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Real hypersurfaces of a complex hyperbolic space

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1. Introduction.

During the last years the study of real hypersurfaces of Kaehlerian manifolds has been an important subject in geometry of submanifolds, especially when the ambient space is a complex space form.

One of the first results in this way (see [**12**]) was to state that any real hypersurface M of a complex space form $\overline{M}(c)$ with holomorphic sectional curvature $c\neq 0$ is not totally umbilical. This is a direct consequence of classical Codazzi's equation for such a hypersurface. From that equation, also one can deduce that there does not exist real hypersurfaces M of $\overline{M}(c)$, $c\neq 0$, with parallel second fundamental form H. So, it seems interesting to describe and characterize real hypersurfaces of $\overline{M}(c)$, $c\neq 0$, with a few principal curvatures or with derivative ∇H of the second fundamental form of short length. These problems have been solved, in the case $c>0$, in [**2**], [**6**], [**10**], [**11**] and other works.

On the other hand, Kon, in [**5**], stated that there are no Einstein real hypersurfaces in $\overline{M}(c)$, $c>0$, and he studied a less restrictive condition for the Ricci tensor of these hypersurfaces: the pseudo-Einstein condition (see also [**6**]). In fact, he classified the pseudo-Einstein real hypersurfaces of the complex projective space using Takagi's works [**10**] and [**11**].

Finally, Cecil and Ryan generalized in [**2**] some results of [**10**] and [**5**]. They described in terms of tubes over complex submanifolds the real hypersurfaces of the complex projective space which appear in the literature.

Now we are interested in these problems when $c<0$, that is, when $\overline{M}(c)$ is the complex hyperbolic space CH^m (for convenience we will assume $c=-4$). So, A. Romero and the author have classified in [**7**] all complete real hypersurfaces of CH^m which admit a S^1-principal bundle which is a parallel Lorentzian hypersurface of the anti-De Sitter space H_1^{2m+1}. These real hypersurfaces have the least length for ∇H as we will show in a forthcoming paper.

In this paper we construct some examples of real hypersurfaces of CH^m (Section 6) and we give a complete classification of the real hypersurfaces of CH^m with at most two principal curvatures at each point. In this classification we obtain tubes over complex and totally real submanifolds of CH^m and a real hypersurface M_m^* of CH^m which has no focal points and which is congruent to all its parallel hypersurfaces. In fact, we will prove

THEOREM 7.4. *If M is a complete real hypersurface of CH^m, $m \geqq 3$, with at most two principal curvatures at each point, then M is congruent to one of the following spaces:*

a) *A geodesic hypersphere.*

b) *A tube of arbitrary radius over a complex hyperbolic hyperplane.*

c) *A "self-tube" M_m^*.*

d) *A tube of radius $\log((1+\sqrt{3})/\sqrt{2})$ over a totally real hyperbolic hyperplane.*

Also, we will obtain the following characterization of the space M_m^*:

COROLLARY 7.5. *The only complete real hypersurface of CH^m, $m \geqq 3$, which has no focal points and which is congruent to all its parallel hypersurfaces and such that $J\xi$ is a principal vector is the space M_m^*, where ξ is a unit vector normal to the hypersurface.*

It is necessary to remark that the real hypersurface appearing in Theorem 7.4, d), has exactly two constant principal curvatures at each point and, however, it is not totally η-umbilical (see [5] for definition). This cannot hold when the ambient space is the complex projective space (see [2]).

In Section 8 we will deal with pseudo-Einstein real hypersurfaces of CH^m. We will state:

COROLLARY 8.2. *There are no Einstein real hypersurfaces in CH^m, $m \geqq 3$.*

In this way, we will prove the following classification result:

THEOREM 8.1. *The only complete real hypersurfaces of CH^m, $m \geqq 3$, which are pseudo-Einstein are (up to congruences) the spaces* a), b) *or* c) *in Theorem* 7.4.

This last result asserts that a real hypersurface of CH^m is pseudo-Einstein if and only if it is totally η-umbilical.

Our main tool in this paper is based on [2]. Given a real hypersurface M of CH^m, we will "displace" it parallelly following a normal direction to obtain a submanifold $\phi_r M$ of CH^m which is complex or anti-holomorphic. Then we will relate the respective second fundamental forms of M and $\phi_r M$.

2. Preliminaries.

The Bergmann metric tensor g and the complex structure J of the complex hyperbolic space CH^m can be obtained as follows (see [4]): we consider the Hermitian form (,) on the complex vector space C^{m+1} given by

$$(z, w) = -z_0\bar{w}_0 + \sum_{j=1}^{m} z_j\bar{w}_j$$

where $z=(z_0, z_1, \cdots, z_m)$, $w=(w_0, w_1, \cdots, w_m) \in \boldsymbol{C}^{m+1}$. The inner product

$$\langle z, w\rangle = \mathrm{Re}(z, w)$$

is an indefinite metric of index 2 on $\boldsymbol{C}^{m+1}$. Then, the hypersurface H_1^{2m+1} of $\boldsymbol{C}^{m+1}$ defined by

$$H_1^{2m+1} = \{z \in \boldsymbol{C}^{m+1} \mid (z, z) = -1\}$$

endowed with the induced metric tensor from $\langle\ ,\ \rangle$ is the well known anti-De Sitter space, which is a Lorentzian manifold of constant sectional curvature -1. Moreover, if $z \in H_1^{2m+1}$, the tangent space $T_z H_1^{2m+1}$ is identifiable with the subspace of $\boldsymbol{C}^{m+1}$

$$\{w \in \boldsymbol{C}^{m+1} \mid \langle z, w\rangle = 0\}.$$

Now, H_1^{2m+1} is a principal S^1-bundle over $\boldsymbol{C}H^m$ with projection map $\pi : H_1^{2m+1} \to \boldsymbol{C}H^m$ such that $\mathrm{Ker}(\pi_*)_z = \mathrm{span}\{V_z\}$ with $V_z = \sqrt{-1}\, z \in T_z H_1^{2m+1}$. So, the tangent space $T_{\pi(z)}\boldsymbol{C}H^m$ can be identified with the subspace of $\boldsymbol{C}^{m+1}$

$$T'_z = \{w \in \boldsymbol{C}^{m+1} \mid (z, w) = 0\}.$$

Now, the complex structure J of $\boldsymbol{C}H^m$ is induced from the multiplication by the imaginary unity $\sqrt{-1}$, that is,

$$JX = (\pi_*)_z(\sqrt{-1}\, X')$$

where $X \in T_{\pi(z)}\boldsymbol{C}H^m$ and $X' \in T'_z$, $(\pi_*)_z(X') = X$ (horizontal lift). Also, the Bergmann metric tensor g of constant holomorphic sectional curvature -4 can be obtained from the relation

$$g(X, Y) = \langle X', Y'\rangle$$

where $X, Y \in T_{\pi(z)}\boldsymbol{C}H^m$.

In this way, the projection $\pi : H_1^{2m+1} \to \boldsymbol{C}H^m$ is a metric submersion in the sense of [8] with fundamental tensor J. So, if ∇' and $\bar{\nabla}$ are the metric connections of H_1^{2m+1} and $\boldsymbol{C}H^m$ respectively, we have

$$\nabla'_{X'}Y' = (\bar{\nabla}_X Y)' + g(JX, Y)V_z \qquad \nabla'_{V_z}X' = \nabla'_{X'}V_z = \sqrt{-1}\, X' \tag{2.1}$$

for all $X, Y \in T_{\pi(z)}\boldsymbol{C}H^m$.

Now, let M be a real hypersurface of $\boldsymbol{C}H^m$ and let ξ be a unit normal field defined near $x = \pi(z) \in M$. Then, if $X \in T_x M$, one has

$$JX = \phi X + f(X)\xi$$

tangent and normal components respectively. So, ϕ is a $(1, 1)$-tensor and f is a 1-form. Moreover, $f(X)=g(X, U)$ with $U=-J\xi$ and (ϕ, f) determines a metric almost contact structure on M (see [**5**] for more details).

We denote by H the Weingarten map on T_xM associated to ξ. Then the Codazzi and Gauss equations for M are (see [**5**], p. 341)

$$(\nabla_X H)Y-(\nabla_Y H)X=-f(X)\phi Y+f(Y)\phi X-2g(X, \phi Y)U \tag{2.2}$$

$$\begin{aligned}R(X, Y)Z=&-g(Y, Z)X+g(X, Z)Y-g(\phi Y, Z)\phi X+g(\phi X, Z)\phi Y\\ &+2g(X, \phi Y)\phi Z+g(HY, Z)HX-g(HX, Z)HY\end{aligned} \tag{2.3}$$

where $X, Y, Z\in T_xM$, ∇ is the metric connection of the induced metric g on M and R is the curvature operator of ∇.

