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On an invariant of homology lens spaces

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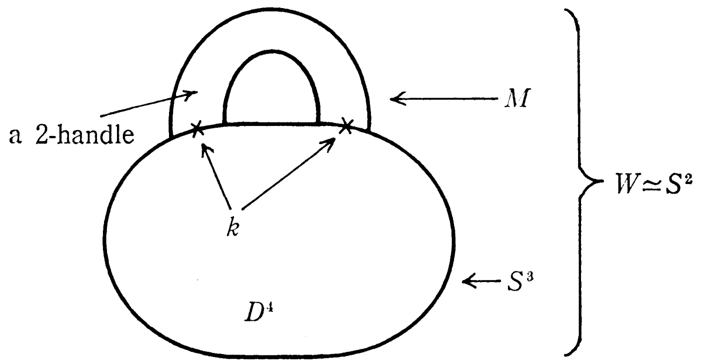
§ 1. Introduction.

In this paper we study problems concerning the following two questions:

QUESTION 1. Which 3-manifold (or lens space) can be obtained from S^3 by integer surgery on a knot?

QUESTION 2. Which 3-manifold (or lens space) bounds a compact 4-manifold that is a homology S^2?

These two questions are related each other. In fact, if a 3-manifold M is obtained from S^3 by integer surgery on some knot k, then M bounds a compact 4-manifold W which is homotopy equivalent to S^2.



Concerning Question 2, Fintushel-Stern [**F-S**] observed that a lens space $L(p, q)$ bounds a homology S^2 only if q is a quadratic residue mod p. They and N. Maruyama showed that, in some special but non-trivial cases of p and q, $L(p, q)$ is obtained from S^3 by integer surgery on a knot. To the best of our knowledge, these are all known results for Questions 1 and 2.

We study the above questions for the 3-manifolds which have the same homology type as $L(p, 1)$. In § 5 a certain new invariant is defined for some class of 3-manifolds and it is proved that vanishing of the invariant is necessary for such manifold to bound a homology S^2. In § 6 we calculate the invariant

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for lens spaces and show that there are infinitely many lens spaces $L(p, q)$ which cannot bound any homology S^2, though each q is a quadratic residue mod p.

In § 7, we shall show that $L(p, q)$ and $L(p, 1)$ bound simply connected 4-manifolds which are homotopy equivalent relative to boundaries when q is a quadratic residue mod p. As a corollary we obtain many examples of compact simply connected 4-manifolds which are homotopy equivalent relative to boundaries but not homeomorphic.

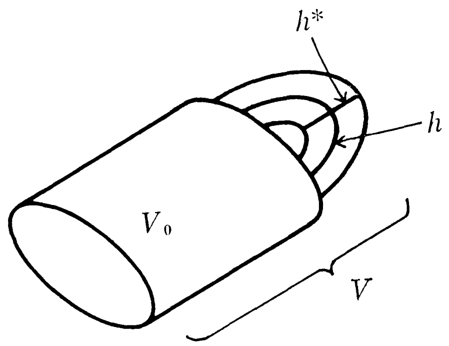
REMARK. The term 'integer surgery' means Dehn surgery with an integral coefficient. But, in the following sections, we use the term 'surgery' instead of 'integer surgery' for brevity.

§ 2. Notations and conventions.

Throughout this paper, we use the following notations and conventions.

A 3-manifold always means a closed, connected and oriented 3-manifold while a 4-manifold is always assumed to be compact, connected and oriented. When W is a 4-manifold and $x \in H_2(W, \partial W)$, $y \in H_2(W)$, then (x, y) means the intersection number of x and y. For 2-chains c_1 and c_2, we also use (c_1, c_2) as their intersection number. Let M be a 3-manifold and x, y be torsion elements of $H_1(M)$, $lk(x, y)$ means the linking number of x and y. For a 1-chain c_1 and a 2-chain c_2 in M, their intersection number is denoted by $\langle c_1, c_2 \rangle$.

W is called a 2-handle body if W is constructed from D^4 by attaching a finite number of 2-handles. Suppose that a 2-handle h is attached to a 4-manifold V_0 on a component of ∂V_0. Let V denote the resulting manifold.



Then we abuse the notation h for the core of h. Also, by ∂h, we denote the boundary of the core of h. Let h^* denote the core of the dual handle for h and ∂h^* denote its boundary. Let M be a component of ∂V such that $M \supset \partial h^*$. Then h^* can be regarded as a relative cycle of (V, M). By $[h^*]$ we denote an element of $H_2(V, M)$ represented by it. Also we denote by $[h]$ the element of $H_2(V, V_0)$ represented by h. When there is a natural isomorphism $j: H_2(V) \rightarrow$

$H_2(V, V_0)$, for example, in the case of $V_0=D^4$, we abuse the notation $[h]$ for $j^{-1}([h])$.

Let $\boldsymbol{C}P(2)$ and $\overline{\boldsymbol{C}P}(2)$ denote complex projective planes. We assume that the intersection form on $H_2(\boldsymbol{C}P(2))$ is (1) while the intersection form on $H_2(\overline{\boldsymbol{C}P}(2))$ is (-1). Let $\deg(f)$ denote a degree of a map f.

Let $\boldsymbol{N}$ denote the natural numbers $\{0, 1, 2, \cdots\}$ and $\boldsymbol{Z}$ denote the integers. Let E_n be a unit matrix of degree n.

We work in the smooth category. The notation $\approx$ stands for 'diffeomorphic to (preserving orientations, if necessary)'.

§ 3. Algebraic lemmas.

In this section we present three algebraic lemmas.

Let $A=(a_{ij})$ be a symmetric integer matrix of degree n. By T_A we denote the linear transformation from $\boldsymbol{Z}^n$ to itself associated with A. Further, since A is symmetric, we can define a symmetric bilinear form $S_A: \boldsymbol{Z}^n \times \boldsymbol{Z}^n \to \boldsymbol{Z}$ by $S_A(e_i, e_j)=a_{ij}$ where $e_i=(0, \cdots, \overset{\overset{i}{\vee}}{1}, \cdots, 0)$. Suppose that A is non-degenerate, that is, $\det(A) \neq 0$. Then $\mathrm{Cok}(T_A)=\boldsymbol{Z}^n/T_A(\boldsymbol{Z}^n)$ is a torsion group. Let π denote the canonical epimorphism from $\boldsymbol{Z}^n$ to $\mathrm{Cok}(T_A)$. A symmetric bilinear form $L_{A^{-1}}: \mathrm{Cok}(T_A) \times \mathrm{Cok}(T_A) \to \boldsymbol{Q}/\boldsymbol{Z}$ is defined by $L_{A^{-1}}(\pi(e_i), \pi(e_j))=\tilde{a}_{ij}$, where $(\tilde{a}_{ij})$ means the inverse matrix of A, and $\tilde{a}_{ij}$ is regarded as an element of $\boldsymbol{Q}/\boldsymbol{Z}$.

We need the following lemma which shows that the linking form of a 3-manifold is derived from the intersection form of its bounding 4-manifold. (Refer to Wall [**W**, p. 286].)

