J. Math. Soc. Japan
Vol. 37, No. 2, 1985

A limit theorem for sums of random number of i. i. d. random variables and its application to occupation times of Markov chains

By Yuji KASAHARA

(Received March 2, 1984)

1. Introduction.

Let $\{X_t : t \geqq 0\}$ be a temporally homogeneous Markov process and $f(x) \geqq 0$ be a bounded measurable function on the state space. Suppose that, as $t \to \infty$, $E_x\left[\int_0^t f(X_s)ds\right]$ is asymptotically equal to a slowly varying function $L(t)$. Then the author [6] proved that the process $(1/\lambda)\int_0^{L^{-1}(\lambda t)} f(X_s)ds$ converges in law to $l(M^{-1}(t))$ as $\lambda \to \infty$, where $l(t)$ and $M(t)$ denote the local time at $x=0$ and the maximum process of a (common) Brownian motion, respectively. A central limit theorem is discussed in [7] and in this case the limiting process is of the form $\tilde{B}(l(M^{-1}(t)))$ where $\tilde{B}(t)$ is a Brownian motion independent of $l(M^{-1}(t))$.

In these papers the main tool was the moment method and therefore the reason why the limiting processes are of the form $l(M^{-1}(t))$ or $\tilde{B}(l(M^{-1}(t)))$ was not clear. The aim of this article is to give a natural explanation to this point from the view point of classical limit theorems for i. i. d. random variables by restricting ourselves to Markov chains with denumerable state space. Our idea is to reduce the problem to the study of sums of random number of i.i.d. random variables, which method has been adopted by Doeblin [4], Dobrusin [3], Chung [1], Kesten [10] and many others.

In section 2 we will give a certain limit theorem for vector-valued i. i. d. random variables and as an application we will prove in section 3 a limit theorem for random sums of i. i. d. random variables, which is our main result. This result may be regarded as the extreme case as $\alpha \to 0$ of the result in [8]. In the last section, we will apply the main theorem to the occupation-time problems of certain class of Markov chains including the usual random walk on the plane. Of course the result is quite similar to that for 2-dimensional Brownian motion given in [6, 7].

This research was partially supported by Grant-in-Aid for Scientific Research (No. 58740091), Ministry of Education.

2. A limit theorem for i. i. d. random vectors.

Let $\{(\xi_i, \tau_i)\}_{i=1}^{\infty}$ be a sequence of i. i. d. (independent, identically distributed) $\boldsymbol{R}\times[0, \infty)$-valued random variables. Thus ξ_n and τ_n are independent of $\{\xi_1, \tau_1, \cdots, \xi_{n-1}, \tau_{n-1}\}$ but τ_n may be even a function of ξ_n. Throughout this section we assume that ξ_1 has finite second moment and let

$$m=E[\xi_1], \qquad \sigma^2=E[(\xi_1-m)^2]. \tag{2.1}$$

Therefore, we have that, as $n\to\infty$,

$$(\xi_1+\xi_2+\cdots+\xi_{[nt]})/n \longrightarrow mt, \qquad \text{a.s.}, \tag{2.2}$$

$$(\xi_1+\xi_2+\cdots+\xi_{[nt]}-mnt)/\sqrt{n} \overset{\mathcal{D}}{\Longrightarrow} \sigma^2 B(t), \tag{2.3}$$

where $B(t)$ is a standard Brownian motion and $\overset{\mathcal{D}}{\Longrightarrow}$ denotes the weak convergence in the Skorohod function space $D([0, \infty))$ endowed with the J_1-topology (see Lindvall [**11**]). As for τ_i we assume that the tail probability varies slowly;

$$P[\tau_1>x] \sim C/L(x) \qquad \text{as } x\to\infty \tag{2.4}$$

for some $C>0$ and continuous, increasing slowly varying function $L(x)$ (i.e., $L(\lambda x)/L(x)\to 1$ as $x\to\infty$ for $\lambda>0$). Here $a(x)\sim b(x)$ means that the ratio converges to 1. In [**8**] we considered the case where

$$P[\tau_1>x] \sim C/(x^{\alpha}L(x)), \qquad x\to\infty \quad (0<\alpha<1). \tag{2.4$'$}$$