Using (2.2) and the fact that $\boldsymbol{C}H^m$ is Kaehlerian, Kon has stated in [**5**], p. 342:

LEMMA 2.1. *Let M be a real hypersurface of $\boldsymbol{C}H^m$ and we suppose that $J\xi$ is a principal vector on M, that is, $HJ\xi=\mu J\xi$. Then, we have*

a) $2\phi=\mu(\phi H+H\phi)-2H\phi H$

b) $X\cdot\mu=(U\cdot\mu)f(X)$ *for all X tangent to M, and* $(U\cdot\mu)(\phi H+H\phi)=0$ *on M.*

Also, from (2.2) we have immediately (see [**12**]):

LEMMA 2.2. *Let M be a real hypersurface of $\boldsymbol{C}H^m$, $m>1$. Then M is not totally umbilical.*

Now, if S is the Ricci tensor of M we have from (2.3) (see [**5**], p. 341):

$$S(X, Y)=-(2m+1)g(X, Y)+3f(X)f(Y)+(\mathrm{tr}H)g(HX, Y)-g(H^2X, Y) \tag{2.4}$$

for all X, Y tangent to M.

Finally, given the real hypersurface M of $\boldsymbol{C}H^m$, one can construct (see [**7**]) a Lorentzian hypersurface M' of H_1^{2m+1} which is a principal S^1-bundle over M with time-like totally geodesic fibers and projection $\pi': M'\to M$ in such a way that the square

$$\begin{array}{ccc} M' & \xrightarrow{\ i'\ } & H_1^{2m+1} \\ \pi'\big\downarrow & & \big\downarrow\pi \\ M & \xrightarrow{\ i\ } & \boldsymbol{C}H^m \end{array}$$

is commutative (i, i' being the respective immersions), and, thus, if $z\in M'$, then $V_z\in T_zM'$ and $\mathrm{Ker}(\pi'_*)_z=\mathrm{span}\{V_z\}$. Moreover, if ξ is a unit field normal to M defined near $x=\pi'(z)$, the horizontal lift ξ' of ξ by π' provides a unit field normal to M' defined near z. If H' denotes the Weingarten map on T_zM' associated to ξ', we have shown in [7] the following relations between the maps H and H':

$$H'X'=(HX)'-f(X)V_z \qquad H'V_z=(U_x)' \tag{2.5}$$

where $X\in T_xM$ and the $'$ denotes horizontal lifts.

So, if M has principal curvatures $\lambda_1,\cdots,\lambda_{2m-2},\mu$ at x and $X_1,\cdots,X_{2m-2},U_x$ is an orthonormal basis of T_xM with $HX_i=\lambda_iX_i$, $i=1,\cdots,2m-2$, and $HU_x=\mu U_x$, then, from (2.5), the vectors $(X_1)',\cdots,(X_{2m-2})',(U_x)',V_z$ form an orthonormal basis of T_zM' with respect to which H' is represented by

$$\left(\begin{array}{ccc|cc} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_{2m-2} & & \\ \hline & & & \mu & 1 \\ & & & -1 & 0 \end{array}\right) \tag{2.6}$$

where the last submatrix corresponds to the restriction of H' to the Lorentz plane span$\{(U_x)',V_z\}\subset T_zM'$.

As an immediate consequence, *H' is diagonalizable if and only if $\mu^2>4$.* If $\mu^2=4$, there exists a null principal direction in T_zM'. If $\mu^2<4$, the Lorentz plane span$\{(U_x)',V_z\}$ contains no principal directions (see examples in Section 6).

3. Focal points of a real hypersurface of CH^m.

Let M be a real hypersurface of CH^m and let $P:NM\to M$ be its normal bundle. We define a map $F:NM\to CH^m$ as follows: if $\eta\in NM$ and $P(\eta)=x\in M$, we call $F(\eta)$ the point of CH^m reached at distance $|\eta|$ along the geodesic of CH^m starting at x with initial direction η. A point $p\in CH^m$ is said to be a focal point of the pair (M,x) with multiplicity $\nu>0$ if $p=F(\eta)$, $\eta\in NM$, $P(\eta)=x$ and $\dim\mathrm{Ker}(F_*)_\eta=\nu$ (see [2]). A point $p\in CH^m$ is said to be a focal point of M if it is a focal point of some pair (M,x).

Now, if $\eta\in NM$, one has $\eta=r\xi$ where $r\in\boldsymbol{R}$ and $\xi\in NM$ is a unit vector. If $P(\eta)=x$ and $w\in H_1^{2m+1}$ with $\pi(w)=x$, F can be determined by

$$F(r\xi)=\pi(\cosh r\, w+\sinh r\, \xi') \tag{3.1}$$

where ξ' is the horizontal lift of ξ to $T_wH_1^{2m+1}$. Moreover (3.1) is independent of the choice of w.

For studying the kernel of $(F_*)_{r\xi}: T_{r\xi}NM \to T_{F(r\xi)}CH^m$, we can suppose from (3.1) $r \geqq 0$, taking $-\xi$ instead of ξ if necessary. Moreover, as one has that $(F_*)_0$ is an isomorphism, we put $r>0$.

On the other hand, if ξ is a unit normal field defined on a neighbourhood W of $x \in M$, we have the following local trivialization of NM, taking into account that $\eta = \lambda\xi$ with $\lambda \in \boldsymbol{R}$ for all $\eta \in NM$

$$TNM|_W = TW \times \mathrm{span}\{\partial/\partial\lambda\}. \tag{3.2}$$

Thus, similar computations as in [**2**] provide us

$$(F_*)_{r\xi}(\partial/\partial\lambda) = (\pi_*)_z(\sinh r\, w + \cosh r\, \xi') \tag{3.3}$$

$$(F_*)_{r\xi}(X) = (\pi_*)_z\{\cosh r\, X' - \sinh r[(HX)' + \langle X', \sqrt{-1}\xi'\rangle\sqrt{-1}\, w]\} \tag{3.4}$$

for all $X \in T_xM$ and where $z = \cosh r\, w + \sinh r\, \xi' \in H_1^{2m+1}$. It is easy to see from (3.3) and (3.4):

PROPOSITION 3.1. *If M is a real hypersurface of $\boldsymbol{C}H^m$ with $HJ\xi = \mu J\xi$ where ξ is a unit normal field defined near $x \in M$, then, with the local trivialization* (3.2), *we have*

a) $(F_*)_{r\xi}(\partial/\partial\lambda) = (\pi_*)_z(\sinh r\, w + \cosh r\, \xi')$

b) $(F_*)_{r\xi}(J\xi) = (\pi_*)_z(\cosh 2r - (1/2)\mu \sinh 2r)(\sinh r \sqrt{-1}\, w + \cosh r \sqrt{-1}\, \xi')$

c) $(F_*)_{r\xi}(X) = (\pi_*)_z(\cosh r\, X' - \sinh r\,(HX)')$

where $X \in T_xM$ with $g(X, J\xi) = 0$ and $z = \cosh r\, w + \sinh r\, \xi'$, $\pi(w) = x$.

REMARK. Computations for getting b) have been made in such a way that the vector on the right side lies in T_z'.

As an immediate consequence we find the focal points of M when $HJ\xi = \mu J\xi$.

PROPOSITION 3.2. *Let M be a real hypersurface of $\boldsymbol{C}H^m$ with $HJ\xi = \mu J\xi$ where the unit normal field ξ is defined near $x \in M$. Then*

a) $\mathrm{Ker}(F_*)_{r\xi} = V_{\coth r}$ *if* $\mu \neq 2\coth 2r$ *at* x, *where $V_{\coth r}$ is the subspace of T_xM consisting of the orthogonal to $J\xi$ principal vectors corresponding to the principal curvature* $\coth r$.

b) $\mathrm{Ker}(F_*)_{r\xi} = V_{\coth r} \oplus \mathrm{span}\{J\xi\}$ *if* $\mu = 2\coth 2r$.

4. Parallel hypersurfaces and focal sets of a real hypersurface of CH^m.

Let M be an orientable real hypersurface of $\boldsymbol{C}H^m$ and let ξ be a unit normal field on M. We suppose that $J\xi$ is a principal vector at each point of M, that is, $HJ\xi = \mu J\xi$. For $r>0$, we define a map $\phi_r: M \to \boldsymbol{C}H^m$ by $\phi_r(x) = F(r\xi(x))$ where F was defined in (3.1).