LEMMA 1. *Let W be a 4-manifold and $M=\partial W$ be a connected 3-manifold. Suppose that $H_2(W)$ is free, $H_1(W)=0$ and $H_1(M)$ is a torsion group. Then, if the intersection form on $H_2(W)$ is represented by S_A for some matrix A, then the linking form on $H_1(M)$ is represented by $L_{A^{-1}}$.*

PROOF. Since $H_1(M)$ is a torsion group, $H_2(M)=0$ follows from Poincaré duality theorem and the universal coefficient theorem. Thus we obtain the following exact sequence:

$$0 \longrightarrow H_2(W) \xrightarrow{h} H_2(W, M) \xrightarrow{\partial} H_1(M) \longrightarrow 0 .$$

Let $e_1, \cdots, e_n$ be a basis for $H_2(W)$. Then, by Poincaré duality theorem, there is a basis $e_1^*, \cdots, e_n^*$ for $H_2(W, M)$ such that $(e_i^*, e_j)=\delta_{ij}$. We determine a matrix $A=(a_{ij})$ by the expression

$$h(e_i)=\sum_{j=1}^{n} a_{ij} e_j^* .$$

Since

$$(e_i, e_j)=(h(e_i), e_j)=\Big(\sum_{k=1}^{n} a_{ik}e_k^*, e_j\Big)=a_{ij},$$

the intersection form on $H_2(W)$ is represented by S_A.

Next we show that the linking form on $H_1(M)$ is represented by $L_{A^{-1}}$. For e_i^*, e_j^*, there are relative chains c_1, c_2 which represent e_i^*, e_j^*. Since $\partial e_j^*=[\partial c_2]$ is a torsion element, for some integer m, $m(\partial c_2)$ is a boundary of some chain c in M. Then $m(c_2)\cup(-c)$ is a cycle in W and represents an element, say z, of $H_2(W)$. Note that $h(z)=m[c_2]=me_j^*$. On the other hand,

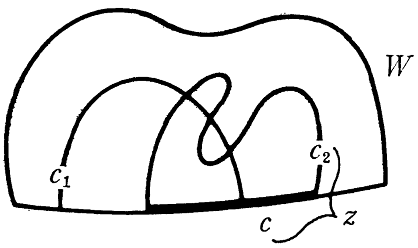
$$h\Big(\sum_{k=1}^{n} \tilde{a}_{jk}e_k\Big)=\sum_{k,l=1}^{n} \tilde{a}_{jk}a_{kl}e_l^*=e_j^*$$

holds (where we regard coefficients of the homology groups as rational numbers). Thus we obtain

$$z=m\Big(\sum_{k=1}^{n} \tilde{a}_{jk}e_k\Big).$$

Now, since $(c_1, z)=m(c_1, c_2)-\langle\partial c_1, c\rangle$,

$$(1/m)(c_1, z)\equiv-(1/m)\langle\partial c_1, c\rangle \mod \boldsymbol{Z}.$$



But, by definition of the linking number,

$$-(1/m)\langle\partial c_1, c\rangle=lk([\partial c_1], [\partial c_2])=lk(\partial e_i^*, \partial e_j^*).$$

(Here we adopt the convention that the sign of linking number is compatible with the equation $-(1/m)\langle\partial c_1, c\rangle=lk([\partial c_1], [\partial c_2])$.)

From $z=m\Big(\sum_{k=1}^{n} \tilde{a}_{jk}e_k\Big)$, we obtain

$$(1/m)(c_1, z)=\sum_{k=1}^{n} \tilde{a}_{jk}(e_i^*, e_k)=\tilde{a}_{ji}=\tilde{a}_{ij}.$$

From these we obtain

$$lk(\partial e_i^*, \partial e_j^*)=\tilde{a}_{ij}.$$

This completes the proof.

Let A, B be symmetric integer matrices. We denote $A\sim B$ when there is a unimodular matrix P such that ${}^tPAP=B$. Hereafter we do not distinguish matrices and bilinear forms associated with them. For torsion group T_i $(i=1, 2)$, two bilinear forms $L_i: T_i \times T_i \to \boldsymbol{Q}/\boldsymbol{Z}$ are called isomorphic and denoted by $L_1 \sim L_2$ when there is an isomorphism $\phi: T_1 \to T_2$ such that $L_2(\phi(x), \phi(y)) = L_1(x, y)$ for any $x, y \in T_1$.

We shall use the following important bilinear forms $(1/p)$ and (p). The bilinear form $(1/p): \boldsymbol{Z}_p \times \boldsymbol{Z}_p \to \boldsymbol{Q}/\boldsymbol{Z}$ is defined by $(1/p)(1, 1) = 1/p$. The bilinear form $(p): \boldsymbol{Z} \times \boldsymbol{Z} \to \boldsymbol{Z}$ is defined by $(p)(1, 1) = p$.

The following was first proved by Kneser-Puppe [**K-P**] and then more generalized version was obtained by Durfee [**D**].

LEMMA 2. *Let A, B be symmetric integer matrices such that $L_{A^{-1}} \sim L_{B^{-1}}$. Then there are $r, s, r', s' \in \boldsymbol{N}$ such that $A \oplus E_r \oplus (-E_s) \sim B \oplus E_{r'} \oplus (-E_{s'})$.*

The following lemma is concerned with an automorphism of the bilinear form $(p) \oplus E_l \oplus (-E_m)$.

LEMMA 3. *Let $\varepsilon = 1$ or -1. Suppose that $p, s \in \boldsymbol{Z}$ $(p > 0)$ satisfy $s^2 \equiv \varepsilon\ (p)$. Then there are $m \in \boldsymbol{N}$ and a unimodular matrix Q which satisfy the following:*

(1) ${}^tQ((p) \oplus (-E_m) \oplus E_m \oplus (-1))Q = (\varepsilon p) \oplus (-E_m) \oplus E_m \oplus (-\varepsilon)$,

(2) *let $Q = (q_{ij})$ and $Q^{-1} = (\tilde{q}_{ij})$, then $q_{11} = s$ and $q_{11}\tilde{q}_{11} \equiv 1\ (p)$ hold.*

PROOF. There is $n \in \boldsymbol{N}$ such that $s^2 - \varepsilon = np$. Let $C = (p) \oplus (-E_n)$ and

$$Q_0 = \left(\begin{array}{c|c} s & \overbrace{1 \cdots\cdots\cdots 1}^{n \text{ times}} \\ \hline p & s \quad s-1 \cdots\cdots s-1 \\ \vdots & \begin{matrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ \vdots & & & \ddots \\ \vdots & 0 & & \end{matrix} \\ p & 0 \qquad\qquad\quad 1 \end{array}\right) = \left(\begin{array}{c|c} s & I \\ \hline R & T \end{array}\right).$$

By easy computation, it can be shown that Q_0 is a unimodular matrix. Consider the following equality

$${}^tQ_0CQ_0 = \left(\begin{array}{c|c} s & {}^tR \\ \hline {}^tI & {}^tT \end{array}\right)\left(\begin{array}{c|c} p & 0 \\ \hline 0 & -E_n \end{array}\right)\left(\begin{array}{c|c} s & I \\ \hline R & T \end{array}\right) = \left(\begin{array}{c|c} ps^2 - {}^tRR & spI - {}^tRT \\ \hline sp{}^tI - {}^tTR & p{}^tII - {}^tTT \end{array}\right).$$

Since $ps^2 - {}^tRR = ps^2 - np^2 = \varepsilon p$ and $spI - {}^tRT = 0$,

$${}^tQ_0CQ_0 = \left(\begin{array}{c|c} \varepsilon p & 0 \\ \hline 0 & D \end{array}\right)$$

where D is a certain symmetric unimodular matrix. As is well known (See, for example, Milnor-Husemoller [**M-H**], Theorem 4.3, Corollary 4.4, pp. 22-23) any integral non-singular quadratic form is stably equivalent. Thus there is $m \in \boldsymbol{N}$ and a unimodular matrix Q_1 such that

$$ {}^tQ_1(D \oplus (-E_{m-n}) \oplus E_m \oplus (-1))Q_1 = (-E_m) \oplus E_m \oplus (-\varepsilon). $$

Let $Q=(Q_0 \oplus E_{2m-n+1})((1) \oplus Q_1)$. Then Q satisfies the condition (2) and the following holds:

$$ ((1) \oplus {}^tQ_1)({}^tQ_0 \oplus E_{2m-n+1})((p) \oplus (-E_m) \oplus E_m(-1))(Q_0 \oplus E_{2m-n+1})((1) \oplus Q_1) $$
$$ =(\varepsilon p) \oplus (-E_m) \oplus E_m \oplus (-\varepsilon). $$

This completes the proof.

