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Steepest descent and differential equations

In memory of my teacher, H. S. Wall

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1. Introduction.

This note is a report on some phenomena suggested by numerical solution of boundary value problems. Two conditions are given. If both hold for a given problem then continuous constrained steepest descent converges to a solution.

Suppose that each of H, K and S is a real Hilbert space, $F: H\to K$, $B: H\to S$ and each of F and B has a locally Lipschitz derivative.

Consider the problem of *constructively* identifying $u\in H$ such that

$$F(u)=0, \qquad B(u)=0. \tag{1}$$

Many boundary value problems in differential equations — ordinary, partial, functional — can be cast as such problems where $F(u)=0$ represents a differential equation and $B(u)=0$ represents boundary conditions.

Denote by P the function on H so that if $x\in H$ then $P(x)$ is the orthogonal projection of H onto $N(B'(x))$, the nullspace of $B'(x)$. It is assumed throughout that P is locally Lipschitz.

Define ϕ on H so that if $x\in H$ then

$$\phi(x)=\|F(x)\|^2/2, \qquad x\in H$$

and denote by $\nabla_B\phi$ the function defined on H so that

$$(\nabla_B\phi)(x)=P(x)(\nabla\phi)(x), \qquad x\in H$$

where $\nabla\phi$ is the gradient function for ϕ. $\nabla_B\phi$ is called the B-gradient of ϕ. The following is intended to justify this terminology:

LEMMA 1. *Suppose $x\in H$ and α_x is the function with domain $N(B'(x))$ so that*

$$\alpha_x(k)=\phi(x+k), \qquad k\in N(B'(x)).$$

Then $(\nabla_B\phi)(x)=(\nabla\alpha_x)(0)$.

LEMMA 2. *If $x\in H$ there is a unique function z from $[0, \infty)$ to H so that*

$$z(0)=x, \qquad z'(t)=-(\nabla_B\phi)(z(t)), \quad t\geqq 0. \tag{2}$$

It will be seen that $-\nabla_B \phi$ generates a nonlinear semigroup which is not necessarily nonexpansive.

The following generalizes a result in [**20**] to a constrained case:

THEOREM 1. *Suppose that* $\Omega \subset H$ *and* $c>0$ *so that*

$$\|(\nabla_B \phi)(x)\| \geqq c\|F(x)\|, \qquad x \in \Omega. \tag{3}$$

Suppose also that $x \in \Omega$, z *satisfies* (2) *and*

$$R(z) \subset \Omega \tag{4}$$

where $R(z)$ *denotes the range of* z. *Then* $u = \lim_{t \to \infty} z(t)$ *exists and satisfies* $F(u)=0$, $B(u)=B(x)$.

Note that if $B(x)=0$, then u satisfies (1).

The following gives some insight into condition (3):

THEOREM 2. *Suppose* $\Omega \subset H$, $M>0$ *and*

(5) *for* $g \in K$, $\|g\|=1$ *and* $x \in \Omega$, *there is* $h \in N(B'(x))$ *such that* $F'(x)h=g$ *and* $\|h\| \leqq M$.

Then (3) *holds with* $c=1/M$.

COROLLARY. *Suppose* $\Omega \subset H$, $M>0$, $B \equiv 0$ *and if* $x \in \Omega$, *then* $F'(x)^{-1} \in L(K, H)$ *and* $|F'(x)^{-1}| \leqq M$. *Then* (3) *is satisfied with* $c=1/M$.

See ([**22**] p. 268) for a discussion of the conditions in Theorem 2 and the Corollary.

THEOREM 3. *Suppose* $\Omega \subset H$ *and*

$$c = \inf_{\substack{x \in \Omega \\ F(x) \neq 0}} \sup_{\substack{h \in N(B'(x)) \\ \|h\|=1}} \langle F'(x)h, F(x) \rangle / \|F(x)\| > 0.$$

Then (3) *is satisfied for this value of* c.