When there are no focal points of M in $\phi_r M$, one has, from Propositions 3.1

and 3.2, that ϕ_r has rank $2m-1$ at each point of M. So, $\phi_r M$ is a real hypersurface of CH^m called parallel hypersurface at oriented distance r from M. If $\phi_r M$ contains some focal point of M, then we need some additional assumptions to guarantee that $\phi_r M$ is a submanifold of CH^m. In fact, we have an analogue to Theorem 1 in [2]:

THEOREM 4.1. *Let M be an orientable real hypersurface of CH^m such that $J\xi$ is a principal vector at each point, corresponding to a constant principal curvature μ. Let $r>0$ and we assume that ϕ_r has constant rank q on M. Then, if $\mu=2\coth 2r$ (resp. if $\mu\neq 2\coth 2r$), for every $x_0\in M$ there exists an open neighbourhood W of x_0 such that $\phi_r W$ is a $q/2$-dimensional complex (resp. q-dimensional anti-holomorphic) submanifold embedded in CH^m. Moreover W lies in a tube of radius r over $\phi_r W$.*

PROOF. Given $x_0\in M$, let W be an open neighbourhood of x_0 such that $\phi_r W=V$ is a q-dimensional real submanifold embedded in CH^m (utilize the inverse function theorem).

If $p\in V$, one has $p=\phi_r(x)=\pi(z)$ with $z=\cosh r\, w+\sinh r\,\xi'$, $x\in W$ and $w\in H_1^{2m+1}$, $\pi(w)=x$. Then $T_pV=\mathrm{span}\{(\phi_r)_*(X)\,|\,X\in T_xM\}$. Since $HJ\xi=\mu J\xi$, $\mathrm{Ker} f_x$ is an H-invariant subspace of T_xM and, so, we can take an orthonormal basis $X_1,\cdots,X_{2m-2},J\xi$ of T_xM which satisfy $HX_i=\lambda_i X_i$, $X_i\in\mathrm{Ker} f_x$, $i=1,\cdots,2m-2$. So, we have

$$T_pV=\mathrm{span}\{(\phi_r)_*(J\xi),\ (\phi_r)_*(X_i),\ i=1,\cdots,2m-2\}.$$

From Proposition 3.1, we get

$$(4.1)\quad T_pV=\mathrm{span}\{(\pi_*)_z(\cosh 2r-(1/2)\mu\sinh 2r)(\sinh r\sqrt{-1}\,w+\cosh r\sqrt{-1}\,\xi'),\ (\pi_*)_z(\cosh r-\lambda_i\sinh r)X_i',\quad i=1,\cdots,2m-2\}$$

where $z=\cosh r\, w+\sinh r\,\xi'\in H_1^{2m+1}$ with $\pi(w)=x$ for each $x\in\phi_r^{-1}(p)$.

Now, we define $\eta:\phi_r^{-1}(p)\to T_pCH^m$ by

$$(4.2)\qquad \eta(x)=(\pi_*)_z(\sinh r\, w+\cosh r\,\xi'_w)$$

where $w\in H_1^{2m+1}$ and $\pi(w)=x$. Then, $\eta(x)$ is a unit vector which is orthogonal to T_pV from (4.1). This map η can be defined for every $p\in V$ and, hence, we have a map $\eta:W\to BV$, where BV is the unit normal bundle over V. On the other hand, if $\psi_r:BV\to CH^m$ is the tube of radius r over V, we have

$$\psi_r(-\eta(x))=(\cosh r(\cosh r\, w+\sinh r\,\xi'_w)-\sinh r\,\eta'(x))=x.$$

So, $\psi_r(BV)\subset W$ and $\psi_r\circ(-\eta)=I_W$. Thus, η is a diffeomorphism from W onto an open set $\eta(W)\subset BV$ and W lies in a tube of radius r over V. Moreover, for $p\in V$, $\eta(W)\cap T_p{}^\perp V$ is open in $T_p{}^\perp V$ and so

(4.3) $T_p{}^\perp V = \mathrm{span}\{(\pi_*)_z(\sinh r\, w + \cosh r\, \xi'_w) \mid w \in H_1^{2m+1},\ \pi(w) = x \in \phi_r^{-1}(p)\}$.

Now, if $\mu = 2\coth 2r$, from (4.3) and (4.1) we get $JT_p{}^\perp V \subset T_p{}^\perp V$ and V is complex. Finally, if $\mu \neq 2\coth 2r$, then from (4.1), the vectors

$$(\pi_*)_z(\sinh r \sqrt{-1}\, w + \cosh r \sqrt{-1}\, \xi'_w) = J(\pi_*)_z(\sinh r\, w + \cosh r\, \xi'_w)$$

with $z = \cosh r\, w + \sinh r\, \xi'$, $\pi(w) = x$ lies in T_pV for every $x \in \phi_r^{-1}(p)$. But these vectors span $JT_p{}^\perp V$ from (4.3). So, $JT_p{}^\perp V \subset T_pV$ and V is anti-holomorphic (generic in the sense of [**13**]).

REMARK. It is important to see that a real hypersurface M of $\boldsymbol{C}H^m$ with $HJ\xi = \mu J\xi$, $\mu \in \boldsymbol{R}$ and $|\mu| \leqq 2$ cannot be a tube over a complex submanifold of $\boldsymbol{C}H^m$ (cf. [**2**]).

A global version of Theorem 4.1 can be obtained by supposing that M is complete from the Palais results in [**9**] exactly as in [**2**]:

THEOREM 4.2. *Let M be a complete real hypersurface of $\boldsymbol{C}H^m$ with $HJ\xi = \mu J\xi$, $\mu \in \boldsymbol{R}$. Let $r > 0$ and we assume that ϕ_r has constant rank q on M. Then, if $\mu = 2\coth 2r$ (resp. $\mu \neq 2\coth 2r$) M is a tube of radius r over the complex (resp. anti-holomorphic) submanifold ϕ_rM of $\boldsymbol{C}H^m$.*

5. Principal curvatures of ϕ_rM.

As in Theorem 4.1, let M be an orientable real hypersurface of $\boldsymbol{C}H^m$ such that $J\xi$ is a principal field corresponding to a constant principal curvature μ. Moreover, we will suppose that ϕ_r has constant rank q on M, $r > 0$.

We take $x \in M$ and we have W and $\phi_rW = V$ associated to x as in Theorem 4.1. If $p \in V$, from (4.3) we can choose $x_1, \cdots, x_{2m-q}$ points of W with $\phi_r(x_i) = p$ and such that $N_i = (\pi_*)_{z_i}(N'_i)$, $N'_i = \sinh r\, w_i + \cosh r\, \xi'_{w_i}$, constitute a basis of unit vectors for $T_p{}^\perp V$, where $w_i \in H_1^{2m+1}$, $\pi(w_i) = x_i$ and $z_i = \cosh r\, w_i + \sinh r\, \xi'_{w_i}$, $i = 1, \cdots, 2m-q$. Now, we distinguish two cases:

A) If $\mu = 2\coth 2r$, for fixed $i \in \{1, \cdots, 2m-q\}$ and using (4.1), we can take q vectors

$$T_j^i = (\pi_*)_{z_i}(\cosh r - \lambda_j \sinh r)X'_j$$

where $X_j \in T_{x_i}M$, $g(X_j, J\xi) = 0$ and $HX_j = \lambda_j X_j$, $\lambda_j \neq \coth r$, $j = 1, \cdots, q$, which form a basis of T_pV.

If we denote by $H_{r,i}$ the Weingarten map on T_pV associated to N_i, we have

$$H_{r,i}T_j^i = \text{tangent component of } -\overline{\nabla}_{T_j^i}N_i. \tag{5.1}$$

Now, by using the O'Neill formulae (2.1) we have

$$\overline{\nabla}_{T_j^i}N_i = (\pi_*)_{z_i}\nabla'_{(\cosh r - \lambda_j \sinh r)X'_j}N'_i. \tag{5.2}$$

It is easy to see that, if $\alpha(t)$ is a curve on H_1^{2m+1} with $\alpha(0)=w_i$ and $\dot{\alpha}(0)=X'_j$, then the curve $\gamma(t)=\cosh r\,\alpha(t)+\sinh r\,\xi'(\alpha(t))$ on H_1^{2m+1} satisfies $\gamma(0)=z_i$ and $\dot{\gamma}(0)=(\cosh r-\lambda_j\sinh r)X'_j$. Moreover, $N'_i(t)=\sinh r\,\alpha(t)+\cosh r\,\xi'(\alpha(t))$ is a tangent field to H_1^{2m+1} along $\gamma(t)$ with $N'_i(0)=N'_i$. Hence

$$\nabla'_{(\cosh r-\lambda_j\sinh r)X'_j}N'_i=\text{tangent to } H_1^{2m+1} \text{ component of } \dot{N}'_i(0)$$
$$=(\lambda_j\cosh r-\sinh r)X'_j$$

as follows from a direct calculation. This, jointly with (5.1) and (5.2), gives us

$$(5.3)\qquad H_{r,i}T_j^i=\frac{\lambda_j\cosh r-\sinh r}{\cosh r-\lambda_j\sinh r}T_j^i=\frac{\lambda_j\coth r-1}{\coth r-\lambda_j}T_j^i.$$

So, T_j^i, $j=1,\cdots,q$, is a diagonalization basis for $H_{r,i}$.