§4. A homology $L(p, 1)$.

From the homological view point, one of the most simple classes of 3-manifolds is the following homology $L(p, 1)$. We shall study this class in the succeeding sections.

DEFINITION 1. A 3-manifold is called a homology $L(p, 1)$ when its linking form is isomorphic to the linking form of lens space $L(p, 1)$ for a positive integer p.

This definition means that a homology $L(p, 1)$ has $\boldsymbol{Z}_p$ as the 1-dimensional homology group and, for some generator x of it, $lk(x, x) = \pm 1/p$.

REMARK 1. If a 3-manifold M bounds a 4-manifold W with $H_*(W)=H_*(S^2)$, then the intersection form on $H_2(W)$ is isomorphic to (p) for some $p \in \boldsymbol{Z}$. Hence, by Lemma 1, the linking form on $H_1(M)$ is isomorphic to $(1/p)$. This means that M is a homology $L(p, 1)$.

REMARK 2. A lens space $L(p, q)$ is a homology $L(p, 1)$ if and only if $\pm q$ is a quadratic residue mod p. This is equivalent to that $L(p, q)$ is homotopy equivalent to $L(p, 1)$.

We show certain characterization of homology $L(p, 1)$.

THEOREM 1. *A 3-manifold M is a homology $L(p, 1)$ if and only if M is obtained from a homology sphere by surgery on a knot with p-framing.*

To prove the theorem, we need the following lemma. Our lemma is some special case of Smale's theory (Refer to Milnor [**Mil**], Theorem 7.6, pp. 92-93).

LEMMA 4. *Suppose that W be a 2-handle body with $W=D^4 \cup h_1 \cup \cdots \cup h_n$ and the intersection form on $H_2(W)$ is represented by a matrix A with respect to the basis $[h_1], \cdots, [h_n]$. Let Q be a unimodular matrix and $B={}^tQAQ$. Then we can find the other handle decomposition $D^4 \cup \bar{h}_1 \cup \cdots \cup \bar{h}_n$ of W such that the* *intersection form is represented by* B *with respect to the basis* $[\bar{h}_1], \cdots, [\bar{h}_n]$.

PROOF OF LEMMA 4. Since, for the case of $n=1$, the statement is obviously true, we assume $n \geqq 2$. As is well known, a unimodular matrix Q is a product of the following Q_1, Q_2, Q_3 and their inverses:

$$Q_1=\left(\begin{array}{c|ccc} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ \hline 1 & 0 & \cdots & 0 \end{array}\right), \quad Q_2=\left(\begin{array}{cc|ccc} 1 & 0 & & 0 & \\ 1 & 1 & & & \\ \hline & & 1 & & 0 \\ & 0 & & \ddots & \\ & & 0 & & 1 \end{array}\right),$$

$$Q_3=\left(\begin{array}{cc|ccc} -1 & 0 & & 0 & \\ 0 & 1 & & & \\ \hline & & 1 & & 0 \\ & 0 & & \ddots & \\ & & 0 & & 1 \end{array}\right).$$

Whenever Q_1, Q_2 and Q_3 appear in the product, by

(1) renumbering the indices of $h_1, \cdots, h_n$,

(2) giving the opposite orientation to some h_i, or

(3) adding a handle h_i to another handle h_j,

we obtain the new handles $\bar{h}_1, \cdots, \bar{h}_n$ satisfying the required condition.

PROOF OF THEOREM 1. Suppose that M is a homology $L(p, 1)$. Let W be a 2-handle body with $\partial W=M$ and A be the intersection form on $H_2(W)$. Then, by Lemmas 1 and 2, we obtain $A \oplus E_r \oplus (-E_s) \sim (p) \oplus E_{r'} \oplus (-E_{s'})$ for some $r, s, r', s' \in \boldsymbol{N}$. Let $W'=W \# r\boldsymbol{C}P(2) \# s\overline{\boldsymbol{C}P}(2)$. Then W' is also a 2-handle body with $\partial W'=M$ and the intersection form on $H_2(W')$ is represented by $A \oplus E_r \oplus (-E_s)$. Applying Lemma 4, we obtain the handle decomposition $D^4 \cup \bar{h}_1 \cup \cdots \cup \bar{h}_m$ of W' such that the intersection form on $H_2(W')$ is represented by $(p) \oplus E_{r'} \oplus (-E_{s'})$ with respect to the basis $[\bar{h}_1], \cdots, [\bar{h}_m]$. Let $V=D^4 \cup \bar{h}_2 \cup \cdots \cup \bar{h}_m$. Then V satisfies the following conditions:

(1) V is a 2-handle body with the intersection form isomorphic to $E_{r'} \oplus (-E_{s'})$,

(2) W' is obtained from V by attaching the handle $\bar{h}_1$.

By (2), we know that $\partial W'=M$ is obtained from ∂V by surgery on a knot. By (1), we know that ∂V is a homology sphere. Thus we can conclude that M is obtained from a homology sphere by surgery on a knot. Other parts of the theorem is obvious and we omit to prove them.

§ 5. Some new invariant.

The μ-invariant plays a crucial role in low dimensional topology. But it is not defined to a manifold which is not a $\boldsymbol{Z}_2$-homology sphere. We try to define some invariant which can be regarded as an extended μ-invariant for a certain class of homology $L(p, 1)$.

DEFINITION 2. A positive integer p is called having property (*) if p satisfies the following two conditions:

$$(*)\qquad \begin{cases} (1) & p \text{ is even and } p \geqq 4, \\ (2) & \text{if } s^2 \equiv \pm 1\,(p) \text{ for } s \in \boldsymbol{Z}, \text{ then } s \equiv \pm 1\,(p). \end{cases}$$

These conditions can be restated as follows.

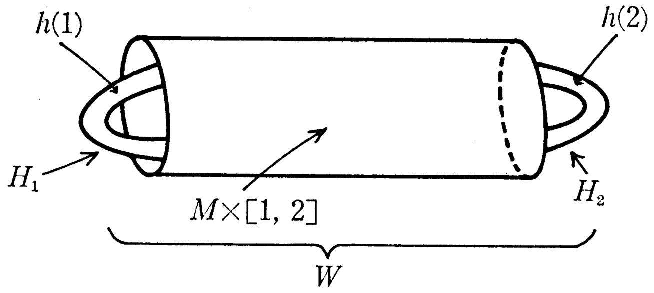
REMARK 3. N. Maruyama showed that, for $p>4$, p has the property (*) if and only if p has the form $p=2q^n$ where q is a prime number such that $q\equiv 3\,(4)$ and n is a positive integer. For the proof of Theorem 4, we need only the fact that if t is a prime number such that $t\equiv 3\,(4)$ then $2t$ has the property (*). Here we demonstrate it.

