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Ergodic affine maps of locally compact groups

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§ 0. Introduction.

Let X be a locally compact group with a left invariant Haar measure μ. Let $f_a = af : X \circlearrowleft$ be a continuous affine map where f is a continuous group automorphism of X and $a \in X$. f_a is said to be *ergodic* under μ if it is measurable and whenever $E \subset X$ is a Borel set such that $f_a(E) = E$ we have either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$. The shift map σ of $\boldsymbol{Z}$ is a translation defined on the discrete group $\boldsymbol{Z}$ of integers by $\sigma(n) = n+1$.

Recently N. Aoki [1] has answered the problem of Halmos (p. 29 of [7]) negatively, i.e., if X is a locally compact totally disconnected group which has an ergodic continuous automorphism with respect to a Haar measure μ, then X is compact. For the affine maps, the problem of Halmos remains an open question when X is totally disconnected.

The purpose of this paper is to prove the following:

THEOREM. *Let X be a locally compact group with a left invariant Haar measure μ and $f_a : X \circlearrowleft$ be a continuous affine map. Let $\sigma : \boldsymbol{Z} \circlearrowleft$ be the shift map. If (X, f_a, μ) is ergodic, then either X is compact or (X, f_a) is homeomorphic to $(\boldsymbol{Z}, \sigma)$.*

In N. Aoki's proof, concepts of the pseudo-orbit tracing property and topological mixing for topological dynamics play an important role. We shall apply his techniques for the proof of Theorem.

REMARK 1. Let X, f_a and μ be as in Theorem. If (X, f_a, μ) is ergodic and if X is discrete, either X is compact or (X, f_a) is homeomorphic to $(\boldsymbol{Z}, \sigma)$. Indeed, if X is finite then X is compact. If X is infinite, then $X = \{f_a^n(x) ; n \in \boldsymbol{Z}\}$ for each $x \in X$ by ergodicity of (X, f_a, μ). We define a homeomorphism φ of $\boldsymbol{Z}$ onto X by $\varphi(n) = f_a^n(x)$ $(n \in \boldsymbol{Z})$, and then we get $\varphi \circ \sigma = f_a \circ \varphi$ on $\boldsymbol{Z}$.

For the subclasses of abelian groups and connected groups, the following results are known.

THEOREM A (N. Aoki and Y. Ito [2]). *Let X be a locally compact abelian group with a left invariant Haar measure μ. If on X there exists an affine map*

$f_a(x)=af(x)$ $(x\in X)$ which is totally ergodic, then X must be compact.

THEOREM B (S.G. Dani [6]). *Let X be a connected locally compact group. Suppose that there exists an affine automorphism f_a of X and $x_0\in X$ such that the orbit $\{f_a^n(x_0); n\in \boldsymbol{Z}\}$ is dense in X. Then X is compact.*

For the proof of Theorem we shall use the definitions and the results in topological groups and topological dynamics for locally compact spaces. If X is a σ-compact group and $f: X\circlearrowleft$ is a bicontinuous automorphism, then there is an f-invariant compact normal subgroup H of X such that X/H is separable and metrizable (see [1]). When X/H is compact, so is X. If (X, f_a, μ) is ergodic then f_a is bicontinuous and μ-measure preserving (Appendix 1) and moreover X is σ-compact. Let V be a compact symmetric neighborhood of the identity e in X. The set $H=\bigcup_{n\geqq 1}V^n$ is a σ-compact open subgroup of X. Since f_a is bicontinuous, the set $K=\bigcup_{j\in \boldsymbol{Z}}f_a^j(H)$ is open σ-compact and $f_a^{-1}(K)=K$. Since $\mu(X\setminus K)=0$, K is dense in X. Put $F=\bigcup_{n\geqq 1}(K\cup K^{-1})^n$, then F is a σ-compact open subgroup of X such that $K\subset F$. Since F is a closed subgroup of X, we have $F=X$ and so X is σ-compact.