THEOREM 4. *Suppose that* Ω *is an open subset of* H, $w \in \Omega$, $F(w)=0$, $B(w)=0$ *and* (3) *holds. There is* $r>0$ *such that if* $\|x-w\|<r$ *and* z *satisfies* (2), *then* $R(z) \subset \Omega$.

If $B(x)=0$ for all $x \in H$ (and hence $P(x)=I$ for all $x \in H$) then problem (1) is called unconstrained. Even in unconstrained cases *linear* boundary conditions may be built into (1) by restricting the domain of F to a suitable translation of a linear subspace of H. Inclusion of $B \not\equiv 0$ gives the possibility of dealing with nonlinear boundary conditions. It may be that extensive computer runs using known techniques (modifications of [**17**], [**18**], for example) will reveal more of the nature of such problems.

The method of steepest descent goes back to Cauchy [**8**]. For additional background in methods related to steepest descent see [**5**] (particularly see 3.2

and 6.5), and the survey [**22**]. The paper [**26**] deals with ϕ strictly convex as does [**6**] under much less restrictive conditions. A sampling of other relevant references is [**15**], [**9**], [**11**], [**7**], [**24**], [**23**], [**10**], [**27**].

In Section 2 there is an example to make more concrete intended use of our results in differential equations. Section 3 contains a discussion of conditions (3) and (4) as well as a discussion of certain invariants. Section 4 contains proofs, Section 5 a further example. Section 6 discusses relationships between semigroups of operators and steepest descent.

2. An example.

A simple example illustrates some intended use of above results (cf. [**1**] as a general reference for Sobolev spaces).

EXAMPLE 1. Take $H=H^2([0, 1])$, $K=L^2([0, 1])$, $S=\boldsymbol{R}^2$. Suppose $p:\boldsymbol{R}\to\boldsymbol{R}$, $p\in C^{(1)}$ and that $R(p')$ is a bounded subset of $[0, \infty)$. Pick $f\in K$ and define

$$F(u)=-u''+p(u)-f\,, \qquad u\in H$$

and $B(u)=(u(0), u(1))$, $u\in H$. Hence $u\in H$ satisfies (1) if and only if

$$u(0)=0=u(1)\,, \qquad -u''+p(u)=f\,.$$

Note that $F'(u)h=-h''+p'(u)h$, $u\in H$, $h\in N(B)=N(B'(u))$. It is known that there is $M>0$ such that if $g\in K$, $u\in H$, then there is $h\in N(B)$ so that

$$-h''+p'(u)h=g \qquad \text{and} \qquad \|h\|\leqq M\|g\|\,.$$

Hence the hypothesis of Theorem 2 is satisfied for $\Omega=H$. Therefore (3) and (4) are satisfied with $\Omega=H$.

Using estimates from ([**2**], sect. 12) one may extend this example to problems of the form

$$-\Delta u+p(u)=f$$

with zero Dirichlet conditions on a bounded region.

It may be gathered upon examination of this example that if the expression $F(u)=0$ represents a system of differential equations and $\Omega\subset H$, then

$$F'(u)h=g\,, \qquad u\in\Omega\,, \quad g\in K\,, \quad \|g\|=1$$

represents a family of *linear* differential equations in $h\in N(B'(u))$. Thus the satisfaction of (3) rests upon a family of linear problems. Vast previous efforts on linear problems (cf. [**2**], [**12**], [**14**], [**28**], [**29**] to cite only a few possibilities) can be brought to bear on the problem of verifying (3) in important instances.

3. Observables and invariants.

In rather general circumstances it is possible to track numerically functions z satisfying (2). Some computational details are given in [**17**], [**16**], [**18**], [**21**], [**13**], [**4**], [**23**], [**11**] for example. Condition (3) was first noticed from extensive computations. Uniform bounds for approximations to $\|(\nabla_B\phi)(z(t))\|/\|F(z(t))\|$ and $\|z(t)\|$ were observed under increasingly fine discretizations and increasingly larger t.