Suppose that t is a prime number such that $t\equiv 3\,(4)$ and s satisfies $s^2\equiv\pm 1\,(2t)$. By well known property of Legendre's symbol, we obtain $\left(\frac{-1}{t}\right)=(-1)^{(t-1)/2}=-1$. This means there is no such s that $s^2\equiv -1\,(t)$. Next consider the equation $s^2\equiv 1\,(2t)$. Since t is an odd prime and s^2-1 is a degree 2 polynomial, $s^2-1\equiv 0\,(t)$ has just two solutions, that is, $s\equiv 1\,(t)$ and $s\equiv -1\,(t)$. Hence, if s satisfies $s^2\equiv 1\,(2t)$, $s\equiv\pm 1$ or $s\equiv\pm 1+t$ should hold. But, since $(\pm 1+t)^2\not\equiv 1\,(2t)$, the equation $s^2\equiv 1\,(2t)$ has just two solutions $s\equiv 1\,(2t)$ and $s\equiv -1\,(2t)$. This means that, if t is a prime number such that $t\equiv 3\,(4)$, then $2t$ has the property (*).

One of our main theorems is:

THEOREM 2. *Suppose that M is a homology $L(p, 1)$ and p has the property (*). Suppose that M is obtained in two ways from homology spheres H_1 and H_2 by surgery on knots. Then $\mu(H_1)=\mu(H_2)$ holds.*

PROOF. Let W be a 4-dimensional cobordism constructed from $M\times[1, 2]$ by attaching 2-handles $h(i)$ $(i=1, 2)$ on $M\times\{i\}$ to result H_1 and H_2 as its boundary components.



As is mentioned in § 2, we abuse the notation $h(i)$ for the core of the 2-handle $h(i)$. Then $[\partial h(i)]$ is regarded as an element of $H_1(M\times\{i\})$. Let

$$j_i : H_1(M\times\{i\}) \longrightarrow H_1(M\times[1, 2]) \qquad (i=1, 2)$$

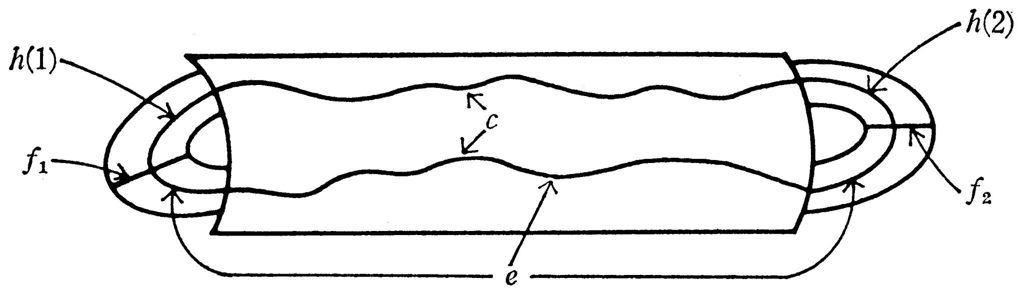
be homomorphisms induced from inclusion maps. Then

$$j_1[\partial h(1)]=sj_2[\partial h(2)]$$

holds for some $s\in \boldsymbol{Z}$. Since $lk([\partial h(i)], [\partial h(i)])=\pm 1/p$, we obtain $s^2/p=\pm 1/p$. This means $s^2\equiv\pm 1\,(p)$. But, since p has the property $(*)$, $s\equiv\pm 1\,(p)$ holds. Hence we obtain

$$j_1[\partial h(1)]=\pm j_2[\partial h(2)]\,.$$

This means that there is a 2-chain c in $M\times[1, 2]$ such that $\partial c=\partial h(1)\cup\pm\partial h(2)$. Let e be an element of $H_2(W)$ which is represented by $h(1)\cup c\cup\pm h(2)$. Let f_i $(i=1, 2)$ denote elements of $H_2(W)$ represented by the cores of dual 2-handles with respect to $h(i)$.

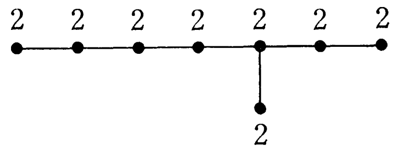


Here we calculate the intersection form on $H_2(W)$ using e and f_i. First we obtain $(e, f_i)=1$ for the appropriate orientations of f_i, and $(f_i, f_i)=\pm p$, $(f_1, f_2)=0$. Next (e, e) is calculated as follows. Since $pe=[ph(1)\cup pc\cup ph(2)]$ $=f_1\pm f_2$, $(e, e)=(1/p^2)(f_1\pm f_2, f_1\pm f_2)=(1/p^2)(\pm p\pm p)$ holds. But, since $p\neq 2$ and $(1/p^2)(\pm p\pm p)$ should be an integer, it is necessary that $(e, e)=0$. Thus the intersection form is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix}$ with respect to the basis e, f_1. Since p is even, this matrix is of even type. Thus we have proved that

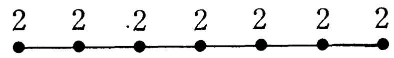
W is a spin manifold and its signature is zero. This means that $\mu(H_1)=\mu(H_2)$.

We present two examples. From these we know the condition that p has the property $(*)$ is essential in Theorem 2.

EXAMPLE 1. Let $P(E_8)$ and $P(A_7)$ denote plumbing 4-manifolds associated with the following diagrams E_8 and A_7 respectively:



E_8,

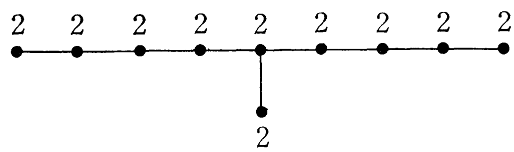


A_7.

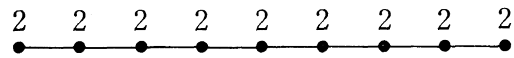
Then $\partial P(E_8)$ is the famous Poincaré homology sphere and $\mu(\partial P(E_8))=1$. Since $P(E_8)$ is obtained from $P(A_7)$ by attaching a 2-handle, we can regard that $\partial P(A_7)$ is obtained from $\partial P(E_8)$ by surgery on a knot. On the other hand, since $\partial P(A_7)\approx L(8, 1)$, it is obtained from S^3 by surgery. Thus $L(8, 1)$ is obtained both from $\partial P(E_8)$ and S^3, but $\mu(\partial P(E_8))\neq\mu(S^3)$.

In this case of $p=8$, p has not the property $(*)$. In fact, for $s=3$, $s^2\equiv 1\ (p)$ holds.

EXAMPLE 2. Like example 1, consider $P(E_{10})$ and $P(A_9)$.



E_{10},



A_9.

Then it is observed that $L(10, 1)$ is obtained both from $\partial P(E_{10})$ and S^3, but $\partial P(E_{10})$ is a homology sphere and $\mu(\partial P(E_{10}))\neq\mu(S^3)$. In this case of $p=10$, for $s=3$, $s^2\equiv -1\ (p)$ holds.