Let Y be a locally compact metric space with a metric function d and g be a homeomorphism from Y onto itself. We recall that (Y, g) is *topologically mixing* iff there is an $M>0$ for any nonempty open sets U and V of Y such that $U\cap g^n(V)\neq\emptyset$ for all $n\geqq M$. If (Y, d) is complete and if (Y, g) is topologically mixing, then (Y, g) has a dense orbit. We say that g is *expansive* under d if there is an $\varepsilon>0$ such that $x\neq y$ implies the existence of $n\in\boldsymbol{Z}$ such that $d(g^n(x), g^n(y))>\varepsilon$ and that ε is an *expansive constant for g*. For $\delta>0$, a sequence $\{x_i\}_{i\in(\alpha,\beta)}$ $(-\infty\leqq\alpha<\beta\leqq\infty)$ of points of Y is called a *δ-pseudo-orbit* under d for g if $d(g(x_i), x_{i+1})<\delta$ for $i\in(\alpha, \beta)$. Given $\varepsilon>0$, a pseudo-orbit $\{x_i\}$ is called to be *ε-traced* under d by a point $x\in Y$ if $d(g^i(x), x_i)<\varepsilon$ for $i\in(\alpha, \beta)$. We say g to have the *pseudo-orbit tracing property* (abbrev. P.O.T.P.) under d if for every $\varepsilon>0$ there is a $\delta>0$ such that every δ-pseudo-orbit under d for g can be ε-traced by some point in Y. Note that P.O.T.P. is defined for continuous maps. Let X be a metric space and $\varphi: Y\to X$ be a homeomorphism for which φ^{-1} is uniformly continuous. If g is expansive then so is $\varphi\circ g\circ\varphi^{-1}$. If in particular φ and φ^{-1} are uniformly continuous and g has P.O.T.P., then $\varphi\circ g\circ\varphi^{-1}$ has P.O.T.P.

Our result will be reduced to the case when X is metrizable and separable and f_a is bicontinuous. Since X has a countable base, the assumption for (X, f_a, μ) to be ergodic will be changed by the assumption that (X, f_a) has a dense orbit (see p. 26 [7]).

The conclusion of Theorem will be obtained in proving the following two propositions.

PROPOSITION 1. *Let X be a locally compact group with a left invariant Haar measure μ and X_0 be the connected component of the identity e in X. Let $f_a : X \circlearrowleft$ be a bicontinuous affine map. If (X, f_a, μ) is ergodic and X/X_0 is compact, then X is compact.*

PROPOSITION 2. *Let X be a locally compact totally disconnected metric group with a left invariant metric function d_0 and $f_a : X \circlearrowleft$ be a bicontinuous affine map. If X is not discrete and (X, f_a) has a dense orbit, then there exist an f-invariant compact subgroup B of X and an f-invariant open subgroup Y of X with $B \subset Y$ such that X/Y is compact, $(Y/B, h)$ is topologically mixing and $(Y/B, h)$ has P.O.T.P. Here h denotes a homeomorphism on X/B defined by $h(xB)=f(x)B$ $(x \in X)$.*

§1. Proof of Proposition 1.

It is enough to show that X_0 is compact. To do this, assuming X_0 is not compact. We see (p. 175, [**10**]) that there exists the maximal compact normal subgroup N of X_0 such that X_0/N is a Lie group. It is easy to see that N is normal in X and invariant under f. Put $Y=X/N$ and $Y_0=X_0/N$. Since Y/Y_0 is homeomorphic to X/X_0, Y/Y_0 is totally disconnected. Since Y_0 is connected and Y/Y_0 is compact, there is a compact normal subgroup K of Y such that Y/K is a Lie group. Let $\bar{f} : Y \circlearrowleft$ be the automorphism induced by $f : X \circlearrowleft$ and put

$$K_n = K\bar{f}(K)\bar{f}^2(K) \cdots \bar{f}^n(K) \qquad \text{for} \quad n \geqq 0 .$$

Since K_n is a compact normal subgroup of Y, Y/K_n is a Lie group. For $n \geqq 0$, Y_0K_n/K_n is open in Y/K_n because the connected component of the identity of a Lie group is open. Therefore Y_0K_n is open and closed in Y. $H=\bigcup_{n\geqq 0} Y_0K_n$ is an open-closed subset of Y and $\bar{f}(H) \subset H$ holds. Since $\bar{f}$ is measure preserving, we have $\bar{f}(H)=H$. Denote by $\bar{f}_a : Y \circlearrowleft$ the affine map induced by $f_a : X \circlearrowleft$ and by $\tilde{f}_a : Y/H \circlearrowleft$ the map induced by $\bar{f}_a : Y \circlearrowleft$. Since (X, f_a, μ) is ergodic, $(Y, \bar{f}_a)$ is ergodic with respect to the induced Haar measure $\bar{\mu}=\mu \circ \pi^{-1}$ where $\pi : X \to X/N$ is the projection. Since H is open in Y, Y/H is discrete. By Remark 1, either $(Y/H, \tilde{f}_a)$ is homeomorphic to $(\boldsymbol{Z}, \sigma)$ or Y/H is compact. If $(Y/H, \tilde{f}_a)$ is homeomorphic to $(\boldsymbol{Z}, \sigma)$, then we can find an element $\bar{x} \in Y$ such that $Y/H=\{\tilde{f}_a^n(\bar{x}H) ; n \in \boldsymbol{Z}\}$ since $(Y/H, \tilde{f}_a)$ has a dense orbit. Hence $\bar{x}H$ is a wandering set of Y for $\bar{f}_a$. Since $\bar{x}H$ is open and closed in Y, we have $\bar{x}H=\{\bar{x}\}$; i.e. $H=\{\bar{e}\}$. Therefore $(X/N, \bar{f}_a)$ is homeomorphic to $(\boldsymbol{Z}, \sigma)$, hence X/N is discrete and N is open and closed. We conclude that $N=X_0$. This contradicts that X_0 is not compact. We now give a proof for the case when Y/H is compact. Since Y/H is discrete, Y/H is finite. Since $\bar{f}^{-1}(K)$ is compact in Y and $\bar{f}^{-1}(K) \subset H=\bigcup_{n\geqq 0} Y_0K_n$, there is an $m>0$ such that $\bar{f}^{-1}(K) \subset Y_0K_{m-1}$. Hence

