''k}^{(s)}\Big),\qquad \text{if}\quad k\geqq m+1,$$

and

(7.28′) $$\beta_n c_{jk}^{(n)}+\cdots+\beta_\kappa c_{jk}^{(\kappa)}=0,\qquad \text{if}\quad k\leqq m.$$

When $j=(m+1)\kappa+j'$, $0\leqq j'<m+1$, and $0\leqq k<m+1$, then by (7.8′) we have

$$c_{jk}^{(l)}=0\qquad\text{for}\quad l\geqq\kappa+1,$$

and by (7.28′)

$$c_{jk}^{(\kappa)}=0\,, \qquad \text{if} \quad j<(m+1)(\kappa+1)\,.$$

On the other hand, by (7.8)

$$c_{j'+(m+1)\kappa,\,k}^{(\kappa)}=\left(\frac{\kappa-j'}{m+1}-\kappa\right)\left(\frac{\kappa-j'}{m+1}-\kappa+1\right)\cdots\left(\frac{\kappa-j'}{m+1}\right)c_{j'k}$$

if $j'\neq 0$. Therefore

(7.29) $$c_{j'k}=0 \qquad \text{for} \quad 0<j'<m+1, \quad 0\leqq k<m+1\,.$$

By (7.28) for $k=m+1$,

(7.30) $$\beta_n c_{j,m+1}^{(n)}+\cdots+\beta_\kappa c_{j,m+1}^{(\kappa)}=\frac{B_m}{c_{00}^m}\Bigg[\sum_{\substack{j'+j''=j\\ j'\geqq(m+1)\kappa}}\bar{\bar{c}}_{j'0}^{(-m)}\bar{c}_{j'',m+1}^{(m)}\Bigg],$$

and

(7.30′) $$\beta_n c_{(m+1)\kappa,\,m+1}^{(n)}+\cdots+\beta_\kappa c_{(m+1)\kappa,\,m+1}^{(\kappa)}=B_m/c_{00}^m\,.$$

Thus, by (7.8′) and (7.10), using (7.30′),

(7.31) $$\beta_\kappa \Gamma c_{00}=B_m/c_{00}^m\,, \qquad \text{i.e.,} \qquad c_{00}^{m+1}=B_m/(\beta_\kappa\Gamma)\neq 0\,,$$

which determines $c_{00}\neq 0$.

By (7.29), $c_{j'0}=0$, $0<j'<m+1$. By (7.30) for $j=(m+1)(\kappa+1)$, we have

(7.32) $$\begin{aligned}&\beta_{\kappa+1}c_{(m+1)(\kappa+1),\,m+1}^{(\kappa+1)}+\beta_\kappa c_{(m+1)(\kappa+1),\,m+1}^{(\kappa)}\\ &=(B_m/c_{00}^m)\big[\bar{\bar{c}}_{(m+1)\kappa,0}^{(-m)}\bar{c}_{m+1,\,m+1}^{(m)}+\cdots+\bar{\bar{c}}_{(m+1)(\kappa+1),0}^{(-m)}\bar{c}_{0,\,m+1}^{(m)}\big]\,.\end{aligned}$$

By (7.31) we see that the coefficients of $c_{m+1,0}$ on the both sides of (7.32) coincide. Hence $c_{m+1,0}$ can be arbitrarily prescribed.

In this way, other c_{jk} are determined successively.

8. Proof of Theorem 3(2). II. Existence of solution.

As in §6, we will prove the existence of solution by the method of Laplace transform, following Harris and Sibuya [2].

Let $V(x)$ be a function, holomorphic in the sector S_0 of (6.1.1) and asymptotically expanded as

(8.1) $$V(x)\sim x^{\kappa/(m+1)}\Big[c_{00}+\sum_{j+k\geqq 1}c_{jk}x^{-j/(m+1)}(\log x)^{(1-k)/(m+1)}\Big]$$

as x tends to ∞ in S_0.

Put

(8.2) $$y(x)=V(x)+z(x)\,,$$

and write the equation (1.1) in the form

(8.3) $$\alpha_n z(x+n)+\cdots+\alpha_1 z(x+1)-B_{-1}z(x)=g(x,\,z(x))\,,$$

where

$$g(x, z)=\sum_{\mu=m}^{\infty}\frac{B_{\mu}}{(V+z)^{\mu}}-[\alpha_{n}V(x+n)+\cdots+\alpha_{1}V(x+1)-B_{-1}V(x)]. \tag{8.4}$$

Arguing as in § 6 by means of inverse Laplace transform, we obtain the existence of the desired solution for (1.1).

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