As an application of Theorems 1 and 2, we obtain the following:

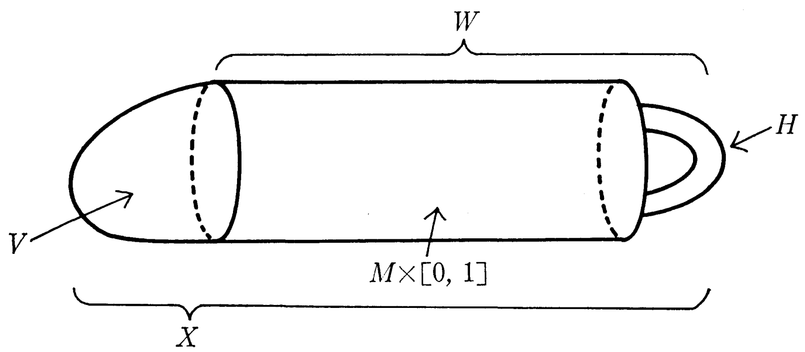
DEFINITION 3. Let M be a homology $L(p, 1)$ where p has the property $(*)$. Then, by Theorem 1, there is a homology sphere H from which M is obtained by surgery on a knot. Define $\bar{\mu}(M)=\mu(H)$. Then, by Theorem 2, $\bar{\mu}(\ )$ is well defined and has a value in $\boldsymbol{Z}_2=\{0, 1\}$.

§ 6. Bounding a homology S^2.

By a homology S^2, we denote a compact 4-manifold with the same homology group as S^2. As for a homology $L(p, 1)$ for p having the property $(*)$, $\tilde{\mu}$-invariant gives us a tool to study whether this manifold bounds a homology S^2 or not.

THEOREM 3. *Suppose that M is a homology $L(p, 1)$ and p has the property $(*)$. Then M bounds a homology S^2 only if $\tilde{\mu}(M)=0$.*

PROOF. This theorem is proved by the argument similar to that of Theorem 2. Suppose that M is obtained from a homology sphere H by surgery on a knot. Let W be a cobordism between M and H constructed from $M\times[0, 1]$ by attaching a 2-handle on $M\times\{1\}$. Let V be a homology S^2 which M bounds.



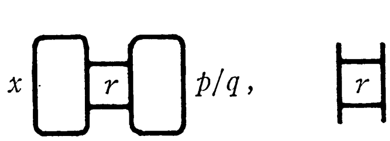
Let X be a 4-manifold constructed from W and V by gluing along ∂V and $M\times\{0\}\subset W$. Then the argument similar to the proof of Theorem 2 is valid and we can show that X is a spin manifold and its signature is zero. Hence

$$\tilde{\mu}(M)=\mu(H)=0 .$$

Next we show that there are many lens spaces having non-vanishing $\tilde{\mu}$-invariant. Our calculation of $\tilde{\mu}$ depends on the following lemma.

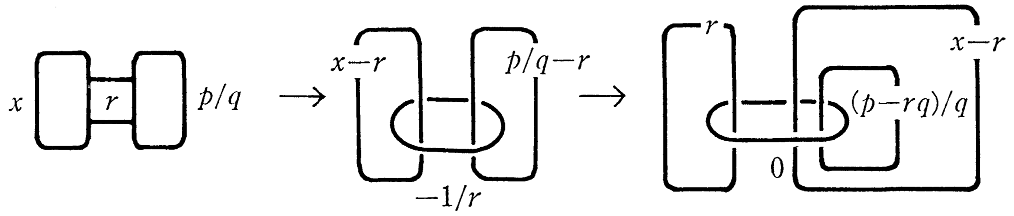
LEMMA 5. *Suppose that $L(p, q)$ is a lens space and $px-r^2q=\pm 1$ holds for some $x, r\in\boldsymbol{Z}$. Then $L(p, q)$ is obtained from Brieskorn homology sphere $\Sigma(|r|, |x-r|, |p-rq|)$ by surgery on a knot.*

PROOF. Consider the following framed link picture:



stands for r times full twist.

The condition $px-r^2q=\pm 1$ implies that the 3-manifold associated with this link is a homology sphere. We apply Kirby-Rolfsen calculus as follows:



The last picture shows that our homology sphere is just Brieskorn homology sphere $\Sigma(|r|, |x-r|, |p-rq|)$ (Refer to Matsumoto [**Mat**]). The first picture shows that $L(p, q)$ is obtained from this homology sphere by surgery on a knot.

THEOREM 4. *There are infinitely many lens spaces $L(p, q)$ which, though each q is a quadratic residue* mod p, *cannot bound a homology S^2.*

PROOF. Let $p=72m+62$ and $q=8m+7$ for $m\in \boldsymbol{N}$. Let $x=1$ and $r=3$. Then $px-r^2q=-1$ holds. By Lemma 5, $L(p, q)$ is obtained from $\Sigma(3, 2, 48m+41)$ by surgery on a knot. By easy computation, we can show

$$\bar{\mu}(\Sigma(3, 2, 48m+41))=1 \quad \text{(See Neumann-Raymond [}\mathbf{N\text{-}R}\text{])}.$$

Next we show that there are infinitely many such p having the property $(*)$. Let $t=p/2$. Then $t=36m+31$ and $t\equiv 3\,(4)$. By Dirichlet's Theorem, there are infinitely many prime numbers among $36+31$, $36\times 2+31, \cdots, 36m+31, \cdots$, because 36 and 31 are relatively prime. Since, if t is a prime number such that $t\equiv 3\,(4)$, $p=2t$ has the property $(*)$ by Remark 3, there are infinitely many p such that $p=72m+62$ for some $m\in \boldsymbol{N}$ and p has the property $(*)$.

Hence we obtain infinitely many lens spaces $L(p, q)$ which satisfy $\bar{\mu}(L(p, q))=1$ and thus cannot bound a homology S^2.

Here we present other examples of lens spaces which cannot bound a homology S^2. For $p\leqq 100$, there are 12 lens spaces up to diffeomorphism which have non-trivial $\bar{\mu}$-invariant. They are $L(22, 3)$, $L(38, 7)$, $L(46, 5)$, $L(54, 5)$, $L(54, 7)$, $L(62, 7)$, $L(86, 11)$, $L(86, 15)$, $L(86, 27)$, $L(94, 13)$, $L(98, 9)$ and $L(98, 19)$ which are obtained from $\Sigma(2, 9, 5)$, $\Sigma(2, 7, 11)$, $\Sigma(2, 3, 31)$, $\Sigma(26, 23, 61)$, $\Sigma(56, 25, 121)$, $\Sigma(2, 3, 41)$, $\Sigma(174, 41, 365)$, $\Sigma(84, 25, 289)$, $\Sigma(27, 11, 211)$, $\Sigma(40, 21, 179)$, $\Sigma(67, 33, 199)$ and $\Sigma(51, 19, 263)$ respectively by surgery on knots. For $p\leqq 500$, there are 273 such lens spaces $L(p, q)$ as $\bar{\mu}(L(p, q))=1$ and thus cannot bound a homology S^2.