$$K\subset\bar{f}(Y_0K_{m-1})\subset Y_0K_m \quad \text{and so} \quad Y_0K_m\subset\bar{f}(Y_0K_m)\,.$$

Since Y_0K_m is open and closed, $\bar{f}(Y_0K_m)=Y_0K_m$. Hence we get $H=Y_0K_m$. Since K_m is a compact normal subgroup of Y, we get $K_m\cap Y_0=\{\bar{e}\}$ because $N=\bar{e}$ and Y_0 contains no compact normal subgroups of Y. We have $H=K_m\times Y_0$. Since the projections of $\bar{f}(K_m)$ and $\bar{f}^{-1}(K_m)$ to Y_0 are compact normal subgroups of Y_0, they must be the trivial subgroup $\{\bar{e}\}$. Therefore we get $\bar{f}(K_m)\subset K_m$ and $\bar{f}^{-1}(K_m)\subset K_m$ so that K_m is invariant under $\bar{f}$. Therefore $\bar{f}(\bar{k}, \bar{y})=(\bar{f}(\bar{k}), \bar{f}(\bar{y}))$ for $(\bar{k}, \bar{y})\in K_m\times Y_0$. We note that $(Y/H, \tilde{f}_a)$ is ergodic with respect to the induced Haar measure $\tilde{\mu}=\bar{\mu}\circ\pi^{-1}$ where $\pi: Y\to Y/H$ is the projection. Since Y/H is finite and $(Y/H, \tilde{f}_a)$ has a dense orbit, there exists a natural number n such that $\bar{f}_a^n(H)=\tilde{f}_a^n(H)=H$ and $Y/H=\{H, \tilde{f}_a(H), \cdots, \tilde{f}_a^{n-1}(H)\}$. Since Y is the disjoint sum of cosets $\bar{f}_a^i(H)$, $0\leqq i\leqq n-1$, and since $(Y, \bar{f}_a)$ has a dense orbit, $(H, \bar{f}_a^n)$ has also a dense orbit. Since $\bar{f}_a^n=a\bar{f}(a)\cdots\bar{f}^{n-1}(a)\bar{f}^n$, there exists a $(b, c)\in K_m\times Y_0$ such that

$$\bar{f}_a^n=(b, c)\bar{f}^n:\ (\bar{k}, \bar{y})\longrightarrow(b\bar{f}^n(\bar{k}), c\bar{f}^n(\bar{y})) \qquad \text{for} \quad (\bar{k}, \bar{y})\in K_m\times Y_0\,,$$

then $b\bar{f}^n: K_m\to K_m$ and $c\bar{f}^n: Y_0\to Y_0$ are affine maps and $(c\bar{f}^n)(Y_0)=Y_0$. Since $(H, \bar{f}_a^n)$ has a dense orbit, $(Y_0, c\bar{f}^n)$ has a dense orbit. Since Y_0 is connected, Y_0 is compact (Theorem B). This contradicts that Y_0 is not compact. The proof is completed.

§ 2. Proof of Proposition 2.

Since X is totally disconnected and not discrete, there is a compact open subgroup B_0 of X. Put $B=\bigcap_{i\in\boldsymbol{Z}}f^i(B_0)$, then B is a compact subgroup of X and $f(B)=B$ holds. Now define a compatible metric function d of the left coset space X/B by

$$d(xB, yB)=\inf\{d_0(xb, yb')\,;\, b, b'\in B\} \qquad (x, y\in X)\,.$$

Define the maps $h: X/B\circlearrowleft$ and $h_a: X/B\circlearrowleft$ by $h(xB)=f(x)B$ and $h_a(xB)=af(x)B$ $(x\in X)$ respectively. Then $(X/B, d)$ is a complete metric space and h is a bicontinuous map on X/B. Since B_0/B is a compact open set of X/B and $\bar{e}=B\in B_0/B$, there exists an $\varepsilon_0>0$ such that $U_{\varepsilon_0}(\bar{e})\subset B_0/B$, where $U_{\varepsilon_0}(\bar{e})=\{\bar{x}\in X/B\,;\, d(\bar{x}, \bar{e})<\varepsilon_0\}$. Then ε_0 is its expansive constant for $(X/B, h_a)$. Indeed, for $\bar{x}=xB$, $\bar{y}=yB\in X/B$, if $d(h_a^n(\bar{x}), h_a^n(\bar{y}))<\varepsilon_0$ for all $n\in\boldsymbol{Z}$, then