It is suggested that (3) and (4) be taken as examples of computationally observable conditions. We do not attempt to define this concept. Rather, for an illustration it is suggested that the Palais-Smale condition (cf. [**22**]):

$$\{\phi(x_i)\}_{i=1}^{\infty} \text{ bounded, } \lim_{i\to\infty}(\nabla_B\phi)(x_i)=0 \Longrightarrow \{x_i\}_{i=1}^{\infty} \text{ has a convergent subsequence}$$

does not seem to meet this criterion.

Suppose for a moment that F is linear and B is identically 0. Denote by Q the orthogonal projection of H onto $N(F)$. Then if $x\in H$ and z satisfies (2),

$$Q(z(t))=Q(x), \qquad t\geqq 0$$

since $(Qz)'(t)=Q(z'(t))=-Q(F'(z(t))^*F(z(t)))=-Q(F^*Fz(t))=0$ inasmuch as $\overline{R(F^*)}=N(F)^{\perp}=\overline{R(Q)}^{\perp}=N(Q)$. Hence Q provides an invariant for steepest descent based upon F. An adequate generalization of this phenomenon to nonlinear cases would have far reaching consequences.

4. Proofs.

PROOF OF LEMMA 1. If $x,\ h\in H$, $\phi'(x)h=\langle F'(x)h,\ F(x)\rangle=\langle h,\ F'(x)^*F(x)\rangle$ and so $(\nabla\phi)(x)=F'(x)^*F(x)$.

Suppose $x\in H$. Then,

$$\begin{aligned}\alpha_x'(k)h&=\langle F'(x+k)h,\ F(x+k)\rangle\\&=\langle h,\ F'(x+k)^*F(x+k)\rangle\\&=\langle h,\ P(x)F'(x+k)^*F(x+k)\rangle, \qquad k,\ h\in N(B'(x)).\end{aligned}$$

In particular

$$\alpha_x'(0)h=\langle h,\ P(x)F'(x)^*F(x)\rangle, \qquad h\in N(B'(x))$$

and so $(\nabla\alpha_x)(0)=P(x)F'(x)^*F(x)=(\nabla_B\phi)(x)$ since $(\nabla_B\phi)(x)$ is the element in $N(B'(x))$ which represents the functional $(\nabla\alpha_x)(0)$.

See [**17**], [**18**], [**19**] for a discussion of calculating adjoints of linear transformations from a Sobolev space to an L_2 space.

LEMMA 3. *Suppose* $x\in H$ *and* z *satisfies* (2). *If* $0\leqq a<b$ *then*

$$\|z(b)-z(a)\| \leqq (b-a)^{1/2}\phi(z(a))^{1/2}.$$

PROOF OF LEMMA 3.

$$\|z(b)-z(a)\|^2 = \left\|\int_a^b z'\right\|^2 \leqq \left(\int_a^b \|z'\|\right)^2 \leqq (b-a)\int_a^b \|z'\|^2.$$

Note that if $t \geqq 0$, $\phi(z)'(t) = \langle(\nabla\phi)(z(t)),\ -P(z(t))(\nabla\phi)(z(t))\rangle = -\|P(z(t))(\nabla\phi)(z(t))\|^2 = -\|(\nabla_B\phi)(z(t))\|^2 = -\|z'(t)\|^2$ and so $\phi(z(a))-\phi(z(b)) = -\int_a^b \phi(z)' = \int_a^b \|z'\|^2$ and hence $\int_a^b \|z'\|^2 \leqq \phi(z(a))$ and therefore $\|z(b)-z(a)\| \leqq (b-a)^{1/2}\phi(z(a))^{1/2}$.