Thus we are interested in the extreme case of [**8**] as $\alpha\downarrow 0$. The reader should notice that (2.4)$'$ is equivalent to that τ_1 belongs to the domain of attraction of stable law with index α. However, under our assumption (2.4), it is *impossible* to choose constants a_n and b_n so that $(\tau_1+\cdots+\tau_n-a_n)/b_n$ has proper limiting distributions (see Darling [**2**]). Instead, using a nonlinear normalization we have the following theorem due to S. Watanabe [**14**]. (Convergence of one-dimensional marginal distributions was first proved by D. A. Darling [**2**].)

THEOREM A. *Under the assumption* (2.4), *we have*

$$(1/\lambda)L(\tau_1+\cdots+\tau_{[nt]}) \overset{\text{f.d.}}{\Longrightarrow} CM(l^{-1}(t)), \qquad \text{as } \lambda\to\infty$$

where $l(t)$ and $M(t)$ are the same as in the previous section;

$$M(t)=\max\{B(s): s\leqq t\},$$

$$l(t)=\lim_{\varepsilon\downarrow 0}(1/2\varepsilon)\int_0^t 1_{[0,\varepsilon)}(B(s))ds,$$

where $B(\cdot)$ is a standard Brownian motion starting at 0.

Throughout this paper, we denote by $\overset{\text{f.d.}}{\Longrightarrow}$ the convergence of all finite-dimensional marginal distributions. The process $M(l^{-1}(t))$ is a temporally homogeneous Markov process and is an *extremal process* in the sense of Dwass. Its inverse process $l(M^{-1}(t))$ has independent increments (see S. Resnick [**13**]).

In this section we consider the joint convergence of

$$(\xi_1+\xi_2+\cdots+\xi_{[\lambda t]}-\lambda mt)/\sqrt{\lambda} \quad \text{and} \quad (1/\lambda)L(\tau_1+\tau_2+\cdots+\tau_{[\lambda t]}).$$

Our assertion is that these two processes are asymptotically independent as $\lambda\to\infty$:

THEOREM 2.1. *Let* $\{(\xi_i, \tau_i)\}_{i=1}^{\infty}$ *be a sequence of* $\boldsymbol{R}\times[0, \infty)$*-valued i.i.d. random variables satisfying* (2.1) *and* (2.4). *Then*

$$((\xi_1+\cdots+\xi_{[\lambda t]}-\lambda mt)/\sqrt{\lambda}, (1/\lambda)L(\tau_1+\cdots+\tau_{[\lambda t]}))$$
$$\overset{\text{f.d.}}{\Longrightarrow} (\sigma\tilde{B}(t), CM(l^{-1}(t))) \qquad \text{as } \lambda\to\infty,$$

where $M(l^{-1}(t))$ *is the same as in Theorem A and* $\tilde{B}(t)$ *is a standard Brownian motion* ($\tilde{B}(0)=0$) *independent of* $M(l^{-1}(t))$.

PROOF. The only thing to be proven is the independence of $\tilde{B}$ and $M(l^{-1}(\cdot))$ because the convergence of the each component is already known (see (2.3) and Theorem A). Observe that we have that the ratio $L(\tau_1+\cdots+\tau_n)/L(\max_{i\leq n}\tau_i)$ converges to 1 in probability as $n\to\infty$ (see Darling [**2**]). ([**2**] assumed that τ_1 has density, which condition can easily be removed.) Thus it suffices to show that

$$((\xi_1+\cdots+\xi_{[\lambda t]}-\lambda mt)/\sqrt{\lambda}, (1/\lambda)\max_{i\leq\lambda t} L(\tau_i))$$
$$\overset{\mathcal{D}}{\Longrightarrow} (\sigma\tilde{B}(t), CM(l^{-1}(t))) \qquad \text{as } \lambda\to\infty.$$

