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Complex abelian Lie groups with finite-dimensional cohomology groups

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Introduction.

Grauert gave an example of a pseudoconvex manifold which admits no nonconstant holomorphic functions (See Narasimhan [10]). Using this Grauert's example, Malgrange [6] constructed an example of 2-dimensional pseudoconvex manifold M whose cohomology group $H^1(M, {}_M\mathcal{O})$ is not Hausdorff.

On the other hand, there exists a noncompact complex Lie group without nonconstant holomorphic functions. Such a Lie group is called an (H, C)-group ([7]) and also called a toroid group ([2], [4]). The first (named) author [3] showed that any complex abelian Lie group is pseudoconvex. The purpose of this paper is to investigate the cohomology groups $H^p(G, \mathcal{O})$, $p>0$ for a complex abelian Lie group G and its structure sheaf $\mathcal{O}$.

In §1 we recall some properties of complex Lie groups and study (complex valued) real analytic functions on complex abelian Lie groups. In §2 we consider the $\bar{\partial}$-problem with respect to real analytic forms on an (H, C)-group and construct formal solutions for the $\bar{\partial}$-problem. In §3, using the formal solutions for the $\bar{\partial}$-problem, we give a condition for an (H, C)-group G to have the finite-dimensional cohomology groups $H^p(G, \mathcal{O})$, $p>0$ (Theorem 3.1). A given (H, C)-group G of dimension n is isomorphic to the quotient group C^n/Γ by a discrete subgroup Γ as a complex Lie group. Theorem 3.1 shows that the condition for $H^p(C^n/\Gamma, \mathcal{O})$, $p>0$ to be finite-dimensional depends on a number theoretical property of the discrete subgroup Γ in C^n. It is well-known that $\dim H^p(T_C^n, \mathcal{O}) = \binom{n}{p}$ for a complex torus T_C^n of dimension n. We give another proof of this fact for a complex torus (Corollary 3.2). Then we can regard Theorem 3.1 as a generalization of this fact. Moreover we construct the family $\{C^n/\Gamma(\alpha)\,;\ \alpha \in R-Q\}$ of n-dimensional noncompact (H, C)-groups where $\Gamma(\alpha)$ is the subgroup of C^n generated by

$$\{e_i, \sqrt{-1}e_j, \sqrt{-1}e_{n-1}+\alpha e_n\ ;\ e_i={}^t(\delta_{1i}, \cdots, \delta_{ni}),\ 1\leqq i\leqq n,\ 1\leqq j\leqq n-2\}.$$

We show that, if α is algebraic, then $\dim H^1(C^n/\Gamma(\alpha), \mathcal{O})=n-1$. Further if α is a kind of Liouville number, then we obtain that $\dim H^1(C^n/\Gamma(\alpha), \mathcal{O})=\infty$.

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1. Preliminaries.

In this section we recall some properties of complex abelian Lie groups and consider real analytic functions on complex abelian Lie groups.

We recall the following theorem proved by Morimoto ([7], [8]) and Remmert (See Kopfermann [4]).

THEOREM 1.1. *Let G be a connected complex Lie group and $G^0 := \{x \in G; f(x)=f(e)$ for all $f \in H^0(G, \mathcal{O})\}$, where e is the unit element of G and $\mathcal{O}$ denotes the structure sheaf of G. Then*

(a) *G^0 is a closed connected abelian complex Lie subgroup of G.*

(b) *Every holomorphic function on G^0 is constant.*

(c) *If G is abelian, then G is isomorphic to $G^0 \times C^{*l} \times C^m$ for some $l, m \geqq 0$ as a complex Lie group.*

A connected complex Lie group G is called an (H, C)-group if every holomorphic function on G is constant.

Let G be an (H, C)-group of dimension n. Since $G=G^0$, by the above theorem G is abelian. Then there exists a discrete subgroup Γ of $C^n := \{{}^t(z_1, \cdots, z_n)\ ;\ z_i \in C\}$ such that G is isomorphic to C^n/Γ as a complex Lie group. The j-th unit vector of C^n for $1 \leqq j \leqq n$ will be denoted by $e_j = {}^t(\delta_{1j}, \delta_{2j}, \cdots, \delta_{nj})$. After a linear change of coordinates of C^n, we may assume that Γ is the discrete subgroup generated by linearly independent vectors $e_1, \cdots, e_n, v_1, \cdots, v_q$ of C^n over R for some q with $1 \leqq q \leqq n$; thus

$$\Gamma = \{\sum_{i=1}^{n} m_i e_i + \sum_{j=1}^{q} m_{n+j} v_j\ ;\ m_l \in Z,\ 1 \leqq l \leqq n+q\}.$$