$$d(f^n(y^{-1}x)B, B)=d(f^n(x)B, f^n(y)B)=d(f_a^n(x)B, f_a^n(y)B)<\varepsilon_0$$

for all $n\in\boldsymbol{Z}$. This implies that $f^n(y^{-1}x)\in B_0$ for all $n\in\boldsymbol{Z}$, hence $\bar{x}=\bar{y}$. If X/B is not discrete then $(X/B, h)$ has P.O.T.P. (see § 2, [**1**]), hence $(X/B, h_a)$ has P.O.T.P. (Appendix 2). We now consider the case when X/B is discrete. Then $(X/B, h_a)$ is homeomorphic to $(\boldsymbol{Z}, \sigma)$ or X/B is compact (by Remark 1). If

$(X/B, h_a)$ is homeomorphic to $(\boldsymbol{Z}, \sigma)$, then $B=\{e\}$ since B is open and (X, f_a) has a dense orbit. Hence (X, f_a) is homeomorphic to $(\boldsymbol{Z}, \sigma)$, but this contradicts nondiscreteness of X. If X/B is compact, X is compact since B is compact. Since f is a continuous automorphism of X, (X, f) has P.O.T.P. (N. Aoki [**4**]). Hence (X, f_a) has P.O.T.P. (Appendix 2). This is enough to give a proof for $(X/B, h_a)$.

Let $\mathrm{Per}(h_a)$ be the set of all periodic points of h_a.

LEMMA 1. *$\mathrm{Per}(h_a)$ is dense in X/B.*

PROOF. Take $\bar{x}\in X/B$ and λ with $0<\lambda<\varepsilon_0$. For this λ, let δ $(0<\delta<\lambda)$ be the number in the definition of P.O.T.P. for $(X/B, h_a)$. Since $(X/B, h_a)$ has a dense orbit, there are $\bar{x}_0\in X/B$ and $m, n\in\boldsymbol{Z}$ $(m>n)$ such that

$$d(h_a^n(\bar{x}_0), \bar{x})<\delta/2 \qquad \text{and} \qquad d(h_a^m(\bar{x}_0), h_a^n(\bar{x}_0))<\delta/2.$$

Put $\bar{z}_i=h_a^{n+k}(\bar{x}_0)$ for $i\equiv k \mod (m-n)$ $(0\leqq k<m-n)$, then $\{\bar{z}_i\}_{i\in\boldsymbol{Z}}$ is a δ-pseudo-orbit for $(X/B, h_a)$. Since $(X/B, h_a)$ has P.O.T.P., there exists $\bar{z}\in X/B$ such that $d(h_a^j(\bar{z}), \bar{z}_j)<\lambda/2$ for all $j\in\boldsymbol{Z}$. Hence

$$d(h_a^j(\bar{z}), h_a^{j+(m-n)}(\bar{z}))\leqq d(h_a^j(\bar{z}), \bar{z}_j)+d(\bar{z}_j, h_a^{j+(m-n)}(\bar{z}))<\lambda$$

for all $j\in\boldsymbol{Z}$. By expansiveness of $(X/B, h_a)$, we have $\bar{z}=h_a^{m-n}(\bar{z})$: i.e. $\bar{z}\in\mathrm{Per}(h_a)$, and

$$d(\bar{z}, \bar{x})\leqq d(\bar{z}, h_a^n(\bar{x}_0))+d(h_a^n(\bar{x}_0), \bar{x})<\lambda.$$

For ε with $0<\varepsilon<\varepsilon_0$ and $\bar{x}=xB\in X/B$, let $W_\varepsilon^s(\bar{x}, h_a)$ and $W_\varepsilon^u(\bar{x}, h_a)$ be the local stable and unstable sets defined by

$$W_\varepsilon^s(\bar{x}, h_a)=\{\bar{y}\in X/B\,;\, d(h_a^j(\bar{y}), h_a^j(\bar{x}))<\varepsilon,\ j\geqq 0\},$$

$$W_\varepsilon^u(\bar{x}, h_a)=\{\bar{y}\in X/B\,;\, d(h_a^{-j}(\bar{y}), h_a^{-j}(\bar{x}))<\varepsilon,\ j\geqq 0\}.$$

Now define the stable and unstable sets $W^s(\bar{x}, h_a)$ and $W^u(\bar{x}, h_a)$ as

$$W^s(\bar{x}, h_a)=\bigcup_{n\geqq 0} h_a^{-n}(W_\varepsilon^s(h_a^n(\bar{x}), h_a)),$$

$$W^u(\bar{x}, h_a)=\bigcup_{n\geqq 0} h_a^{n}(W_\varepsilon^u(h_a^{-n}(\bar{x}), h_a)).$$