Here we have used that $L(u)$ is monotone and that the limit process has no fixed discontinuities. The problem of this kind has already been studied and the above assertion is a special case of [**9**]. We refer to [**9**] for details but we emphasize that the independence of $\tilde{B}$ and $M(l^{-1})$ can be reduced to the well-known fact that a Brownian motion and a Poisson point process on the same probability space with a common reference family of sub σ-field are necessarily independent. To make sure, we will explain how to apply the results of [**9**]. Observe that $P[L(\tau_1)>x]\sim 1/x$ as $x\to\infty$ and that hence (4.2) of [**9**] is satisfied with $g(x)=x$, $C_n=0$, $B_n=1/n$. Thus we see that the assumptions of Theorem 5.1 of [**9**] is satisfied. It is easy to check that the limiting process $J^{(1)}\xi(t)$ equals $M(l^{-1}(t))$ if we look at these processes from the view point of Poisson point processes (see Watanabe [**14**]). However, we already know the convergence of the both components and the only problem was the independence of $\tilde{B}$ and $M(l^{-1})$ and so we omit the details.

3. Sums of random number of i. i. d. random variables.

Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of i. i. d. random variables. We are interested in the asymptotic behavior of random variables of the form

$$S(t)=\xi_1+\xi_2+\cdots+\xi_{T(t)}, \qquad t\geqq 0 \tag{3.1}$$

as $t\to\infty$, where $T(t)$ is an integral valued nondecreasing process ($T(0)=1$). In this section we consider the case where $T(t)$ is defined as the inverse process of $U(t)=\tau_1+\cdots+\tau_{[t]}$;

$$T(t)=\inf\left\{n\geqq 0: \sum_{i=1}^{n}\tau_i>t\right\}, \qquad t\geqq 0 \tag{3.2}$$

where $\{\tau_i\}$ are nonnegative random variables.

THEOREM 3.1. *Let* $\{(\xi_i, \tau_i)\}_{i=1}^{\infty}$ *be as in Theorem* 2.1 *and let* $S(t)$ *be the process defined by* (3.1) *and* (3.2). *Then we have*

(i) *the law of* $S(t)/L(t)$ *converges weakly to an exponential distribution with expectation* m/C, *as* $t\to\infty$, *and*

(ii) *if in addition* $m=0$, *the law of* $S(t)/\sqrt{L(t)}$ *converges weakly to a bilateral exponential distribution with variance* σ^2/C.

Bilateral exponential distribution with variance v^2 has density $(\sqrt{2}v)^{-1}\cdot\exp\{-(\sqrt{2}/v)|x|\}$, $-\infty<x<\infty$. Since this theorem will be proven in a generalized form later (Theorem 3.2) we omit the proof. (This theorem can also be regarded as a special case of Kesten [**10**].) The aim of this article is to study the convergence of the process $t\mapsto S(\lambda t)/L(\lambda)$ (or $S(\lambda t)/\sqrt{L(\lambda)}$ if $m=0$) as $\lambda\to\infty$. However, as is pointed out in [**6**], since $L(\lambda t)/L(\lambda)\to 1$ as $\lambda\to\infty$, it is not difficult to see that the limiting processes, if exist in some sense, are constant random variables and do not depend on t. Therefore, it would be interesting to consider another normalization; $(1/\lambda)S(L^{-1}(\lambda t))$ (or $(1/\sqrt{\lambda})S(L^{-1}(\lambda t))$ if $m=0$). The author learned this kind of normalization from D. Stroock. Our main result is as follows.