We put ${}^t(v_{1j}, \cdots, v_{nj}) := v_j$, $\mathrm{Re}\, v_j := {}^t(\mathrm{Re}\, v_{1j}, \cdots, \mathrm{Re}\, v_{nj})$ and $\mathrm{Im}\, v_j := {}^t(\mathrm{Im}\, v_{1j}, \cdots, \mathrm{Im}\, v_{nj})$, $1 \leqq j \leqq q$. Since $\mathrm{Im}\, v_1, \cdots, \mathrm{Im}\, v_q$ are linearly independent over R, we may assume $\det[\mathrm{Im}\, v_{ij}\ ;\ 1 \leqq i, j \leqq q] \neq 0$ and then $\mathrm{Im}\, v_1, \cdots, \mathrm{Im}\, v_q, e_{q+1}, \cdots, e_n$ are linearly independent over R. We put $v_j := \sqrt{-1}\, e_j$, $q+1 \leqq j \leqq n$. Then C^n is spanned by $\{e_1, \cdots, e_n, v_1, \cdots, v_n\}$ over R. We put $[v_{ij}] := [v_1, \cdots, v_n]$, $\alpha_{ij} := \mathrm{Re}\, v_{ij}$ and $\beta_{ij} := \mathrm{Im}\, v_{ij}$, $1 \leqq i, j \leqq n$. Since the $n \times n$-matrix $\beta := [\beta_{ij}]$ is nonsingular, we have the inverse matrix $\gamma = [\gamma_{ij}] := \beta^{-1}$. Since $v_j = \sqrt{-1}\, e_j$, $q+1 \leqq j \leqq n$, we have $\alpha_{ij} = 0$, $\beta_{ij} = \delta_{ij}$, $\gamma_{ij} = \delta_{ij}$, $1 \leqq i \leqq n$, $q+1 \leqq j \leqq n$ and $\gamma_{ij} = -\sum_{k=1}^{q} \beta_{ik} \gamma_{kj}$, $q+1 \leqq i \leqq n$, $1 \leqq j \leqq q$. For $z = {}^t(z_1, \cdots, z_n) = \sum_{i=1}^{n} z_i e_i \in C^n$ with $z_i = x_i + \sqrt{-1}\, y_i$, $x_i, y_i \in R$, $1 \leqq i \leqq n$, we have a unique vector $t={}^t(t_1, \cdots, t_{2n})\in R^{2n}$ satisfying

$$z=\sum_{i=1}^{n} t_i e_i+\sum_{i=1}^{n} t_{n+i} v_i .$$

Then we obtain a real linear isomorphism $C^n\ni z\mapsto t\in R^{2n}$, which induces the isomorphisms $C^n/\Gamma\cong R^{2n}/\Gamma\cong T^{n+q}\times R^{n-q}$ as a real Lie group, where

$$T^{n+q}:=\{{}^t(\exp 2\pi\sqrt{-1}t_1, \cdots, \exp 2\pi\sqrt{-1}t_{n+q})\in C^{*n+q}\ ;\ {}^t(t_1, \cdots, t_{n+q})\in R^{n+q}\}.$$

Treating real analytic functions on C^n/Γ, we shall sometimes identify C^n/Γ with $T^{n+q}\times R^{n-q}$ under the above isomorphism. By the definition of the isomorphism $C^n\ni z\mapsto t\in R^{2n}$, we have $t_i=x_i-\sum_{j=1}^{n}(\sum_{k=1}^{n}\alpha_{ik}\gamma_{kj})y_j$, $t_{n+i}=\sum_{j=1}^{n}\gamma_{ij}y_j$ and

$$(1.1)\qquad \frac{\partial}{\partial\bar{z}_i}:=\frac{1}{2}\Big(\frac{\partial}{\partial x_i}+\sqrt{-1}\frac{\partial}{\partial y_i}\Big)$$
$$=\frac{1}{2}\Big\{\frac{\partial}{\partial t_i}+\sqrt{-1}\Big(-\sum_{j=1}^{n}\sum_{k=1}^{n}\alpha_{jk}\gamma_{ki}\frac{\partial}{\partial t_j}+\sum_{j=1}^{n}\gamma_{ji}\frac{\partial}{\partial t_{n+j}}\Big)\Big\}$$

for $1\leqq i\leqq n$.

Let $t={}^t(t_1, \cdots, t_{2n})\in R^{2n}$ and $m={}^t(m_1, \cdots, m_{n+q})\in Z^{n+q}$. We put $t'={}^t(t_1, \cdots, t_{n+q})$, $t''={}^t(t_{n+q+1}, \cdots, t_{2n})$ and $\langle m, t'\rangle:=m_1t_1+m_2t_2+\cdots+m_{n+q}t_{n+q}$. Let f be a (complex valued) real analytic function on C^n/Γ. Then we have the Fourier expansion of f:

$$(1.2)\qquad f(t)=\sum_{m\in Z^{n+q}} a^m(t'')\exp 2\pi\sqrt{-1}\langle m, t'\rangle$$

for $t=\begin{pmatrix} t' \\ t'' \end{pmatrix}\in R^{2n}$.