B) If $\mu\neq 2\coth 2r$, for fixed $i\in\{1,\cdots,2m-q\}$, we have that $JN_i=(\pi_*)_{z_i}(\sqrt{-1}N'_i)$ lies in T_pV as we have seen in Theorem 4.1 from (4.1). Moreover, there exists $q-1$ vectors

$$T_j^i=(\pi_*)_{z_i}(\cosh r-\lambda_j\sinh r)X'_j$$

of T_pV with $X_j\in T_{x_i}M$, $g(X_j, J\xi)=0$, $HX_j=\lambda_jX_j$ and $\lambda_j\neq\coth r$, $j=1,\cdots,q-1$, which form an orthogonal basis of the orthogonal complement of the line $\operatorname{span}\{JN_i\}$ in T_pV.

Now, we will evaluate

$$(5.4)\qquad H_{r,i}JN_i = \text{tangent component of } -\bar{\nabla}_{JN_i}N_i.$$

By utilizing the O'Neill equalities (2.1), one has

$$(5.5)\qquad \bar{\nabla}_{(\cosh 2r-(1/2)\mu\sinh 2r)JN_i}N_i$$
$$=(\pi_*)_{z_i}\nabla'_{(\cosh 2r-(1/2)\mu\sinh 2r)\sqrt{-1}N'_i}N'_i$$
$$+(\pi_*)_{z_i}(\cosh 2r-(1/2)\mu\sinh 2r)(\cosh r\sqrt{-1}\,w_i+\sinh r\sqrt{-1}\,\xi'_{w_i}).$$

But $(\cosh 2r-(1/2)\mu\sinh 2r)(\sqrt{-1}N'_i)=L_i+\langle L_i, V_{z_i}\rangle V_{z_i}$ with $L_i=(\cosh r-\mu\sinh r)\sqrt{-1}\xi'_{w_i}-\sinh r\sqrt{-1}\,w_i\in T_{z_i}H_1^{2m+1}$. Hence, the first addend on the right side of (5.5) is

$$(5.6)\qquad (\pi_*)_{z_i}\{\nabla'_{L_i}N'_i+\langle L_i, V_{z_i}\rangle\nabla'_{V_{z_i}}N'_i\}.$$

Now, if $\alpha(t)$ is a curve on H_1^{2m+1} with $\alpha(0)=w_i$ and $\dot{\alpha}(0)=\sqrt{-1}\xi'_{w_i}$, then the curve $\gamma(t)=\cosh r\,\alpha(t)+\sinh r\,\xi'(\alpha(t))$ on H_1^{2m+1} satisfies $\gamma(0)=z_i$ and $\dot{\gamma}(0)=L_i$. So, $N'_i(t)=\sinh r\,\alpha(t)+\cosh r\,\xi'(\alpha(t))$ is a tangent to H_1^{2m+1} field which extends N'_i along $\gamma(t)$. Evaluating $\dot{N}'_i(0)$, taking its tangent to H_1^{2m+1} component and using (2.1), one has that (5.6) becomes

$$(5.7)\qquad (\pi_*)_{z_i}\{(\sinh r-\mu\cosh r)\sqrt{-1}\,\xi'_{w_i}-\cosh r\sqrt{-1}\,w_i\}$$
$$+(\pi_*)_{z_i}(\mu\cosh 2r-2\sinh 2r)\sqrt{-1}N'_i\,.$$

Finally, (5.4), (5.5) and (5.7) give us

$$(5.8)\qquad H_{r,i}JN_i=2\frac{\mu\coth 2r-2}{2\coth 2r-\mu}JN_i\,.$$

Moreover, in the same way as in the case A), we have

$$(5.9)\qquad H_{r,i}T_j^i=\frac{\lambda_j\coth r-1}{\coth r-\lambda_j}T_j^i\,.$$

The equations (5.3), (5.8) and (5.9) relate the principal curvatures $\lambda_j\neq\coth r$, μ of the real hypersurface M and their corresponding of the focal submanifold $\phi_r M$.

6. Examples.

EXAMPLE 6.1 (see [7]). If p, q are integers with $p+q=m-1$ and $t\in\boldsymbol{R}$ with $0<t<1$, we consider the Lorentz hypersurface $M_{p,q}(t)$ of H_1^{2m+1} defined by the equations

$$-|z_0|^2+\sum_{j=1}^{m}|z_j|^2=-1\qquad t\Big(-|z_0|^2+\sum_{j=1}^{p}|z_j|^2\Big)=-\sum_{k=p+1}^{m}|z_k|^2\,.$$

It is easy to see that $M_{p,q}(t)$ is isometric to the product

$$H_1^{2p+1}(1/(t-1))\times S^{2q+1}(t/(1-t))$$

where $1/(t-1)$ and $t/(1-t)$ are the respective squares of the radii.

If $z\in M_{p,q}(t)$, one can see that

$$\xi'(z)=-\frac{1}{\sqrt{t}}(tz_0,\cdots,tz_p,z_{p+1},\cdots,z_m)$$

is a unit vector normal to $T_zM_{p,q}(t)$. So, if $a=(a_0,\cdots,a_m)$ lies in $T_zM_{p,q}(t)=\{a\in\boldsymbol{C}^{m+1}\,|\,\langle z,a\rangle=0,\ \langle\xi'(z),a\rangle=0\}$ and H' denotes the Weingarten map associated to $\xi'(z)$, we have

$$H'a=-\nabla'_a\xi'(z)=-D_a\xi'(z)$$

where D is the usual connection of $\boldsymbol{C}^{m+1}$. Hence

$$(6.1)\qquad H'a=\frac{1}{\sqrt{t}}(ta_0,\cdots,ta_p,a_{p+1},\cdots,a_m)\,.$$

Now, if we put $M_{p,q}^h(t)=\pi(M_{p,q}(t))$, we have a real hypersurface of $\boldsymbol{C}H^m$. Since $M_{p,q}(t)$ is S^1-invariant, $\xi_{\pi(z)}=(\pi_*)_z\xi'(z)$ provides a unit field normal to $M_{p,q}^h(t)$. If we denote by H its associated Weingarten map, we have by using

(6.1) and (2.5)

$$(HU_{\pi(z)})'=H'(-\sqrt{-1}\xi'(z))+V_z=-(\sqrt{t}+(1/\sqrt{t}))\sqrt{-1}\xi'(z)$$

and so $U=-J\xi$ is a principal field corresponding to the principal curvature $\sqrt{t}+(1/\sqrt{t})$. Then, from (2.5) and (2.6), we know that the principal curvatures of H on the orthogonal complement of the line span$\{U_{\pi(z)}\}$ agree with those of H' in the orthogonal complement of the Lorentz plane span$\{V_z, \sqrt{-1}\xi'(z)\}$. Now, from (6.1) one can see that these principal curvatures are $\sqrt{t}$ and $1/\sqrt{t}$ with respective multiplicities $2p$ and $2q$.

So, $M_{p,q}^h(t)$ has constant principal curvatures $\tanh r$, $\coth r$ and $2\coth 2r$ with multiplicities $2p$, $2q$ and 1 respectively and where we have put $\tanh r=\sqrt{t}$. It is necessary to remark that, from (2.6), the Weingarten map H' of $M_{p,q}(t)$ is diagonalizable because $2\coth 2r>2$.

Now, the map $\phi_r: M_{p,q}^h(t)\rightarrow \boldsymbol{C}H^m$, $r=\arg\tanh\sqrt{t}$, defined in Section 4, has constant rank $2(m-q-1)$ from Proposition 3.2. So, Theorem 4.2 asserts that $M_{p,q}^h(t)$ is a tube of radius r over a $(m-q-1)$-dimensional complex submanifold of $\boldsymbol{C}H^m$. Moreover, from (5.3), this submanifold is totally geodesic. In fact, $M_{p,q}^h(t)$ is a tube of radius r over a space $\boldsymbol{C}H^{m-q-1}$ embedded in $\boldsymbol{C}H^m$ in a totally geodesic way.