THEOREM 3.2. *Under the assumptions of Theorem* 3.1., *we have*

(i) $$(1/\lambda)S(L^{-1}(\lambda t))\xrightarrow{\text{f.d.}}(m/C)l(M^{-1}(t)) \quad \textit{as } \lambda\to\infty,$$

(ii) *if, in addition,* $m=0$ *then*

$$\lambda^{-1/2}S(L^{-1}(\lambda t))\xrightarrow{\text{f.d.}}(\sigma^2/C)^{1/2}\tilde{B}(l(M^{-1}(t))) \quad \textit{as } \lambda\to\infty$$

where $\tilde{B}(t)$ *and* $l(M^{-1}(t))$ *are the same as in Theorem* 2.1.

PROOF. Since $T(t)$ is the inverse process of $U(t)=\sum_{i\leq t}\tau_i$, $(1/\lambda)T(L^{-1}(\lambda t))$ is the inverse of $L(U(\lambda t))/\lambda$, which converges to $CM(l^{-1}(t))$ as $\lambda\to\infty$ by Theorem 2.1. Keeping in mind that $M(l^{-1})$ has no fixed discontinuities, we have from

Theorem 2.1 that, for arbitrary $0 \leqq t_1 < t_2 < \cdots < t_n$, $n=1, 2, \cdots$, $(\lambda^{-1/2}(\xi_1 + \cdots + \xi_{[\lambda t]} - \lambda m t)$, $(1/\lambda)T(L^{-1}(\lambda t_1)), \cdots, (1/\lambda)T(L^{-1}(\lambda t_n)))$ converges weakly to $(\sigma \tilde{B}(t)$, $l(M^{-1}(t_1/C)), \cdots, l(M^{-1}(t_n/C)))$ in the product topology of $D([0, \infty)) \times \boldsymbol{R}^n$. On the basis of the well known continuity theorem we have the following by plugging $(1/\lambda)T(L^{-1}(\lambda t_j))$ to t in the first component.

$(\lambda^{-1/2}S(L^{-1}(\lambda t_1)), \cdots, \lambda^{-1/2}S(L^{-1}(\lambda t_n)))$ converges weakly to $\sigma(\tilde{B}(l(M^{-1}(t_1/C))), \cdots, \tilde{B}(l(M^{-1}(t_n/C))))$ provided that $m=0$. Since $\{l(M^{-1}(t/C))\}$ and $\{l(M^{-1}(t))/C\}$ are identical in law (see Remark) so are $\{\tilde{B}(l(M^{-1}(t/C)))\}$ and $\{C^{-1/2}\tilde{B}(l(M^{-1}(t)))\}$. Thus (ii) is proved. (i) can be proved in a similar way using (2.2).

REMARKS. (i) The finite-dimensional marginals of $Z(t)=l(M^{-1}(t))$ are as follows (cf. Watanabe [**14**]).

$$P[Z(t_1)>x_1, \cdots, Z(t_n)>x_n]$$
$$=\exp[-x_1/t_1-(x_2-x_1)/t_2- \cdots -(x_n-x_{n-1})/t_n],$$

for $0 \leqq t_1 < \cdots < t_n$ and $0 \leqq x_1 < \cdots < x_n$, $n=1, 2, \cdots$.

(ii) In the proof of Theorem 3.2 (ii), we have actually proven that

$$\lambda^{-1/2}\{S(L^{-1}(\lambda t))-mT(L^{-1}(\lambda t))\} \overset{\text{f.d.}}{\Longrightarrow} (\sigma^2/C)^{1/2}\tilde{B}(l(M^{-1}(t))), \quad \text{as } \lambda \to \infty. \tag{3.3}$$

(iii) Define $S^-(t)=\sum_{i<T(t)}\xi_i$. Then with the obvious modification of the proof we see that an analogue of (3.3) holds for S^-, and hence it is easy to see that

$$\lambda^{-1/2}\{S(L^{-1}(\lambda t))-S^-(L^{-1}(\lambda t))\} \overset{\text{f.d.}}{\Longrightarrow} 0, \quad \lambda \to \infty. \tag{3.4}$$

We will use this fact in the next section.