PROOF OF LEMMA 2. Since $\nabla_B\phi$ is a locally Lipschitz function it follows from basic ordinary differential equation theory that there is $\delta>0$ for which there is a unique solution y on $[0, \delta)$ to $y(0)=x$, $y'(t)=-(\nabla_B\phi)(y(t))$, $0\leqq t<\delta$. Suppose the conclusion to the lemma is false. Then the set of all δ above has a finite least upper bound α. Clearly there is a unique solution y to $y(0)=x$, $y'(t)=-(\nabla_B\phi)(y(t))$, $0\leqq t<\alpha$. But for such y, $r=\lim_{t\to\alpha} y(t)$ exists by virtue of Lemma 3. Hence there is $\delta_1>0$ and w uniquely on $[\alpha, \delta_1)$ such that $w(\alpha)=r$, $w'(t)=-(\nabla_B\phi)(w(t))$, $\delta\leqq t<\alpha+\delta_1$. Define z on $[0, \alpha+\delta_1)$ by

$$z(t)=\begin{cases} y(t), & 0\leqq t<\alpha \\ r, & t=\alpha \\ w(t), & \alpha<t<\alpha+\delta_1. \end{cases}$$

Then $z(0)=x$, $z'(t)=-(\nabla_B\phi)(z(t))$, $0\leqq t<\alpha+\delta_1$, and z is unique with this property. This is a contradiction and so the lemma is established. Proofs of Lemmas 2 and 3 are nearly identical to arguments in the unconstrained case given in [**5**], section 3.2 c.

PROOF OF THEOREM 1. Note first that if $t\geqq 0$, $B(z)'(t)=B'(z(t))z'(t)=0$ since $z'(t)=-P(z(t))(\nabla\phi)(z(t))\in N(B'(z(t)))$ inasmuch as $P(z(t))$ is the orthogonal projection of H onto $N(B'(z(t)))$. Hence $B(z)$ is constant at $B(x)=B(z(0))$.

Suppose now that $\phi(z(t_0))=0$ for some $t_0\geqq 0$. Since $R(\phi(z))\subset[0, \infty)$ and $\phi(z)$ is nondecreasing it follows $\phi(z(t))=0$, $t\geqq t_0$ and hence $0=\phi'(z(t))z'(t)=-\|(\nabla_B\phi)(z(t))\|^2$, $(\nabla_B\phi)(z(t))=0$, $z'(t)=0$, $t\geqq t_0$. Therefore z is constant on $[t_0, \infty)$ so $u=\lim_{t\to\infty} z(t)$ exists and satisfies $F(u)=0$, $B(u)=B(x)$.

On the other hand suppose $\phi(z(t))>0$ for all $t\geqq 0$. Then

$$\phi(z)'(t)=-\|(\nabla_B\phi)(z(t))\|^2\leqq -c^2\|F(z(t))\|^2=-2c^2\phi(z(t)),$$

and hence

$$\frac{\phi(z)'(t)}{\phi(z)(t)}\leqq -2c^2, \qquad t\geqq 0.$$

So

$$\ln(\phi(z(t))/\phi(z(0)))\leqq -2c^2t$$

and

$$\phi(z(t))\leqq\phi(z(0))e^{-2c^2t}, \qquad t\geqq 0.$$

Therefore $\lim_{t\to\infty}\phi(z(t))=0$.

From Lemma 3, if n is a positive integer,

$$\int_n^{n+1}\|z'\|\leqq\phi(z(n))^{1/2}\leqq\phi(x)^{1/2}e^{-c^2n}$$

and hence

$$\int_0^\infty\|z'\|=\sum_{n=0}^\infty\int_n^{n+1}\|z'\|\leqq\phi(x)^{1/2}\sum_{n=0}^\infty e^{-c^2n}=\phi(x)^{1/2}/(1-e^{-c^2}).$$

Therefore $\|z'\|\in L_1([0,\infty))$ and hence $u=\lim_{t\to\infty}z(t)$ exists.

Now $\phi(u)=0$ and hence $F(u)=0$ since $0=\lim_{t\to\infty}\phi(z(t))$. Since $B(z(t))=B(x)$, $t\geqq 0$, it follows that $B(u)=B(x)$.