Then for every $\bar{x}\in X/B$ we obtain (see [**1**]) that

$$W^s(\bar{x}, h_a)=\{\bar{y}\in X/B\,;\, \lim_{n\to\infty} d(h_a^n(\bar{x}), h_a^n(\bar{y}))=0\},$$

$$W^u(\bar{x}, h_a)=\{\bar{y}\in X/B\,;\, \lim_{n\to\infty} d(h_a^{-n}(\bar{x}), h_a^{-n}(\bar{y}))=0\}.$$

REMARK 2. Since d is left invariant for X/B, we have that

$$W^s(\bar{x}, h_a)=W^s(\bar{x}, h)=\{\bar{y}\in X/B\,;\, \lim_{n\to\infty} d(h^n(\bar{x}), h^n(\bar{y}))=0\},$$

$$W^u(\bar{x}, h_a)=W^u(\bar{x}, h)=\{\bar{y}\in X/B\,;\, \lim_{n\to\infty} d(h^{-n}(\bar{x}), h^{-n}(\bar{y}))=0\}.$$

Hereafter we denote by $\bar{E}$ the closure of a subset E.

LEMMA 2. *For* $\bar{p}\in \mathrm{Per}(h_a)$, *put* $W_{\bar{p}}^{s}(h_a)=\overline{W^{s}(\bar{p}, h_a)}$ *and* $W_{\bar{p}}^{u}(h_a)=\overline{W^{u}(\bar{p}, h_a)}$. *Then* $W_{\bar{p}}^{s}(h_a)$ *and* $W_{\bar{p}}^{u}(h_a)$ *are open in* X/B.

PROOF. For $\lambda>0$ with $0<\lambda<\varepsilon_0$, let $\delta>0$ be the number in the definition of P.O.T.P. for $(X/B, h_a)$. Put

$$U_{\delta/2}(W_{\bar{p}}^{s}(h_a))=\{\bar{y}\in X/B\,;\ d(\bar{y}, W_{\bar{p}}^{s}(h_a))<\delta/2\}.$$

Since $\mathrm{Per}(h_a)$ is dense in X/B, it is enough to see that if $\bar{q}\in \mathrm{Per}(h_a)\cap U_{\delta/2}(W_{\bar{p}}^{s}(h_a))$ then $\bar{q}\in W_{\bar{p}}^{s}(h_a)$. Now take $\bar{x}=xB\in W^{s}(\bar{p}, h_a)$ with $d(\bar{x}, \bar{q})<\delta$ and put $\bar{y}_j=h_a^{j}(\bar{x})$ for $j\geqq 0$ and $\bar{y}_j=h_a^{j}(\bar{q})$ for $j\leqq -1$. Then $\{\bar{y}_j\}_{j\in \boldsymbol{Z}}$ is a δ-pseudo-orbit for $(X/B, h_a)$. Hence there is $\bar{y}\in X/B$ such that

$$d(h_a^{j}(\bar{x}), h_a^{j}(\bar{y}))<\lambda \quad \text{for } j\geqq 0 \qquad \text{and} \qquad d(h_a^{j}(\bar{q}), h_a^{j}(\bar{y}))<\lambda \quad \text{for } j\leqq -1.$$

This implies that $\bar{y}\in W^{s}(\bar{x}, h_a)\cap W^{u}(\bar{q}, h_a)$. Since $\bar{p}$ and $\bar{q}$ are periodic points of h_a, let $h_a^{m}(\bar{p})=\bar{p}$ and $h_a^{n}(\bar{q})=\bar{q}$. Since $\bar{x}\in W^{s}(\bar{p}, h_a)$, $W^{s}(\bar{x}, h_a)=W^{s}(\bar{p}, h_a)$ and $h_a^{-kmn}(\bar{y})\in h_a^{-kmn}(W^{s}(\bar{x}, h_a))=h_a^{-kmn}(W^{s}(\bar{p}, h_a))=W^{s}(\bar{p}, h_a)$ for all $k>0$. Hence

$$\lim_{k\to\infty} d(h_a^{-kmn}(\bar{y}), \bar{q})=\lim_{k\to\infty} d(h_a^{-kmn}(\bar{y}), h_a^{-kmn}(\bar{q}))=0\,.$$

Therefore $\bar{q}\in \overline{W^{s}(\bar{p}, h_a)}=W_{\bar{p}}^{s}(h_a)$ and $W_{\bar{p}}^{s}(h_a)$ is open in X/B. Similarly, $W_{\bar{p}}^{u}(h_a)$ is open in X/B.