4. Occupation times of Markov chains.

Let $\{X_n\}_{n=0}^{\infty}$ be a recurrent, irreducible Markov chain with state space $S=\{a_0, a_1, \cdots\}$. By $P_{ij}(k)$ we denote the k-step transition probability $P[X_k=a_j \mid X_0=a_i]$, and by $E_0[\cdot]$, we denote the expectation with respect to $P[\cdot \mid X_0=a_0]$. Define $V_0=0$, $V_n=\min\{i>V_{n-1}: X_i=a_0\}$, $n=1, 2, \cdots$. Thus V_n denotes the time of nth visit to a_0.

LEMMA 4.1. *Let $L(t)$ be a slowly varying function and C be a positive constant. Then the following conditions are equivalent.*

(i) $$E_0\left[\sum_{k=0}^{n} 1_{\{a_0\}}(X_k)\right] \sim (1/C)L(n), \quad \textit{as } n \to \infty.$$

(ii) $$\sum_{k=0}^{n} P_{00}(k) \sim (1/C)L(n), \quad \textit{as } n \to \infty.$$

(iii) $$\sum_{k=0}^{\infty} z^k P_{00}(k) \sim (1/C)L(1/(1-z)), \qquad \text{as } z \uparrow 1.$$

(iv) $$P_0[V_1>t] \sim C/L(t), \qquad \text{as } t\to\infty.$$

PROOF. The equivalence of (i) and (ii) is obvious and (ii) is equivalent to (iii) by the well known Tauberian theorem (see page 447 of [**5**]). Define $P(s) =\sum_{k=0}^{\infty} s^k P_{00}(k)$ and $F(s)=\sum_{k=0}^{\infty} s^k P[V_1=k]$, $(|s|<1)$. Then we have $P(s)= 1/(1-F(s))$ and also $1-F(s)=(1-s)\sum_{k=0}^{\infty} s^k P[V_1>k]$. Therefore, (iii) is equivalent to

(4.1) $$\sum_{k=0}^{\infty} s^k P_0[V_1>k]\,(=1/(1-s)P(s)) \sim C/\{(1-s)L(1/(1-s))\}, \qquad \text{as } s\uparrow 1.$$

Applying the Tauberian theorem again we see that each of (4.1) and (iv) implies the other.

Let $f(x)$ be a nontrivial function on the state space vanishing outside a finite set. The random variables $\sum_{k=0}^{n} f(X_k)$, $n=1, 2, \cdots$ are called occupation times, and the limiting distributions with suitable normalizations have been investigated by many authors as we mentioned in Introduction and one of the ideas adopted by them is as follows: Let ξ_n be the occupation time during the nth excursion, i.e., $\xi_n=\sum_{k=V_{n-1}}^{V_n-1} f(X_k)$, $n=1, 2, \cdots$. Notice that $\{\xi_n\}_n$ is a sequence of i.i.d. random variables if $X_0=a_0$ as a consequence of strong Markov property. Thus the occupation time can be represented as a sum of random number of i.i.d. random variables;

(4.2) $$\sum_{k=0}^{n} f(X_k)=\xi_1+\cdots+\xi_{T(n)}-\varepsilon_n$$

where $T(t)$ denotes the number of visit to a_0 up to time t and ε_n is the obvious error term. More precisely,

(4.3) $$T(t)=\inf\{n\geqq 1: V_n>t\}, \qquad t\geqq 0$$

and

(4.4) $$\varepsilon_n=\sum_{k=n+1}^{V_{T(n)}-1} f(X_k), \qquad n=1, 2, \cdots.$$

Although $T(t)$ and ξ_n are dependent, $\{(\xi_n, V_n-V_{n-1})\}_n$ are i.i.d. random vectors and therefore we can apply the results in the previous section. Before we state our assertion we define

$$m_i=E_0\left[\sum_{k=0}^{V_1-1} 1_{\{a_i\}}(X_k)\right], \qquad i=0, 1, \cdots.$$

Clearly $m_0=1$ and it is well known that m_i is finite. (Indeed, we can easily reduce to the simplest case where the state space consists of only two elements a_0 and a_i.) It should be noticed that $E_0[\xi_1]$ may be expressed as $\sum f(a_i)m_i$.