Only in the cases $p=0$ or $q=0$ (geodesic hypersphere or tube over a complex hyperbolic hyperplane) $M_{p,q}^h(t)$ is totally η-umbilical (see [**3**], p. 341 for definition) and only in these cases the Ricci tensor S of $M_{p,q}^h(t)$ is of the form

$$S(X, Y)=ag(X, Y)+bf(X)f(Y)$$

(pseudo-Einstein condition) for some functions a, b. In fact, from (2.4), one can see that $a=-2m+(2m-2)\coth^2 r$, $b=2m$ if $p=0$ and $a=-2m+(2m-2)\tanh^2 r$, $b=2m$ if $q=0$.

EXAMPLE 6.2 (see [**7**]). For fixed $t\in\boldsymbol{R}$, with $t>0$, let $N(t)$ be the Lorentz hypersurface of H_1^{2m+1} given by

$$-|z_0|^2+\sum_{j=1}^{m}|z_j|^2=-1 \qquad |z_0-z_1|^2=t.$$

Then $N(t)$ is clearly S^1-invariant. Moreover, if $\alpha(s)=(\alpha_0(s), \cdots, \alpha_m(s))$ is a curve on $N(t)$ with $\alpha(0)=z\in N(t)$ and $\dot{\alpha}(0)=a=(a_0, \cdots, a_m)$, we have

$$\langle z, a\rangle=0 \qquad \mathrm{Re}(\bar{z}_0-\bar{z}_1)(a_0-a_1)=0$$

where $\langle\ ,\ \rangle$ is the indefinite inner product on $\boldsymbol{C}^{m+1}$ defined in Section 2. Hence, the tangent space $T_zN(t)\subset T_zH_1^{2m+1}$ is identifiable with

$$\{a\in\boldsymbol{C}^{m+1} \mid \langle a, z\rangle=0, \langle a, \eta(z)\rangle=0\}$$

where we have put $\eta(z)=(z_0-z_1, z_0-z_1, 0, \cdots, 0)$. So, the vector

$$\xi'(z)=\frac{1}{t}\eta(z)-z$$

satisfies the equalities $\langle\xi'(z), z\rangle=0$ and $\langle\xi'(z), \xi'(z)\rangle=1$. Hence, we have

$$T_zN(t)=\{a\in T_zH_1^{2m+1} \mid \langle a, \xi'(z)\rangle=0\}.$$

Then, $\xi'(z)$ is a unit vector normal to $N(t)$ at z. Now, if H' denotes the associated Weingarten map, we have for each $a\in T_zN(t)$

$$H'a=-\nabla'_a\xi'(z)=-D_a\xi'(z)$$

where D is the usual connection on $\boldsymbol{C}^{m+1}$. So

$$H'a=-\frac{1}{t}(a_0-a_1, a_0-a_1, 0, \cdots, 0)+a. \tag{6.2}$$

Now, if $M_m^*(t)=\pi(N(t))$ is the corresponding real hypersurface of $\boldsymbol{C}H^m$, $\xi_{\pi(z)}=(\pi_*)_z\xi'(z)$ provides a unit field normal to $M_m^*(t)$. If H is the associated Weingarten map, we get by using (6.2) and (2.5)

$$(HU_{\pi(z)})'=H'(-\sqrt{-1}\xi'(z))+V_z=-2\sqrt{-1}\xi'(z).$$

Hence, $U=-J\xi$ is a principal field corresponding to the principal curvature 2. Moreover, since from (6.2) H' is the identity map on the orthogonal complement of the Lorentz plane span$\{V_z, \sqrt{-1}\xi'(z)\}\subset T_zN(t)$ (note that this orthogonal complement is given by $\{a\in T_zN(t)\mid a_0=a_1\}$), then, using the relations (2.5), we have that H is also the identity map on the orthogonal complement of the line span$\{U_{\pi(z)}\}\subset T_{\pi(z)}M_m^*(t)$.

So, we have seen that $HX=X+f(X)U$ for all X tangent to $M_m^*(t)$, and so $M_m^*(t)$ is totally η-umbilical. By using Corollary 5.3 of [7], we have that $M_m^*(t)$ is congruent to $M_m^*(1)=M_m^*$ for each $t>0$. In [7], we have shown that $M_m^*(t)$ is a homogeneous space with isometry group neither semisimple nor soluble and that it is diffeomorphic to $\boldsymbol{R}^{2m-1}$.

From Proposition 3.2, the map $\phi_r: M_m^*(t)\to\boldsymbol{C}H^m$ has constant rank $2m-1$ for all $r>0$. So, every ϕ_r is an immersion. Moreover, the real hypersurface $\phi_rM_m^*(t)$ is also totally η-umbilical with principal curvatures 1 and 2 as follows from (5.8) and (5.9). Again by using Corollary 5.3 of [7], we have that $\phi_rM_m^*(t)$ is congruent to M_m^* (in fact, one can easily prove that $\phi_rM_m^*(t)=M_m^*(te^{2r})$) and, so, it is also congruent to $M_m^*(t)$. For these arguments we will say that M_m^* is a "self-tube".

The equation (2.4) and the previous considerations show that M_m^* is pseudo-Einstein with $a=-2$ and $b=2m$.

REMARK 1. It is easy to note that $M_m^*(t)$, $t>0$, provides an isoparametric family of hypersurfaces of $\boldsymbol{C}H^m$, in such a way that all hypersurfaces of this family have constant mean curvature $2m$. Moreover, from (2.6), this same occurs for the family $N(t)$ of Lorentz hypersurfaces of H_1^{2m+1}. Again from (2.6) the Weingarten map of each $N(t)$ is not diagonalizable because the Lorentz plane span$\{V_z, \sqrt{-1}\xi'(z)\} \subset T_z N(t)$ is irreducible for each $z \in N(t)$. So, one cannot use for $N(t)$ the Cartan results in [**1**]. That is, $N(t)$ is an isoparametric family of hypersurfaces of H_1^{2m+1} which have not an analogue in a Riemannian space form of negative curvature.

REMARK 2. We will characterize M_m^* in Corollary 7.5 as the only self-tube among all complete real hypersurfaces of $\boldsymbol{CH}^m$ such that $J\xi$ is principal.

EXAMPLE 6.3. We take $t \in \boldsymbol{R}$ with $t>1$ and let $M(t)$ be the Lorentz hypersurface of H_1^{2m+1} defined by

$$-|z_0|^2 + \sum_{j=1}^m |z_j|^2 = -1 \qquad \left|-z_0^2 + \sum_{j=1}^m z_j^2\right|^2 = t$$

which is clearly S^1-invariant. Taking curves on $M(t)$ one can easily see in an analogous way to the Example 6.2 that

$$\xi'(z) = \frac{1}{\sqrt{t(t-1)}}[Q(z)\bar{z} + tz]$$

where $Q(z) = -z_0^2 + \sum_{j=1}^m z_j^2$, is a unit vector normal to $M(t)$ at z and we can identify

$$T_z M(t) = \{a \in \boldsymbol{C}^{m+1} \mid \langle a, z\rangle = \langle a, Q(z)\bar{z}\rangle = 0\}.$$

Now, if H' is the Weingarten map associated to $\xi'(z)$, we have for all $a \in T_z M(t)$

$$H'a = -\nabla'_a \xi'(z) = -D_a \xi'(z)$$

where D is the usual connection of $\boldsymbol{C}^{m+1}$. So

$$H'a = -\frac{1}{\sqrt{t(t-1)}}[2Q(z, a)\bar{z} + Q(z)\bar{a} + ta] \tag{6.3}$$

with $Q(z, a) = -a_0 z_0 + \sum_{j=1}^m a_j z_j$.

Let $M^h(t) = \pi(M(t))$ the corresponding real hypersurface of $\boldsymbol{CH}^m$. Then, since $M(t)$ is S^1-invariant, $\xi_{\pi(z)} = (\pi_*)_z \xi'(z)$ is a unit vector normal to $M^h(t)$ at $\pi(z)$. If H denotes its associated Weingarten map, we have, by using (6.3) and (2.5)

$$(HU_{\pi(z)})' = H'(-\sqrt{-1}\xi'(z)) + V_z = -2\frac{\sqrt{t-1}}{\sqrt{t}}\sqrt{-1}\xi'(z).$$

So, we have that $U = -J\xi$ is a principal field corresponding to the principal curvature $2\sqrt{(t-1)/t}$. Moreover, from (6.3), one can see that $M(t)$ has two principal curvatures $(\sqrt{t}-1)/\sqrt{t-1}$ and $(\sqrt{t}+1)/\sqrt{t-1}$ both with multiplicities

$m-1$ on the orthogonal complement of the Lorentz plane span$\{V_z, \sqrt{-1}\xi'(z)\} \subset T_z M(t)$ for all $z \in M(t)$. Hence, from (2.6) and the previous considerations, we have that $M^h(t)$ has constant principal curvatures $\tanh r$, $\coth r$, $2\tanh 2r$ with multiplicities $m-1$, $m-1$ and 1 respectively, where we have put $\cosh^2 2r = t$.