Since $h_a^{m}(\bar{p})=\bar{p}$ and $h_a(W_{\bar{p}}^{s}(h_a))=W_{h_a(\bar{p})}^{s}(h_a)$, we have $h_a^{m}(W_{\bar{p}}^{s}(h_a))=W_{\bar{p}}^{s}(h_a)$. Since $(X/B, h_a)$ has a dense orbit, there is m' $(1\leqq m'\leqq m)$ such that

$$X/B=W_{\bar{p}}^{s}(h_a)\cup h_a(W_{\bar{p}}^{s}(h_a))\cup \cdots \cup h_a^{m'-1}(W_{\bar{p}}^{s}(h_a))$$

is a disjoint union. Similarly,

$$X/B=W_{\bar{p}}^{u}(h_a)\cup h_a(W_{\bar{p}}^{u}(h_a))\cup \cdots \cup h_a^{m'-1}(W_{\bar{p}}^{u}(h_a))$$

is a disjoint union. Since $W_{\bar{p}}^{s}(h_a)$ and $W_{\bar{p}}^{u}(h_a)$ are open in X/B and $\bar{p}\in W_{\bar{p}}^{s}(h_a)\cap W_{\bar{p}}^{u}(h_a)$ holds, there is a δ $(0<\delta<\varepsilon_0)$ such that

$$U_{\delta}(\bar{p})\subset W_{\bar{p}}^{s}(h_a)\cap W_{\bar{p}}^{u}(h_a)$$

where $U_{\delta}(\bar{p})=\{\bar{x}\in X/B\,;\ d(\bar{x}, \bar{p})<\delta\}$. We note that

$$W_{\delta}^{u}(\bar{p}, h_a)\subset U_{\delta}(\bar{p}) \qquad \text{and} \qquad W_{\bar{p}}^{u}(h_a)=\bigcup_{j\geqq 0} h_a^{jm}(W^{u}(\bar{p}, h_a))\,.$$

Then every h_a^{m}-invariant closed set which contains $U_{\lambda}(\bar{p})$ coincides with $W_{\bar{p}}^{u}(h_a)$. Similarly, each h_a^{-m}-invariant closed set which contains $U_{\lambda}(\bar{p})$ coincides with $W_{\bar{p}}^{s}(h_a)$. Hence $W_{\bar{p}}^{s}(h_a)=W_{\bar{p}}^{u}(h_a)$. We write

$$W_{\bar{e}}(h)=W_{\bar{e}}^{s}(h)=W_{\bar{e}}^{u}(h)\,.$$

LEMMA 3. $(W_{\bar{e}}(h), h)$ *has P.O.T.P.*

PROOF. Since $W_{\bar{e}}(h)$ is open in X/B, there is a $\lambda>0$ such that $U_{\lambda}(\bar{e})=\{\bar{x}\in X/B\,;\ d(\bar{x}, \bar{e})<\lambda\}\subset W_{\bar{e}}(h)$. For $\lambda/2$, let δ $(0<\delta<\lambda/2)$ be the number in the definition of P.O.T.P. for $(X/B, h)$. If $\{\bar{x}_i\}_{i\in(a,b)}$ is a δ-pseudo-orbit for $(W_{\bar{e}}(h), h)$, then there exist $\bar{z}_a\in W^u(\bar{e}, h)$ and $n>0$ such that $d(h(\bar{z}_a), \bar{x}_{a+1})<\delta$ and $d(h^{-n}(\bar{z}_a), \bar{e})<\delta$. Put $\bar{y}_k=h^{-n+k}(\bar{z}_a)$ for $0\leqq k\leqq n$ and $\bar{y}_k=\bar{x}_{a+(k-n)}$ for $n+1\leqq k\leqq b-a+1$. Then $\{\bar{y}_k\}_{k\in(-1,b-a+n)}$ is a δ-pseudo-orbit for $(X/B, h)$. Since $(X/B, h)$ has P.O.T.P., there is an $\bar{x}\in X/B$ such that $d(h^j(\bar{x}), \bar{y}_j)<\lambda/2$ for $0\leqq j\leqq b-a+n$ and in particular, $d(\bar{x}, \bar{e})<\lambda$. Hence $\bar{x}\in W_{\bar{e}}(h)$. Put $\bar{z}=h^{n-a}(\bar{x})$. Since $W_{\bar{e}}(h)$ is h-invariant, $\bar{z}=h^{n-a}(\bar{x})\in W_{\bar{e}}(h)$ and $\bar{z}$ is a λ-tracing point for $\{\bar{x}_i\}_{i\in(a,b)}$. Therefore $(W_{\bar{e}}(h), h)$ has P.O.T.P.