THEOREM 4.2. *Suppose one (hence all) of* (i)-(iv) *in Lemma* 4.1 *is satisfied with continuous, increasing slowly varying function* $L(t)$. *Then with respect to* P_0 *we have*

(i) $$(1/\lambda)\sum_{k=0}^{L^{-1}(\lambda t)} f(X_k) \stackrel{\text{f.d.}}{\Longrightarrow} (\bar{f}/C)l(M^{-1}(t)) \qquad as\ \lambda\to\infty$$

where $l(M^{-1}(t))$ *is the same as in Theorem A and* $\bar{f}=\sum f(a_i)m_i$.

(ii) *If, in addition,* $\bar{f}=0$, *then*

$$\lambda^{-1/2}\sum_{k=0}^{L^{-1}(\lambda t)} f(X_k) \stackrel{\text{f.d.}}{\Longrightarrow} \sqrt{\langle f\rangle/C}\,\tilde{B}(l(M^{-1}(t))) \qquad as\ \lambda\to\infty$$

where $\tilde{B}(l(M^{-1}(t)))$ *is the same as in Theorem* 2.1 *and*

$$\langle f\rangle=E_0\Big[\Big(\sum_{k=0}^{V_1-1} f(X_k)\Big)^2\Big].$$

PROOF. In view of (4.2), our assertion is immediate from Theorem 3.2 if we show that $\varepsilon_n/\sqrt{n}$ converges to 0 in distribution as $n\to\infty$. Since we assumed that $f(x)$ has finite support, this fact is well known. However, it may also be proven easily from (3.4) as follows. If $f(x)$ is nonnegative, we see that ε_n is dominated by $S(n)-S^-(n)$. Thus we have that $\varepsilon_n/\sqrt{n}$ converges to 0 by (3.4). The general case can be proven in a similar way by considering the positive and the negative part of $f(x)$.

REMARK. The convergence of one-dimensional marginal distribution is a special case of Kesten's result. This is why we did not give much explanation to constants $\bar{f}$ and $\langle f\rangle$. See [**10**] for more information about these constants.

EXAMPLE. (2-dimensional random walk.) Let $\{X_n\}$ be the simplest random walk on $\boldsymbol{Z}^2$; $X_n=X_0+\eta_1+\cdots+\eta_n$, where η_n is a sequence of i.i.d. random vectors such that $P[\eta_1=(1,0)]=P[\eta_1=(-1,0)]=P[\eta_1=(0,1)]=P[\eta_1=(0,-1)]=1/4$. Since $P_{00}(2n)=4^{-2n}\binom{2n}{n}^2$, we have that $P_{00}(2n)\sim 1/(\pi n)$ as $n\to\infty$. Therefore, (ii) of Lemma 4.1 is satisfied with $L(x)=\log x$, $C=\pi$ (cf. Spitzer [**12**] page 167). Thus by Theorem 4.2, we obtain that

$$\lambda^{-1}\sum_{k=0}^{[e^{\lambda t}]} f(X_k) \stackrel{\text{f.d.}}{\Longrightarrow} (\bar{f}/\pi)l(M^{-1}(t)) \qquad \text{as } \lambda\to\infty,$$

where $\bar{f}=\sum_{x\in\boldsymbol{Z}^2} f(x)$. Here, we have used the following fact;