It is necessary to remark that $2\tanh 2r \neq \tanh r$ for all $r>0$, but that $2\tanh 2r = \coth r$ if and only if $r=\log((1+\sqrt{3})/\sqrt{2})$, that is, $t=4$. Hence, $M^h(4)$ has two constant principal curvatures $\sqrt{3}$ and $1/\sqrt{3}$ with multiplicities m and $m-1$ respectively. If $t \neq 4$, $M^h(t)$ has three constant principal curvatures with multiplicities $m-1$, $m-1$ and 1. So, $M^h(t)$ is not totally η-umbilical for each $t>1$. Moreover, from (2.4) and the comments above, $M^h(t)$ is not pseudo-Einstein.

Now, if $r=(1/2)\arg\cosh\sqrt{t}$, Proposition 3.2 assures us that $\phi_r : M^h(t) \to \boldsymbol{C}H^m$ has constant rank m. By using Theorem 4.2, $M^h(t)$ is a tube of radius r over a totally real m-dimensional submanifold of $\boldsymbol{C}H^m$. A convenient use of (5.8) and (5.9) shows that this submanifold is totally geodesic and, so, it is a real hyperbolic space $\boldsymbol{R}H^m$ embedded in $\boldsymbol{C}H^m$.

On the other hand, let $SO^1(m+1)$ be the identity component of the subgroup of $GL(m+1, \boldsymbol{R})$ which preserves the Lorentzian form $-x_0^2+x_1^2+\cdots+x_m^2$ on $\boldsymbol{R}^{m+1}$. One can see that $SO^1(m+1)$ acts transitively on $M^h(t)$ and that $M^h(t)$ is diffeomorphic to the homogeneous space $SO^1(m+1)/SO(m-1)$. Since $SO^1(m+1)$ has maximal compact subgroups isomorphic to $SO(m)$, $M^h(t)$ has the same homotopy type as a totally geodesic submanifold which is isometric to the symmetric space $SO(m)/SO(m-1)$, that is, to a unit sphere S^{m-1}.

REMARK 3. The relation (2.6) between the Weingarten maps H and H' of $M^h(t)$ and $M(t)$ respectively, asserts that H' is not diagonalizable. So, $M(t)$ is an isoparametric family of hypersurfaces of H_1^{2m+1} which has not an analogue in a Riemannian space form of negative curvature (see [**1**]).

REMARK 4. The tube $M^h(4)$ of radius $\log((1+\sqrt{3})/\sqrt{2})$ over $\boldsymbol{R}H^m \subset \boldsymbol{C}H^m$ provides an example of real hypersurface of $\boldsymbol{C}H^m$ with two constant principal curvatures which is not totally η-umbilical. This fact is impossible when the ambient space is the complex projective space (see [**10**] and [**2**]).

7. Real hypersurfaces of CH^m with at most two principal curvatures at each point.

Let M be a real hypersurface of $\boldsymbol{C}H^m$ with at most two principal curvatures at each point. The remark after Proposition 5.2 of [**2**] states

LEMMA 7.1. *Let M be a real hypersurface of $\boldsymbol{C}H^m$, $m \geqq 3$, with exactly two principal curvatures at each point. Then $J\xi$ is a principal vector.*

For real hypersurfaces of the complex projective space a well-known result of Maeda, [**6**], assures that, if $J\xi$ is principal, then the corresponding principal curvature is locally constant. In this way, we have

LEMMA 7.2. *Let M be a real hypersurface of CH^m, $m\geqq 3$, with at most two principal curvatures at each point. Then the principal curvature corresponding to $J\xi$ is locally constant.*

PROOF. From Lemma 7.1, we have $HJ\xi=\mu J\xi$. Let $x\in M$ such that $\phi H+H\phi=0$ at x. Then, from Lemma 2.1, a), one has $\phi=-H\phi H$ at x. Hence, x is not umbilical and, so, there exists $X\in T_xM$ with $g(X, J\xi)=0$ and $HX=\lambda X$. Then, $H\phi X=-\lambda\phi X$ and $H\phi X=-(1/\lambda)\phi X$. Thus we get $\lambda^2=1$ and $\mu=-\lambda$. On the other hand, in the open set consisting of the points of M where $\phi H+H\phi\neq 0$ we have, from Lemma 2.1, b), that μ is locally constant. So, μ is locally constant on M.

Now, we need a result that Takagi has shown when the ambient space is the complex projective space. Slight modifications in the computations of Lemma 3.4 in [**11**] provide us

LEMMA 7.3. *If M is a real hypersurface of CH^m, $m\geqq 3$, with exactly three constant principal curvatures x, y, z at each point where the line* span$\{J\xi\}$ *is the eigenspace associated to z, then, we have one of the following possibilities:*

a) $\phi V_x\subset V_x$, $\phi V_y\subset V_y$, $x+y=z$ *and* $xy=1$.

b) $\phi V_x\subset V_y$, $\phi V_y\subset V_x$, $x+y=4/z$ *and* $xy=1$.

Now, we can state

THEOREM 7.4. *Let M be a complete and connected real hypersurface of CH^m, $m\geqq 3$, with at most two principal curvatures at each point. Then, M is congruent to one of the following spaces:*

a) *A geodesic hypersphere* $M^h_{0,m-1}(\tanh^2 r)$ *of radius* $r>0$.

b) *A tube* $M^h_{m-1,0}(\tanh^2 r)$ *of radius* $r>0$ *over a complex hyperbolic hyperplane.*

c) *A self-tube* M^*_m.

d) *A tube* $M^h(4)$ *of radius* $\log((1+\sqrt{3})/\sqrt{2})$ *over a totally real hyperbolic hyperplane.*

PROOF. From Lemmas 7.1 and 7.2, we know that $HJ\xi=\mu J\xi$ with $\mu\in\boldsymbol{R}$ for each unit local field ξ normal to M. We will distinguish three cases:

A) We suppose $\mu^2>4$. In this case M is orientable and we choose an orientation for M such that the associated principal curvature μ is greater than 2. Then we can put $\mu=2\coth 2r$ for some $r>0$.

Let $\phi_r: M\to CH^m$ be as in Section 4. We denote by ν the least multiplicity on M of the principal curvature $\coth r$. Then, from Proposition 3.2, the set $\Omega=\{x\in M \mid \coth r$ has multiplicity ν at $x\}$ consists of the points of M where ϕ_r has maximum rank $2m-2-\nu$ and, so, Ω is a non-empty open set of M.

If $x\in\Omega$, from Theorem 4.1, there exists an open neighbourhood W of x such that $\phi_r W=V$ is a complex submanifold embedded in CH^m. If we have $0<\nu<2m-2$, then, since $\mu=2\coth 2r=\tanh r+\coth r\neq\coth r$, we have two principal curvatures at x, namely, $\mu=2\coth 2r$ and $\lambda=\coth r$. Moreover $\dim V>0$ and the discussions in Section 5 say that there exists a basis of unit vectors $N_1, \cdots, N_{\nu+2}$ of $T_p^{\perp}V$, $p=\phi_r(x)$, in such a way that their associated Weingarten maps $H_{r,i}$, $i=1, \cdots, \nu+2$, are related with H as in (5.3). From this and since μ is the only principal curvature of M at x with associated eigenspace orthogonal to $J\xi$, as it follows from the assumption $\nu<2m-2$, we have that $H_{r,i}=(\mu\coth r-1)/(\coth r-\mu)I$ where $i=1, \cdots, \nu+2$ and I is the identity map. But V is complex and, so, $H_{r,i}=0$. Hence, $\mu=\tanh r$ which is impossible.

So, if $\nu>0$, then $\nu=2m-2$. In this case, as $\mu\neq\coth r$ and ν is the least multiplicity of $\coth r$, we have that M has two constant principal curvatures $2\coth 2r$ and $\coth r$ with multiplicities 1 and $2m-2$ at each point. Now, from Proposition 3.2, $\phi_r: M\to CH^m$ has constant rank zero and M is a geodesic hypersphere $M^h_{0,m-1}(\tanh^2 r)$ from Theorem 4.2.