LEMMA 4. *$(W_{\bar{e}}(h), h)$ is topologically mixing.*

PROOF. Let U and V be nonempty open sets of $W_{\bar{e}}(h)$. Then there exist $\bar{x}\in W^u(\bar{e}, h)\cap U$ and $\bar{y}\in W^s(\bar{e}, h)\cap V$ and $\lambda>0$ such that $U_\lambda(\bar{x})\subset U$ and $U_\lambda(\bar{y})\subset V$. For λ, let δ $(0<\delta<\lambda)$ be the number in the definition of P.O.T.P. for $(W_{\bar{e}}(h), h)$. Then there exists an $n_0>0$ such that $d(\bar{e}, h^{-n}(\bar{x}))<\delta/2$ and $d(\bar{e}, h^n(\bar{y}))<\delta/2$ for $n\geqq n_0$. For $n\geqq n_0$ and $j\geqq 0$, since the finite sequence

$$\{\bar{y}, h(\bar{y}), \cdots, h^{n+j}(\bar{y}), \bar{e}, h^{-n}(\bar{x}), \cdots, h^{-1}(\bar{x}), \bar{x}\}$$

is a δ-pseudo-orbit for $(W_{\bar{e}}(h), h)$, there is a $\bar{z}\in W_{\bar{e}}(h)$ such that $d(\bar{y}, \bar{z})<\delta$ and $d(h^{2(n+1)+j}(\bar{z}), \bar{x})<\delta$. Put $M=2(n_0+1)$, then $\bar{z}\in U_\lambda(\bar{y})\subset V$ and $h^n(\bar{z})\in U_\lambda(\bar{x})\subset U$ for $n\geqq M$. This implies that $h^n(V)\cap U\neq\emptyset$ for all $n\geqq M$.

LEMMA 5 (N. Aoki [**1**]). *Let Y be a locally compact totally disconnected metric group with a left invariant metric function d and g be a bicontinuous automorphism of Y. If (Y, g) is topologically mixing and has P.O.T.P. under d, then Y is compact.*

Let $\pi: X\to X/B$ be the projection. Put $Y=\pi^{-1}(W_{\bar{e}}(h))$. Then Y is open in X since $W_{\bar{e}}(h)$ is open in X/B. It is easy to see that Y is a subgroup of X. Indeed, for xB and $yB\in W^u(\bar{e}, h)$, since $d(h^{-j}(xB), \bar{e})\to 0$ and $d(h^{-j}(yB), \bar{e})\to 0$ as $j\to\infty$, we have

$$\begin{aligned} d(h^{-j}(y^{-1}xB), \bar{e}) &= d(f^{-j}(x)B, f^{-j}(y)B) \\ &\leqq d(h^{-j}(xB), \bar{e})+d(h^{-j}(yB), \bar{e})\to 0 \qquad \text{as } j\to\infty \end{aligned}$$

and so $y^{-1}xB\in W^u(\bar{e}, h)$. This implies that Y is a group since $W^u(\bar{e}, h)$ is dense in $W_{\bar{e}}(h)$ and $Y=\pi^{-1}(W_{\bar{e}}(h))$. It is easy to see that the left coset space X/Y is compact. Indeed, since $\sharp(X/Y)=\sharp((X/B)/W_{\bar{e}}(h))\leqq m'$ (the notation $\sharp(E)$ means the cardinality of a set E), X/Y is finite. Moreover, since $Y/B=W_{\bar{e}}(h)$, (Y, f) is topologically mixing and it has P.O.T.P. (see § 2, [**1**]). Therefore the conclusion of Theorem is obtained by Proposition 2 and Lemma 5.

§ 3. Appendices.

In this section, we prove some properties of ergodic affine maps of locally compact groups.

APPENDIX 1. *Let X be a locally compact group with a left invariant Haar measure μ and $f_a: X \circlearrowleft$ be a continuous affine map. If (X, f_a, μ) is ergodic, then*

(1) *f_a is bicontinuous, and*

(2) *f_a is μ-measure preserving.*

PROOF OF (1). As the assertion is obvious if X is discrete, we assume that X is not discrete. If f_a is not bicontinuous, then f is not bicontinuous. Thus there exists an open σ-compact subgroup H of X such that $f(H) \subset H$ and $f^{-1}(H)$ is not σ-compact. Let F be the subgroup of X generated by the σ-compact set $H \cup f_a(H)$. Since f is continuous, the sets $f^j(F)$ $(j=0, 1, 2, \cdots)$ are σ-compact. The subgroup K of X generated by $\bigcup_{j \geqq 0} f^j(F)$ is open and σ-compact. Clearly $f(K) \subset K$. Since $a \in f_a(H) \subset K$, we see that $f_a^{-1}(K) = f^{-1}(a^{-1}) f^{-1}(K) = f^{-1}(a^{-1}K) = f^{-1}(K)$. Put $P = f^{-1}(K) \setminus K$. Since $f^{-1}(K)$ is not σ-compact, P is a nonempty open-closed subset of X and $f_a^k(P) \cap f_a^j(P) = \emptyset$ whenever $k \neq j$. Since X is not discrete, there is a compact subset C such that $\mu(C) > 0$ and $\mu(P \setminus C) > 0$. The set $W = \bigcup_{j \in \boldsymbol{Z}} f_a^j(C)$ is a Borel set of X satisfying $f_a^{-1}(W) = W$. However, $\mu(W) > 0$ and $\mu(X \setminus W) \geqq \mu(P \setminus C) > 0$ because P is a wandering set. This contradicts the ergodicity of f_a.