(4.5) $$E_x\Big[\sum_{k=0}^{V_1-1} f(X_k)\Big]=\sum_y\{\phi(x,y)-\phi(0,y)\}f(y)+\sum_y f(y)1_{\{0\}}(x)$$

where

(4.6) $$\phi(x,y)=\sum_{k=0}^{\infty}\{P_{x,y}(k)-P_{0,0}(k)\}$$

$$=(2\pi)^{-2}\iint_{[0,2\pi)^2}\frac{e^{\sqrt{-1}(\theta,x-y)}-1}{1-(\cos\theta_1+\cos\theta_2)/2}d\theta.$$

From Theorem 4.2 (ii) it also follows that if, in addition, $\bar{f}=0$, then

$$\lambda^{-1/2}\sum_{k=0}^{[e^{t\lambda}]}f(X_k)\overset{\text{f.d.}}{\Longrightarrow}\sqrt{\langle f\rangle/\pi}\,\tilde{B}(l(M^{-1}(t))),\qquad \text{as } \lambda\to\infty.$$

The constant $\langle f\rangle$ may be expressed as follows;

$$\langle f\rangle=2\sum_{x,y\in Z^2}f(x)\phi(x,y)f(y)-\sum_{x\in Z^2}f(x)^2. \tag{4.7}$$

To see this fact let $g(x)=E_x[\sum_{k=0}^{V_1-1}f(X_k)]$ $(=\sum_y\{\phi(x,y)-\phi(0,y)\}f(y))$. Then, by the strong Markov property we have that

$$\langle f\rangle=E_0\left[\left\{\sum_{k=0}^{V_1-1}f(X_k)\right\}^2\right]$$
$$=2E_0\left[\sum_{k=0}^{V_1-1}f(X_k)g(X_k)\right]-E_0\left[\sum_{k=0}^{V_1-1}f(X_k)^2\right]$$
$$=2\sum f(x)g(x)-\sum f(x)^2.$$

Thus we have (4.7) using $\bar{f}=0$.

References

[1] K. L. Chung, Contributions to the theory of Markov chains II, Trans. Amer. Math. Soc., **76** (1954), 397-419.

[2] D. A. Darling, The influence of the maximum term in the addition of independent random variables, ibid., **73** (1952), 95-107.

[3] R. L. Dobrusin, Two limit theorems for the simplest random walk on a line, Uspehi Mat. Nauk, **10** no 3 (1955), 139-146.

[4] W. Doeblin, Sur deux problèmes de M. Kolmogoroff concertant les chaines dénombrables, Bull. Soc. Math. France, **66** (1938), 210-220.

[5] W. Feller, An introduction to probability theory and its applications, Vol. 2, 2nd ed., Wiley, New York, 1971.

[6] Y. Kasahara, A limit theorem for slowly increasing occupation times, Ann. Probability, **10** (1982), 728-736.

[7] Y. Kasahara, Another limit theorem for slowly increasing occupation times, J. Math. Kyoto Univ., **24** (1984), 507-520.

[8] Y. Kasahara, Limit theorems for Lévy processes and Poisson point processes and their applications to Brownian excursions, Ibid., 521-538.

[9] Y. Kasahara, A note on sums and maxima of independent, identically distributed random variables, Proc. Japan Acad. Ser. A, **60** (1984), 353-356.

[10] H. Kesten, Occupation times for Markov and semi-Markov chains, Trans. Amer. Math. Soc., **103** (1962), 82-112.

[11] T. Lindvall, Weak convergence of probability measures and random functions in the function space $D[0,\infty)$, J. Appl. Probability, **10** (1973), 109-121.

[12] F. Spitzer, Principles of random walk, D. Van Nostrand, Princeton, 1964.
[13] S. I. Resnick, Inverses of extremal processes, Advances in Appl. Probability, **6** (1974), 392-406.
[14] S. Watanabe, A limit theorem for sums of non-negative i. i. d. random variables with slowly varying tail probabilities, Proc. 5th International Conf. Multivariate Anal., 1980, 249-261.

Yuji KASAHARA
Institute of Mathematics
University of Tsukuba
Sakura-mura, Ibaraki 305
Japan