PROOF OF THEOREM 4. Suppose $w\in\Omega$ and $F(w)=0$. Denote by s a positive number so that if $\|v-w\|<s$ then $v\in\Omega$. Denote by r a member of $(0, s)$ so that if $\|x-w\|\leqq r$ then $r+d\phi(x)^{1/2}<s$ where $d\equiv(1-e^{-c^2})^{-1}$. Suppose $\|x-w\|\leqq r$, z satisfies (2) and suppose there is $t>0$ such that $\|z(t)-w\|=s$. Denote by t_0 the least such number t. Then for $t\in[0, t_0)$, $z(t)\in\Omega$ and $\|z(t)-w\|\leqq\|x-w\|+\|z(t)-x\|$. As in the proof of Theorem 1, if $t\in[0, t_0)$, $\|z(t)-x\|\leqq\int_0^t\|z'\|\leqq d\phi(x)^{1/2}$ with d as above and so $\|z(t)-w\|\leqq r+d\phi(x)^{1/2}<s$. Therefore $\|z(t_0)-w\|\leqq r+d\phi(x)^{1/2}<s$, a contradiction. Hence $\|z(t)-w\|<s$ for all $t\geqq 0$, i.e., $R(z)\subset\Omega$ and the theorem is established.

PROOF OF THEOREM 2. Note that if $x, h\in H$ then $\langle(\nabla\phi)(x), h\rangle=\phi'(x)h=\langle F'(x)h, F(x)\rangle=\langle h, F'(x)^*F(x)\rangle$ so that $(\nabla\phi)(x)=F'(x)^*F(x)$. Hence if $x\in H$, $h\in N(B'(x))$, $\langle(\nabla_B\phi)(x), h\rangle=\langle P(x)(\nabla\phi)(x), h\rangle=\langle(\nabla\phi)(x), h\rangle=\langle F'(x)h, F(x)\rangle$. Suppose $x\in\Omega$ and $F(x)\neq 0$. Denote $\|F(x)\|^{-1}F(x)$ by g and denote by h_0 an element of $N(B'(x))$ so that $F'(x)h_0=g$ and $\|h_0\|\leqq M$. Then

$$\langle(\nabla_B\phi)(x), h_0\rangle=\langle F'(x)h_0, F(x)\rangle=\langle g, F(x)\rangle=\|F(x)\|.$$

Hence for $h_1=\|h_0\|^{-1}h_0$,

$$\begin{aligned}\|(\nabla_B\phi)(x)\|&=\sup_{\substack{h\in H\\ \|h\|=1}}\langle h, (\nabla_B\phi)(x)\rangle\geqq\langle h_1, (\nabla_B\phi)(x)\rangle\\ &=\langle h_0, (\nabla_B\phi)(x)\rangle/\|h_0\|=\|F(x)\|/\|h_0\|\geqq\|F(x)\|/M.\end{aligned}$$

This completes a proof of the theorem.

PROOF OF THEOREM 3. If $x\in H$ then

$$\|(\nabla_B\phi)(x)\|=\sup_{\substack{h\in H\\ \|h\|=1}}\langle h, (\nabla_B\phi)(x)\rangle$$

$$= \sup_{\substack{h \in H \\ \|h\|=1}} \langle h,\ P(x)F'(x)^* F(x) \rangle$$

$$= \sup_{\substack{h \in N(B'(x)) \\ \|h\|=1}} \langle F'(x)h,\ F(x) \rangle .$$

Hence if $y \in \Omega$ and $F(y) \neq 0$, $\|(\nabla_B \phi)(y)\| / \|F(y)\| \geqq \inf_{\substack{x \in \Omega \\ F(x) \neq 0}} \|(\nabla_B \phi)(x)\| / \|F(x)\| \geqq c$, using the hypothesis of the theorem.