As a corollary to Theorem 4, we obtain:

COROLLARY 1. *There are infinitely many lens spaces $L(p, q)$ which, though each q is a quadratic residue* mod p, *cannot be obtained from S^3 by surgery on knots.*

It is of interest to compare this corollary with the results of Fintushel-Stern and N. Maruyama. They discovered many examples which are obtained from S^3 by surgery on knots (Fintushel-Stern [**F-S**], N. Maruyama (unpublished)). The following are the examples which are obtained from S^3 by surgery:

$$\left.\begin{array}{l} L(pq+1,\ q^2), \\ L(4pq+1,\ 4q^2), \\ L(9n,\ 3n+1), \end{array}\right\} \text{(due to Moser, Fintushel-Stern, Gordon)}$$

$$\left.\begin{array}{l} L(16n-2,\ 6n-1), \\ L(4n^2,\ 4n+1), \\ L(4n^2,\ 4n-1), \\ L(m(2n+1)^2,\ 2m(2n+1)+1). \end{array}\right\} \text{(due to N. Maruyama)}$$

§7. A homology $L(p, 1)$ and a 2-handle body.

There is close relation between a 3-manifold and its bounding 4-manifold. The following was obtained (in implicit form) by Kirby [**K**] and Melvin [**Mel**].

THEOREM (K-M). *Let $f: M \to N$ be a diffeomorphism between 3-manifolds M and N. Let V and W be 2-handle bodies such that $\partial V = M$ and $\partial W = N$. Then there is a diffeomorphism $h: V\# pCP(2)\# q\overline{CP}(2) \to W\# rCP(2)\# s\overline{CP}(2)$ for some $p, q, r, s \in \boldsymbol{N}$ such that $h|\partial = f$.*

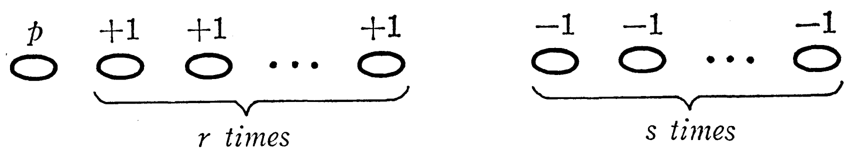
Is the statement true when f is replaced by a homotopy equivalence and h is replaced by a homotopy equivalence relative to boundary? We propose the following conjecture which is regarded as a homotopy equivalence version of Theorem (K-M).

CONJECTURE. Let $f: M \to N$ be a homotopy equivalence between 3-manifolds M and N. Then there are 2-handle bodies V, W and a homotopy equivalence as pairs $h: (V, \partial V) \to (W, \partial W)$ such that $\partial V = M$, $\partial W = N$ and $h|\partial V = f$.

In Theorem 5 we show that the conjecture is true under the assumption that M and N are homotopy $L(p, 1)$. We need several lemmas to obtain it.

LEMMA 6. *Let M be a homology $L(p, 1)$. Then there are 2-handle bodies V, W and a map $g: (V, \partial V) \to (W, \partial W)$ which satisfy the following conditions:*

(1) *$\partial V = M$, and W is a 2-handle body associated with the following framed link picture*



for some $r, s \in \boldsymbol{N}$,

(2) $\deg(g)=\pm 1$ *and* g *induces an isomorphism*

$$g_*: H_2(V) \longrightarrow H_2(W).$$

SUBLEMMA 1. *Let* k *be a knot in a homology sphere* H. *Let* k_0 *be an unknot in a 3-sphere* S^3. *Let* $N(k)$ *and* $N(k_0)$ *be tubular neighbourhoods of* k *and* k_0 *respectively. Then there is a map* $f: H \to S^3$ *which satisfies the following:*

(1) $\deg(f)=\pm 1$,

(2) $f^{-1}(N(k_0))=N(k)$ *and* $f|N(k): N(k) \to N(k_0)$ *is a bundle isomorphism which preserves* 0-*framing.*

This sublemma follows from elementary obstruction theory and we omit the proof.

SUBLEMMA 2. *Suppose that* H *is a homology sphere and* k *is a knot in* H. *Let* V *be a* 4-*manifold constructed from* $H \times [0, 1]$ *by attaching a* 2-*handle along* $N(k) \times \{1\} \subset H \times \{1\}$ *with* p-*framing. Let* k_0 *be an unknot in* S^3 *and* W *be a* 4-*manifold constructed from* $S^3 \times [0, 1]$ *by attaching a* 2-*handle along* $N(k_0) \times \{1\} \subset S^3 \times \{1\}$ *with* p-*framing. Then there is a map*

$$g: (V; H \times \{0\}, \partial V - H \times \{0\}) \longrightarrow (W; S^3 \times \{0\}, \partial W - S^3 \times \{0\})$$

such that g *induces an isomorphism* $g_*: H_2(V) \to H_2(W)$.

PROOF OF SUBLEMMA 2. Let $f: H \to S^3$ be a map which satisfies the conditions (1) and (2) in Sublemma 1. Then $f \times \mathrm{id}_{[0,1]}: H \times [0, 1] \to S^3 \times [0, 1]$ can be extended naturally to $g: V \to W$ so as to satisfy the required conditions.

SUBLEMMA 3. *Let* M *be a* 3-*manifold and* $f, g: M \to S^3$ *be maps. Then* f *is homotopic to* g *if and only if* $\deg(f)=\deg(g)$.

This also follows from elementary obstruction theory. To construct a homotopy between f and g, the only obstruction lies in $H^4(M \times I, M \times \partial I; \pi_3(S^3)) \approx \boldsymbol{Z}$ and it corresponds to $\deg(f)-\deg(g)$.

PROOF OF LEMMA 6. Since M is a homology $L(p, 1)$, M bounds a 2-handle body

$$V=D^4 \cup h_0 \cup h_1 \cup \cdots \cup h_r \cup h_{r+1} \cup \cdots \cup h_{r+s}$$

such that the intersection form on $H^2(V)$ is represented by $(p) \oplus E_r \oplus (-E_s)$ with respect to the basis $[h_0], [h_1], \cdots, [h_r], [h_{r+1}], \cdots, [h_{r+s}]$. We decompose V as

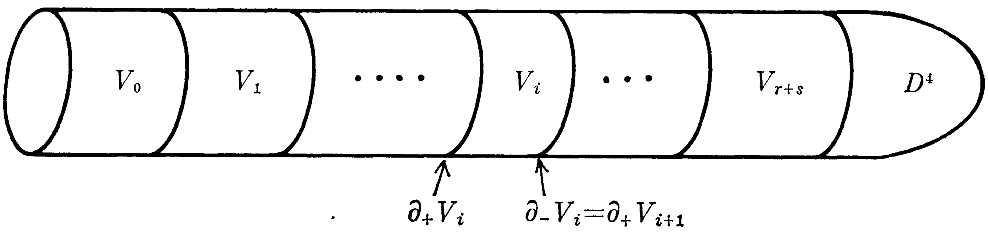
$$V=V_0 \cup V_1 \cup \cdots \cup V_r \cup V_{r+1} \cup \cdots \cup V_{r+s} \cup D^4.$$

This decomposition is assumed to correspond to the above handle decomposition and to satisfy the following conditions:

(1) each V_i has two components in ∂V_i, say $\partial_+ V_i$ and $\partial_- V_i$,

(2) $\partial_- V_i$ and $\partial_+ V_{i+1}$ are identified in V, also $\partial_- V_{r+s}$ and ∂D^4 are identified in V,

(3) $\partial_- V_i$ is a homology sphere and V_i is constructed from $(\partial_- V_i)\times[0, 1]$ by attaching a 2-handle along some knot in $(\partial_- V_i)\times\{1\}$ with 1-framing $(1\leqq i\leqq r)$ or with (-1)-framing $(r+1\leqq i\leqq r+s)$ or with p-framing $(i=0)$ respectively.