On the other hand, if $\nu=0$, then V is a complex hypersurface of CH^m. If the multiplicity of μ is greater than 1 at x, we could have chosen W with the same property at each point. Hence, using (5.3), V would have exactly two principal curvatures at each point. Since V is complex, V would be a complex Einstein hypersurface of CH^m with a principal curvature $(\mu\coth r-1)/(\coth r-\mu)$ as follows from (5.3). The Chern result in [**3**] asserts that V is totally geodesic and, so, this principal curvature is zero, that is, $\mu=\tanh r$, which is impossible.

As conclusion, if $\nu=0$, then the multiplicity of μ is 1 on Ω. If we denote by λ the other principal curvature on W, we have from (5.3) that the complex hypersurface V of CH^m has one principal curvature $(\lambda\coth r-1)/(\coth r-\lambda)$ at each point. Since V is complex, then $\lambda=\tanh r$. Hence, Ω has two constant principal curvatures $2\coth 2r$ and $\tanh r$ with multiplicities 1 and $2m-2$ respectively. So, Ω is closed in M and $\Omega=M$. Now, from Proposition 3.2, Theorem 4.2 and (5.3) we have that M is a tube $M^h_{m-1,0}(\tanh^2 r)$ over a space CH^{m-1} embedded in CH^m as a complex totally geodesic hypersurface.

B) We suppose $\mu^2=4$. As in the above case, M is orientable and we choose an orientation for M such that the associated principal curvature μ is 2. We denote by ν the least multiplicity on M of the principal curvature 2. We know that $\nu\geqq 1$ and $\nu\leqq 2m-2$ from Lemma 2.2.

If $\nu>1$, we take r such that $\coth r=2$, that is, $r=\log\sqrt{3}$. Then, from Proposition 3.2, the set $\Omega=\{x\in M \mid 2 \text{ has multiplicity } \nu \text{ at } x\}$ consists of the points of M where the map $\phi_r: M\to CH^m$ has maximum rank $2m-\nu$ (note that $\mu=2\neq 2\coth 2r=5/2$). So, Ω is open in M.

By using Theorem 4.1, if $x\in\Omega$, then there exists an open neighbourhood W

of x such that $\phi_r W=V$ is an anti-holomorphic submanifold embedded in CH^m. Since $\nu \leqq 2m-2$, let λ be another principal curvature of M on W. From Lemma 2.1, a), it is easy to see that $\lambda=1$. Now, from discussions in Section 5, (5.8) and (5.9), there exists a basis of unit vectors $N_1, \cdots, N_\nu$ of $T_p{}^\perp V$, $p=\phi_r(x)$, such that their corresponding Weingarten maps satisfy $H_{r,i}JN_i=2JN_i$ and $H_{r,i}$ is the identity map on the orthogonal complement of JN_i in T_pV, $i=1, \cdots, \nu$. So, since we suppose $\nu>1$, we take $i\neq j$ and we have

$$H_{r,j}JN_i=JN_i+g(N_i, N_j)JN_j .$$

Now, from Lemma 2.1 of [**13**], one has $H_{r,j}JN_i=H_{r,i}JN_j$. Hence, $g(N_i, N_j)=1$ which is not possible because N_i, N_j are linearly independent.

As conclusion, we get $\nu=1$ and, so, $\Omega=\{x\in M \mid HX=X+f(X)U \text{ at } x\}$ is closed in M. Then $\Omega=M$ and M has two constant principal curvatures 2 and 1 with respective multiplicities 1 and $2m-2$. From Corollary 5.3 of [**7**], we conclude that M is congruent to a self-tube M_m^*. It is convenient to remark that, from Proposition 3.2, (5.8) and (5.9), the map $\phi_r: M\to CH^m$, $r>0$, is always an immersion and that $\phi_r M$ has the same principal curvatures as M.

C) Finally, we suppose $\mu^2<4$. If $\mu=0$ at some $x\in M$, then, from Lemma 2.1, a), we have $\phi=-H\phi H$. So, we would have three principal curvatures 0, λ and $-(1/\lambda)$ at x, which is impossible from our hypothesis. Hence, we can take a unit normal field ξ such that its corresponding principal curvature μ is $2\tanh 2r$ for some $r>0$ and M is orientable.

Now, Lemma 2.2, a) asserts that, if α is a principal curvature on M corresponding to a principal vector X with $g(X, J\xi)=0$, then $\alpha'=(1-\alpha\tanh 2r)/(\tanh 2r-\alpha)$ is another principal curvature corresponding to ϕX. But $\alpha'=\alpha$ implies the inequality $\tanh^2 2r\geqq 1$ which is absurd. So, since M has at most two principal curvatures at each point, there are two principal curvatures α, β with $\alpha'=\beta$, $\beta'=\alpha$ and $\phi V_\alpha=V_\beta$ on the orthogonal complement of the line span$\{J\xi\}$ at each point of M. Moreover, from our hypothesis, we have $\alpha=\mu$ or $\beta=\mu$. So, we can put $\alpha=\mu$ and, hence

$$(7.1)\qquad \beta=2\tanh 2r-\coth 2r \qquad \mathrm{mult}(\beta)=m-1 \qquad \mathrm{mult}(\mu)=m .$$

Now, if $\beta=\coth r$, then one has $\coth^2 r=1$ or $\coth^2 r=1/3$, which is impossible. Hence, from Proposition 3.2, the map $\phi_r: M\to CH^m$ has constant rank either m (if $2\tanh 2r=\coth r$, that is, $r=\log((1+\sqrt{3})/\sqrt{2})$) or $2m-1$ (if $2\tanh 2r\neq\coth r$). But, if ϕ_r has constant rank $2m-1$, then, from (5.8), (5.9) and (7.1), $\phi_r M$ would be a real hypersurface of CH^m with three constant principal curvatures 0, $(\beta\coth r-1)/(\coth r-\beta)$, $(\mu\coth r-1)/(\coth r-\mu)$ and this is impossible from Lemma 7.3.

Hence, if $\mu^2<4$, then we have $\mu=2\tanh 2r$ with $r=\log((1+\sqrt{3})/\sqrt{2})$. So,

$\mu=\coth r=\sqrt{3}$, $\beta=\tanh r=1/\sqrt{3}$ and ϕ_r has constant rank m. Using Theorem 4.2, (5.8) and (5.9), we have that M is a tube of radius r over a space $\boldsymbol{R}H^m$ embedded in $\boldsymbol{C}H^m$ as a totally real and totally geodesic submanifold.

This last theorem and discussions in Section 5 enable us to prove the announced characterization for the space M_m^* defined in Example 6.2.

COROLLARY 7.5. *The only complete and connected self-tube of* $\boldsymbol{C}H^m$, $m\geqq 3$, *such that* $J\xi$ *is principal is the space* M_m^*.

REMARK. We will call "self-tube" a real hypersurface of $\boldsymbol{C}H^m$ without focal points and such that all its parallel hypersurfaces are congruent to itself.

PROOF OF COROLLARY. We have $HJ\xi=\mu J\xi$ for some function μ where ξ is a local unit field normal to M. Since M has no focal points, from Proposition 3.2, we have $|\mu|\leqq 2$ and all the remaining principal curvatures λ of H satisfy $|\lambda|\leqq 1$. Moreover, $\phi_r M$ is always a real hypersurface of $\boldsymbol{C}H^m$ for each $r>0$, which has the same principal curvatures at a point $\phi_r(x)$ as M at the point x. By using the relations (5.8) and (5.9) between the principal curvatures of M and $\phi_r M$, it is easily seen that $\mu^2=4$ and $\lambda^2=1$ for another principal curvature of M different from μ.

We choose on M a unit normal field ξ such that $HJ\xi=2J\xi$ at each point. So, M is orientable. Moreover, from Lemma 2.1, a), we have that $\lambda=1$ is a principal curvature at each point of M. Also, taking into account Lemma 7.3, there are no points of M where -1 is a principal curvature. Then one concludes that M has two principal curvatures 2 and 1 with respective multiplicities 1 and $2m-2$ at each point. From the last theorem, one has that $M=M_m^*$.

8. Pseudo-Einstein real hypersurfaces of $\boldsymbol{C}H^m$.

A real hypersurface M of $\boldsymbol{C}H^m$ is called pseudo-Einstein when its Ricci tensor S satisfies the equation

$$S(X, Y)=ag(X, Y)+bf(X)f(Y) \tag{8.1}$$

for all X, Y tangent to M and some functions a, b (see [6]). From (2.3), if M is pseudo-Einstein, then we have

$$H^2X-\alpha HX+(a+2m+1)X+(b-3)f(X)U=0 \tag{8.2}$$

for all X tangent to M, where H is the Weingarten map associated to a unit normal vector $\xi=JU$ and $\alpha=\mathrm{tr}H$.