5. A further example.

EXAMPLE 2. Suppose that m is a positive integer, $a, b \in \boldsymbol{R}$, $a < b$ and G is a $C^{(1)}$ function from $\boldsymbol{R}^m$ to $\boldsymbol{R}^m$. Take H to be the H^1 Sobolev space of $\boldsymbol{R}^m$ valued functions on $[a, b]$ and take K to be the corresponding L_2 space and take $S = \boldsymbol{R}^m$. Pick $r \in \boldsymbol{R}^m$ and consider the problem of finding $u \in H$ such that

$$u(a) = r , \qquad u' = G(u) .$$

Define $F: H \to K$ by $F(u) = u' - G(u)$, $u \in H$. Define $B: H \to S$ by $B(u) = u(a) - r$, $u \in H$. Suppose Ω is a bounded subset of H. Then the closure of Ω in $C([a, b])$ is compact. Note that $F'(u)h = h' - G'(u)h$, $u \in H$, $h \in N(B'(u))$. Now Ω regarded as a subset of $C([a, b])$ is bounded and equicontinuous and hence $\{G'(u) : u \in \Omega\}$ has the same property. From elementary considerations of ordinary differential equations one has that (5) holds. Thus existence of a solution $u \in H$ of $u' = G(u)$, $u(a) = r$ is equivalent to existence of $x \in H$, $x(a) = r$ so that z satisfying (2) has bounded range.

To see that $R(z)$ need not be bounded consider the following: Take $[a, b] = [0, 4]$, $H = H^1([0, 4])$, $S = \boldsymbol{R}$, $F(u) = u' - (1 + u^2)$, $B(u) = u(0)$, $u \in H$. Then (3) holds for any bounded region Ω of H. But there is no solution $u \in H$ to $F(u) = 0$, $B(u) = 0$. Hence (4) is violated for all $x \in H$, (with $x(a) = r$) so that the corresponding z satisfies (2).

6. Semigroups.

Using Lemma 2, define a function T on $[0, \infty)$ such that if $r \geqq 0$ then $T(r)$ is the transformation from H to H so that if $x \in H$ then

$$T(r)x = z(r) \qquad \text{where } z \text{ satisfies (2).}$$

The existence and uniqueness given by Lemma 2 assures one that

$$T(0) = I$$

$$T(t)T(s) = T(t+s) , \qquad t,\ s \geqq 0 ,$$

i.e., that T gives a semigroup of (usually nonlinear) transformations on H.

It is known that if ϕ is convex (cf. [6]) then T is nonexpansive. In this case there is a theory of asymptotic convergence (cf. [25], [3]). In [26] there is the condition that $\phi \in C^{(2)}$ and for some $\varepsilon > 0$, $\phi''(x)(h, h) \geqq \varepsilon \|h\|^2$, $x, h \in H$. Under this condition ϕ is convex and, for any $x \in H$, $u = \lim_{t\to\infty} T(t)x$ exists and satisfies $(\nabla\phi)(u) = 0$.

For T nonexpansive it is known that the set of all fixed points of T is convex (for $\nabla\phi$ Lipschitz, x is a fixed point of T if and only if $(\nabla\phi)(x) = 0$).

Semigroups T derived from the general construction of this note need not be nonexpansive. To see this, take $H = H^1([0, 1])$. $F(u) = u' + u^2$, $B(u) = 0$, $u \in H$. If T derived from this choice of F and B were nonexpansive, then the set $FP \equiv \{x \in H : T(t)x = x,\ t \geqq 0\}$ would be convex. But also $FP = \{x \in H : (\nabla_B \phi)(x) = 0\} = F^{-1}(0)$. Now $F^{-1}(0) = \{f \in H : \exists c > -1$ such that $f(t) = c/(1+ct),\ t \in [0, 1]\}$, clearly not a convex set.

Finally it is mentioned that a semigroup T from our general construction need not be extendable to a group. For an example start with $F(x) = e^{-x}$, $B(x) = 0$, $x \in \boldsymbol{R}$.

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