Similarly we consider W_i $(i=0, \cdots, r+s)$ and $W=W_0\cup\cdots\cup W_{r+s}\cup D^4$. But, this time, we assume the following instead of (3):

(3′) W_i is constructed from $S^3\times[0, 1]$ by attaching a 2-handle along an unknot in $S^3\times\{1\}$ with 1-framing $(1\leqq i\leqq r)$ or with (-1)-framing $(r+1\leqq i\leqq r+s)$ or with p-framing $(i=0)$.

Note that $\partial W=L(p, 1)$, $\partial_+ W_i=S^3$ for $i=1, \cdots, r+s$ and $\partial_- W_0=S^3$.

Now we construct a map g. By Sublemma 2, we obtain a map

$$g_i : (V_i\,;\, \partial_+ V_i, \partial_- V_i) \longrightarrow (W_i\,;\, \partial_+ W_i, \partial_- W_i)$$

such that g_i induces an isomorphism $(g_i)_* : H_2(V_i)\to H_2(W_i)$. Note that $\partial_- V_i=\partial_+ V_{i+1}$ and, by Sublemma 3, $g_i|\partial_- V_i : \partial_- V_i\to S^3$ and $g_{i+1}|\partial_+ V_{i+1} : \partial_+ V_{i+1}\to S^3$ are homotopic. Further, since $g_{r+s}|\partial_- V_{r+s} : \partial_- V_{r+s}\to\partial_- W_{r+s}$ can be extended to a map from D^4 to D^4, by pasting g_i for $i=0, \cdots, r+s$, we obtain a map $g : (V, \partial V)\to(W, \partial W)$ such that g induces an isomorphism $g_* : H_2(V)\to H_2(W)$. This completes the proof.

REMARK 4. In Lemma 6 the fact that $g(\partial V)\subset\partial W$ is significant. If it is removed, the lemma will be trivial.

The following lemma might be interesting in itself.

LEMMA 7. *Let $f : L(p, 1)\to L(p, 1)$ be a homotopy equivalence. Then there are a 2-handle body V and a map $g : V\to V$ which satisfies the following:*

(1) *$\partial V=L(p, 1)$ and $g|\partial V=f$,*

(2) *g induces an isomorphism $g_* : H_2(V)\to H_2(V)$.*

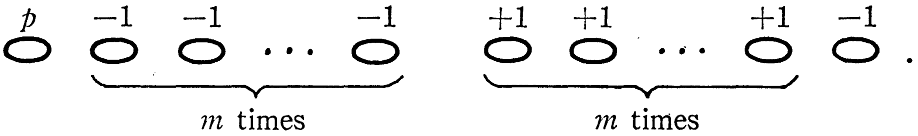
PROOF. We use Olum's result [O] about homotopy equivalences of lens spaces. It asserts that homotopy classes of self homotopy equivalences of $L(p, 1)$ correspond to $x\in \boldsymbol{Z}_p$ such as $x^2\equiv\pm 1\,(p)$ bijectively.

For a homotopy equivalence h, h corresponds to x when $h_* : H_1(L(p, 1))\to H_1(L(p, 1))$ is a multiplication by x.

Now suppose that $f_*: H_1(L(p, 1)) \to H_1(L(p, 1))$ is a multiplication by s. Then $s^2 \equiv \varepsilon\,(p)$ holds where $\varepsilon=1$ or -1. By Lemma 3, for such s, there is a unimodular matrix Q such that, for $Q^{-1}=(\tilde{q}_{ij})$, $s\tilde{q}_{11} \equiv 1\,(p)$ and

$$(**) \qquad {}^tQ((p)\oplus(-E_m)\oplus E_m\oplus(-1))Q=(\varepsilon p)\oplus(-E_m)\oplus E_m\oplus(-\varepsilon)\,.$$

Let V be the 2-handle body which is associated with the framed link picture



Let $h_0, h_1, \cdots, h_m, h_{m+1}, \cdots, h_{2m}, h_{2m+1}$ be corresponding 2-handles. Here we apply Lemma 4 for $V=D^4\cup h_0\cup\cdots\cup h_{2m+1}$ and Q. Then we obtain the second handle decomposition $V=D^4\cup\bar{h}_0\cup\cdots\cup\bar{h}_{2m+1}$.

Let $[h_i]$, $[\bar{h}_i]\in H_2(V)$ denote the homology classes represented by the cores of the 2-handles h_i, $\bar{h}_i$. Let $[h_i^*]$, $[\bar{h}_i^*]\in H_2(V, \partial V)$ denote the homology classes represented by the cores of the dual 2-handles of h_i, $\bar{h}_i$. For $Q=(q_{ij})$ and $Q^{-1}=(\tilde{q}_{ij})$ $(i, j=1, \cdots, 2m+2)$, the following holds by the construction of $\bar{h}_0, \cdots, \bar{h}_{2m+1}$:

$$[\bar{h}_{i-1}]=\sum_{j=1}^{2m+2} q_{ji}[h_{j-1}]\,.$$

Let $[\bar{h}_{i-1}^*]=\sum_{j=1}^{2m+2}\alpha_{ji}[h_{j-1}^*]$ and $A=(\alpha_{ij})$. Then

$$\delta_{ij}=([\bar{h}_{i-1}], [\bar{h}_{j-1}^*])=\Big(\sum_{k=1}^{2m+2} q_{ki}[h_{k-1}], \sum_{n=1}^{2m+2}\alpha_{nj}[h_{n-1}^*]\Big)=\sum_{k=1}^{2m+2} q_{ki}\alpha_{kj}$$

holds. Thus A is the inverse matrix of ${}^tQ$. Hence

$$[\bar{h}_{i-1}^*]=\sum_{j=1}^{2m+2}\tilde{q}_{ij}[h_{j-1}^*]\,.$$

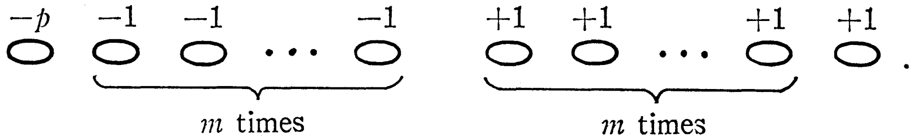
In paticular,

$$[\bar{h}_0^*]=\tilde{q}_{11}[h_0^*]+\tilde{q}_{12}[h_1^*]+\cdots+\tilde{q}_{1\,2m+2}[h_{2m+1}^*]$$

holds. Since $s\tilde{q}_{11}\equiv 1\,(p)$ and $[\partial h_i^*]=0$ for $i=1, \cdots, 2m+1$, we obtain the following:

$$(***) \qquad s[\partial\bar{h}_0^*]=[\partial h_0^*]\,.$$

Note that the intersection form on $H_2(V)$ represented by the basis $[\bar{h}_0], \cdots, [\bar{h}_{2m+1}]$ equals $(\varepsilon p)\oplus(-E_m)\oplus E_m\oplus(-\varepsilon)$ because of $(**)$. As in the proof of Lemma 6, we consider the decompositions of V in two ways. For the first, $V=V_0\cup V_1\cup\cdots\cup V_{2m+1}\cup D^4$ which corresponds to the handle decomposition $V=D^4\cup\bar{h}_0\cup\cdots\cup\bar{h}_{2m+1}$, and for the second, $V=W_0\cup W_1\cup\cdots\cup W_{2m+1}\cup D^4$ which corresponds to the handle decomposition $V=D^4\cup h_0\cup\cdots\cup h_{2m+1}$. When $\varepsilon=1$, just as in the proof of Lemma 6, we obtain a map $g: V\to V$ such that $g(V_i)\subset W_i$ for each i and g induces an isomorphism $g_*: H_2(V)\to H_2(V)$. When $\varepsilon=-1$, let consider the 2-handle body V' associated with the following framed link picture:

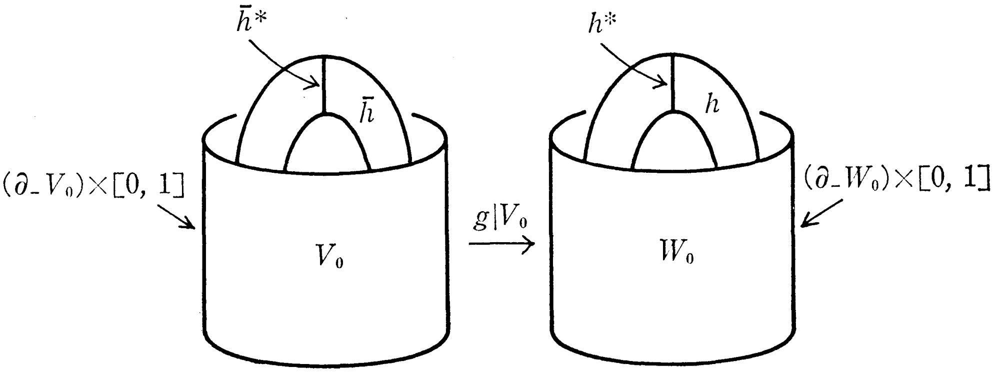


Let $h_0', h_1', \cdots, h_m', h_{m+1}', \cdots, h_{2m}', h_{2m+1}'$ be corresponding 2-handles and $V'=W_0'\cup W_1'\cup\cdots\cup W_{2m+1}'\cup D^4$ be the decomposition of V' as in the proof of Lemma 6. But there is the natural (orientation reversing) diffeomorphism $h: V'\to V$ such that $h(h_0')=h_0$, $h(h_i')=h_{i+m}$ and $h(h_{i+m}')=h_i$ for $i=1, \cdots, m$, $h(h_{2m+1}')=h_{2m+1}$. Then, composing a map from V to V' as in the proof of Lemma 6 and h, we obtain a map $g: V\to V$ such that $\deg(g)=-1$ and g induces an isomorphism $g_*: H_2(V)\to H_2(V)$.

Next we shall see that the homomorphism

$$(g|\partial V)_*: H_1(\partial V)\longrightarrow H_1(\partial V)=\boldsymbol{Z}_p$$

is a multiplication by s. By definition of V_0 and W_0, V_0 (respectively W_0) is constructed from $(\partial_- V_0)\times[0, 1]$ (respectively $(\partial_- W_0)\times[0, 1]$) by attaching a 2-handle, say $\bar{h}$ (respectively h). Let $\bar{h}^*$ (respectively h^*) be the core of the dual 2-handle for $\bar{h}$ (respectively h). Note that, as homology classes, $[\bar{h}^*]=[\bar{h}_0^*]$ and $[h^*]=[h_0^*]$ hold.



Hence we obtain

$$(****)\qquad [\partial\bar{h}^*]=[\partial\bar{h}_0^*] \quad\text{and}\quad [\partial h^*]=[\partial h_0^*].$$

By the construction of g (in the proof of Lemma 6), $g|\bar{h}: \bar{h}\to h$ is a homeomorphism. Hence we obtain $(g|\partial V)_*[\partial\bar{h}^*]=[\partial h^*]$. On the other hand, by (***) and (****), we obtain $s[\partial\bar{h}^*]=[\partial h^*]$. This means that $(g|\partial V)_*$ is a multiplication by s. Finally, by Olum's result, $g|\partial V$ is homotopic to f, thus f itself has such an extension as g. This completes the proof of Lemma 7.

Now we can prove the following theorem by the aid of the above lemmas.

THEOREM 5. *Let* $f: M \rightarrow N$ *be a homotopy equivalence between* M *and* N *which are homotopy* $L(p, 1)$. *Then there are* 2*-handle bodies* V *and* W, *and a map* $h: V \rightarrow W$ *such that:*

(1) $\partial V = M$ *and* $\partial W = N$,

(2) $h: (V, \partial V) \rightarrow (W, \partial W)$ *is a homotopy equivalence as pairs.*

We mean that a homotopy $L(p, 1)$ is a 3-manifold which is homotopy equivalent to $L(p, 1)$.

PROOF. It is sufficient to prove this theorem in the case of $N = L(p, 1)$. By Lemma 6, there are 2-handle bodies V and W, and a map $g: V \rightarrow W$ such that:

(1) $\partial V = M$ and $\partial W = N$,

(2) g induces an isomorphism $g_*: H_2(V) \rightarrow H_2(W)$.

Let $(g|\partial V)^{-1}$ denote a homotopy inverse of $g|\partial V$. For $f \circ (g|\partial V)^{-1}$, we apply Lemma 7. Then we obtain a 2-handle body W_0 and a map $g_0: W_0 \rightarrow W_0$ such that:

(1) $\partial W_0 = N$ and $g_0|\partial W_0 = f \circ (g|\partial V)^{-1}$,

(2) g_0 induces an isomorphism $(g_0)_*: H_2(W_0) \rightarrow H_2(W_0)$.

But, by Theorem (K-M), we can assume that $W = W_0$ without loss of generality. Let $h = g_0 \circ g: (V, \partial V) \rightarrow (W, \partial W)$. Then h satisfies the following:

(1) $h|\partial V$ is homotopic to f,

(2) h induces an isomorphism $h_*: H_2(V) \rightarrow H_2(W)$.

Since V and W are homotopy equivalent to a bouquet of S^2, h is a homotopy equivalence by well known Whitehead's Theorem. Furthermore, since $h|\partial V$ is also a homotopy equivalence, $h: (V, \partial V) \rightarrow (W, \partial W)$ is a week homotopy equivalence as pairs and thus a homotopy equivalence as pairs. This means Theorem 5. We finish this paper by presenting the following theorem which is an application of Theorem 5.

THEOREM 6. *For any* $n \in \boldsymbol{N}$ *there are* n *compact simply connected* 4*-manifolds with boundaries which are homotopy equivalent relative to boundaries, but not homeomorphic. In fact their boundaries are not homeomorphic.*

PROOF. For given $n \in \boldsymbol{N}$, choose a sufficiently large number p so as to obtain n numbers $q_1, \cdots, q_n$ which are quadratic residues mod p and $q_i \not\equiv \pm q_j^{\pm 1}\,(p)$ for $i \neq j$. By using Theorem 5 and Theorem (K-M) repeatedly, we obtain simply connected 4-manifolds V_i $(i=1, \cdots, n)$ such that $\partial V_i = L(p, q_i)$, and $(V_i, \partial V_i)$ are homotopy equivalent to $(V_j, \partial V_j)$ as pairs $(i, j=1, \cdots, n)$. But, as is well known, $L(p, q_i)$ is not homeomorphic to $L(p, q_j)$ if $q_i \not\equiv \pm q_j^{\pm 1}\,(p)$. This completes the proof.

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