Now, from (8.2), it is easily seen that, at those points of M where $b\neq 3$, the operator $K=H^2-\alpha H$ has two eigenvalues $-(a+2m+1)$ and $-(a+b+2m-2)$ and the line span$\{J\xi\}$ is the eigenspace corresponding to the last one. Where $b=3$, K has an only eigenvalue $-(a+2m+1)$. Hence, if $x\in M$ and $\lambda_1, \lambda_2, \cdots, \lambda_{2m-1}$ are the principal curvatures of M at x, we have that $\lambda_i^2-\alpha\lambda_i$ is an eigenvalue of K at x. So

$$\lambda_i^2-\alpha\lambda_i+(a+2m+1)=0 \qquad i=1, \cdots, 2m-1 \qquad \text{if} \quad b(x)=3, \tag{8.3}$$

$$\begin{aligned} &\lambda_i^2-\alpha\lambda_i+(a+2m+1)=0 \qquad i=2, \cdots, 2m-1 \qquad \text{and} \\ &\lambda_1^2-\alpha\lambda_1+(a+b+2m-2)=0 \qquad V_{\lambda_1}=\text{span}\{J\xi\} \qquad \text{if} \quad b(x)\neq 3. \end{aligned} \tag{8.4}$$

After these observations we can state:

THEOREM 8.1. *Let M be a complete and connected real hypersurface of CH^m, $m\geqq 3$, which is pseudo-Einstein. Then M is congruent to one of the following spaces:*

a) *A geodesic hypersphere $M_{0,m-1}^h(\tanh^2 r)$ of radius $r>0$.*

b) *A tube $M_{m-1,0}^h(\tanh^2 r)$ of radius $r>0$ over a complex hyperbolic hyperplane.*

c) *A self-tube M_m^*.*

PROOF. From (8.3) and (8.4), we know that M has at most three principal curvatures at each point. If M has at most two principal curvatures at each point, we conclude from Theorem 7.4 and the fact that the tube $M^h(4)$ defined in Example 6.3 is not pseudo-Einstein. Thus we will suppose that the set Σ consisting of the points of M where there are exactly three principal curvatures $\lambda_1, \lambda_2, \lambda_3$ is open and non-empty. From (8.3) and (8.4), we have $b\neq 3$ on Σ and, so, $V_{\lambda_1}=\text{span}\{J\xi\}$.

Now, if $x\in\Sigma$ and $\phi H+H\phi=0$ at x, then, from Lemma 2.1, a), one has $\phi=-H\phi H$ and, hence, $\lambda_i^2=1$, $i=2, 3$, $\lambda_2=-\lambda_3$ and $\phi V_{\lambda_2}=V_{\lambda_3}$ at x. So, $\alpha(x)=\text{tr}H_x=(m-1)(\lambda_2+\lambda_3)+\lambda_1=\lambda_1$. Moreover, from (8.4), we have $\alpha(x)=\lambda_2+\lambda_3=0$ and, so, we have $\lambda_1=0$ at those points of M where $\phi H+H\phi=0$. Since, from Lemma 2.1, b), λ_1 is locally constant on the open set of M where $\phi H+H\phi\neq 0$, we conclude that λ_1 is locally constant on Σ.

Let $y\in\Sigma$ and Σ_0 the component of Σ with $y\in\Sigma_0$. We know that λ_1 is constant on Σ_0. We will suppose $\lambda_1\geqq 0$ by reversing the orientation if necessary. Let Ω denote the subset of Σ_0 consisting of the points where the principal curvature $\coth r$ appears with its least multiplicity ν, for some $r>0$. Since λ_1 is constant on Σ_0, from Proposition 3.2, we have $\Omega=\{x\in\Sigma_0 \mid \phi_r \text{ has maximum rank}\}$ and, so, Ω is open and non-empty. Now, we will distinguish three cases:

A) If $\lambda_1>2$, we take $r>0$ with $\lambda_1=2\coth 2r$. Let $x\in\Omega$ and let W be as in Theorem 4.1. If $\nu=0$, from Proposition 3.2, we have that $\phi_r W=V$ is a complex hypersurface of CH^m. Moreover, using (5.3), we have that V has at each point two principal curvatures $(\lambda_i\coth r-1)/(\coth r-\lambda_i)$, $i=2, 3$. Hence, V is

Einstein and the Chern result in [3] asserts that V is totally geodesic, that is, $\lambda_2=\lambda_3=\tanh r$. But it is impossible.

Hence, we have $0<\nu<2m-2$ and we can put $\lambda_2=\coth r$. Again, from (5.3), the $(m-1-\nu/2)$-dimensional complex submanifold V has at each point one principal curvature $(\lambda_3\coth r-1)/(\coth r-\lambda_3)$. Since V is complex, this principal curvature vanishes and, so, $\lambda_3=\tanh r$. Thus, there are on Ω three principal curvatures $2\coth 2r$, $\coth r$ and $\tanh r$ with multiplicities 1, ν and $2m-2-\nu$ respectively. As in Theorem 7.4, Ω is closed in M and, so, $\Omega=M$. By using Theorem 4.2 and (5.3), we conclude that M is the tube $M^h_{m-1-\nu/2,\nu/2}(\tanh^2 r)$, $0<\nu/2<m-1$. But any tube of this tube is not pseudo-Einstein.

B) If $\lambda_1=2$ on Σ_0, from Lemma 2.1, a), we have $\phi=H\phi+\phi H-H\phi H$. Now, if $x\in\Sigma_0$ and $X\in T_xM$ with $HX=\lambda_iX$, $i=2, 3$, we get $(1-\lambda_i)H\phi X=(1-\lambda_i)\phi X$. So, either $\lambda_i=1$ or ϕX is a principal vector corresponding to the principal curvature 1. In any case, we can put $\lambda_2=1$ and $\phi V_{\lambda_3}\subset V_{\lambda_2}$ at x.

Let p denote the multiplicity of λ_3 at x. We have $0<p\leqq m-1$. So, $\alpha(x)=\mathrm{tr}H_x=p\lambda_3+2m-p$. On the other hand, from (8.4), one has $\alpha(x)=1+\lambda_3$. Hence, $(p-1)\lambda_3=1+p-2m$. Since $m\geqq 3$, then $p\neq 1$ and, finally, one has $\lambda_3=(1+p-2m)/(p-1)$. So, λ_3 is locally constant and we can apply Lemma 7.3. In this way, one sees that $\lambda_1=2$ is also impossible.

C) Finally, if $\lambda_1<2$, then, by using Lemma 2.1, a), it can be easily seen that, for $x\in\Sigma_0$, $X\in T_xM$ with $HX=\lambda_iX$, $i=2, 3$, ϕX is a principal vector corresponding to the principal curvature $\lambda_i'=(2-\lambda_1\lambda_i)/(\lambda_1-2\lambda_i)$. But $\lambda_i'=\lambda_i$ implies $\lambda_1^2\geqq 4$ which is not possible. Hence, we have

$$\lambda_3=(2-\lambda_1\lambda_i)/(\lambda_1-2\lambda_i) \quad \text{and} \quad \phi V_{\lambda_2}=V_{\lambda_3}. \tag{8.5}$$

Now, from (8.4) and (8.5), we have $(m-2)(\lambda_2+\lambda_3)=-\lambda_1$. So, $\lambda_2+\lambda_3$ is constant on Σ_0. Again from (8.5), we get $2\lambda_2\lambda_3=\lambda_1(\lambda_2+\lambda_3)-2$ and $\lambda_2\lambda_3$ is also constant on Σ_0. Then, we can use Lemma 7.3 and we can put $\lambda_1=2\tanh 2r$, $\lambda_2=\coth r$ and $\lambda_3=\tanh r$ for some $r>0$. Moreover, $\nu=m-1$ and $\Omega=\Sigma_0$ is closed in M. Thus, $\Omega=M$. Now, we apply Theorem 4.2, (5.8) and (5.9) and we have that M is the tube $M^h(\cosh^2 2r)$ defined in Example 6.3. But any tube of this type is pseudo-Einstein. So, the proof is concluded.

We found in Section 6 that the function b of (8.1) is the constant $2m$ for all spaces appearing in Theorem 8.1. So, it is immediate

COROLLARY 8.2. *There are no Einstein real hypersurfaces in* CH^m, $m\geqq 3$.

Moreover, taking into account Corollary 5.3 of [7], we can state

COROLLARY 8.3. *A complete and connected real hypersurface of* CH^m, $m\geqq 3$, *is pseudo-Einstein if and only if it is totally* η*-umbilical.*

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