PROOF OF (2). Since f_a is bicontinuous and μ is a left invariant Haar measure, there is a $\delta > 0$ such that $\mu(f_a(E)) = \mu(af(E)) = \mu(f(E)) = \delta\mu(E)$ and $\mu(f_a^{-1}(E)) = \mu(f^{-1}(a^{-1}) f^{-1}(E)) = \mu(f^{-1}(E)) = \delta^{-1}\mu(E)$ for all Borel sets $E \subset X$. If f_a is not μ-measure preserving, then $\delta \neq 1$ and X is not compact. If $\delta > 1$, then we show that the ergodicity of f_a does not hold. For $\lambda > 0$, there is a nonempty open subset U such that $\mu(U) < \lambda$. Now let V be a compact neighborhood of the identity e of X. Put $W = \bigcup_{n \geqq 1} f_a^{-n}(V)$. Then $\mu(W) \leqq \sum_1^\infty \mu(f_a^{-n}(V)) = \sum_1^\infty (\delta^{-n}) \mu(V) = (1/(\delta-1))\mu(V) < \infty$. Clearly, $f_a(W) \supset W$ and $f_a^n(X \setminus W) \cap W = \emptyset$ for $n=0, 1, 2, \cdots$. Since W is open and σ-compact, there is a σ-compact open subgroup H of X such that $W \subset H$. Therefore there exists a Borel subset E of X such that $E \subset X \setminus W$ and $0 < \mu(E) < ((\delta-1)/2)\mu(V)$. Then

$$\mu\Big(\bigcup_{n \geqq 1} f_a^{-n}(E)\Big) \leqq \sum_1^\infty (\delta^{-n}) \mu(E) < \mu(V)/2 .$$

Put $F = \bigcup_{n \in \boldsymbol{Z}} f_a^n(E)$. Then $f_a^{-1}(F) = F$ and $\mu(F) > 0$. Since $f_a^n(E) \cap V = \emptyset$ for $n=0, 1, 2, 3, \cdots$,

$$\mu(X \setminus F) \geqq \mu\Big(V \setminus \bigcup_{n \in \boldsymbol{Z}} f_a^n(E)\Big) = \mu\Big(V \setminus \bigcup_{n \geqq 1} f_a^{-n}(E)\Big) \geqq \frac{1}{2}\mu(V) > 0 .$$

This contradicts the ergodicity of f_a. For the case when $\delta < 1$, f_a^{-1} is not ergodic since $\delta^{-1}>1$. In any case f_a must be μ-measure preserving.

APPENDIX 2. *Let X be a locally compact metric group with a left invariant metric function d and $f: X \circlearrowleft$ be a bicontinuous automorphism. Let $f_a: X \circlearrowleft$ be a bicontinuous affine map defined by $f_a(x)=af(x)$ $(x \in X)$. If (X, f) has P.O.T.P., then (X, f_a) has P.O.T.P.*

PROOF. For $\varepsilon>0$, let $\delta>0$ be the number in the definition of P.O.T.P. for (X, f). Let $\{x_i\}_{i \in \boldsymbol{Z}}$ be a δ-pseudo-orbit for (X, f_a). Now put

$$z_n = f^{n-1}(a^{-1}) f^{n-2}(a^{-1}) \cdots f(a^{-1}) a^{-1} x_n \qquad (n \in \boldsymbol{Z}),$$

then

$$\begin{aligned} d(f(z_n), z_{n+1}) &= d(f^n(a^{-1}) \cdots f(a^{-1}) f(x_n), f^n(a^{-1}) \cdots f(a^{-1}) a^{-1} x_{n+1}) \\ &= d(f(x_n), a^{-1} x_{n+1}) = d(f_a(x_n), x_{n+1}) < \delta \qquad (n \in \boldsymbol{Z}). \end{aligned}$$

Hence $\{z_n\}_{n \in \boldsymbol{Z}}$ is a δ-pseudo-orbit for (X, f). Since (X, f) has P.O.T.P., there exists a $z \in X$ such that $d(f^n(z), z_n) < \varepsilon$ $(n \in \boldsymbol{Z})$. Hence

$$d(f_a^n(z), x_n) = d(f_a^n(z), af(a) \cdots f^{n-1}(a) z_n) = d(f^n(z), z_n) < \varepsilon$$

for all $n \in \boldsymbol{Z}$. The proof is completed.

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