We regard $C^n/\Gamma\cong T^{n+q}\times R^{n-q}$ as a real analytic submanifold of $C^{*n+q}\times C^{n-q}$ under the natural inclusion $T^{n+q}\times R^{n-q}\subset C^{*n+q}\times C^{n-q}$.

LEMMA 1.2. *Let $\{a^m(t'')\ ;\ m\in Z^{n+q}\}$ be a sequence of real analytic functions on R^{n-q}. Then the following statements are equivalent.*

(a) *There exists a real analytic function $f(t)$ on $T^{n+q}\times R^{n-q}$ such that*

$$f(t)=\sum_{m\in Z^{n+q}} a^m(t'')\exp 2\pi\sqrt{-1}\langle m, t'\rangle$$

for $t=\begin{pmatrix} t' \\ t'' \end{pmatrix}\in R^{2n}$.

(b) *There are an open neighbourhood V of R^{n-q} in C^{n-q} and a holomorphic function $a_*^m(\eta)$ in V for each $m\in Z^{n+q}$ such that $a_*^m|_{R^{n-q}}=a^m$ and to every compact subset K of V correspond positive numbers C and ε satisfying*

$$\sup\{|a_*^m(\eta)|\ ;\ \eta\in K\}\leqq C\exp(-\varepsilon|m|),$$

where $|m|=\sqrt{m_1^2+m_2^2+\cdots+m_{n+q}^2}$.

PROOF. Assume (a) holds. For every point $p\in T^{n+q}\times R^{n-q}$, there exist a neighbourhood U_p of p in $C^{*n+q}\times C^{n-q}$ and a holomorphic function h_p in U_p such that

$$f|_{U_p\cap(T^{n+q}\times R^{n-q})}=h_p|_{U_p\cap(T^{n+q}\times R^{n-q})}.$$

From the uniqueness of analytic continuation we have $h_p=h_q$ in the connected components of $U_p\cap U_q$ that intersect with $T^{n+q}\times R^{n-q}$ for $p, q\in T^{n+q}\times R^{n-q}$. Thus there exist a neighbourhood V of R^{n-q} in C^{n-q}, a neighbourhood W of $T^{n+q}\times R^{n-q}$ in $C^{*n+q}\times C^{n-q}$ with $T^{n+q}\times V\subset W$ and a holomorphic function h in W which satisfies $f=h|_{T^{n+q}\times R^{n-q}}$. For each compact subset K of V, there exists $\delta>0$ such that h has a Laurent expansion $h(\zeta, \eta)=\sum_{m\in Z^{n+q}} a_*^m(\eta)\zeta_1^{m_1}\cdots\zeta_{n+q}^{m_{n+q}}$ for

$$(\zeta, \eta)\in\{\zeta={}^t(\zeta_1, \cdots, \zeta_{n+q})\ ;\ 1-\delta<|\zeta_i|<1+\delta\}\times V_K,$$

where V_K is an open neighbourhood of K in V and $a_*^m(\eta)$ are holomorphic in V. From the uniqueness of the Fourier expansion we have $a_*^m|_{R^{n-q}}=a^m$. On the other hand

$$h_*(\xi, \eta):=\sum_{m\in Z^{n+q}} a_*^m(\eta)\exp 2\pi\sqrt{-1}\langle m, \xi\rangle$$

is holomorphic in

$$\{(\xi_1, \cdots, \xi_{n+q})\in C^{n+q}\ ;\ |\mathrm{Im}\,\xi_i|<\varepsilon\}\times V_K$$

for some $\varepsilon>0$, where $\langle m, \xi\rangle=m_1\xi_1+\cdots+m_{n+q}\xi_{n+q}$. We put

$$C:=\sup\{|a_*^m(\eta)|\exp 2\pi\langle m, \xi\rangle\ ;\ m\in Z^{n+q},\ |\xi_i|\leqq\varepsilon/2\pi,\ \xi\in R^{n+q},\ \eta\in K\}<+\infty.$$

Then we have

$$\sup\{|a_*^m(\eta)|\ ;\ \eta\in K\}\leqq C\exp(-\varepsilon|m|).$$

Suppose (b) holds. Then it is shown that

$$g(\zeta, \eta):=\sum_{m\in Z^{n+q}} a_*^m(\eta)\zeta_1^{m_1}\cdots\zeta_{n+q}^{m_{n+q}}$$

converges in a neighbourhood of $T^{n+q}\times R^{n-q}$ in $C^{*n+q}\times C^{n-q}$. Since $g|_{T^{n+q}\times R^{n-q}}$ is real analytic on $T^{n+q}\times R^{n-q}$, then

$$f(t):=\sum_{m\in Z^{n+q}} a^m(t'')\exp 2\pi\sqrt{-1}\langle m, t'\rangle$$

is a real analytic function on C^n/Γ.

Suppose f is a real analytic function on C^n/Γ. We write as in (1.2):

$$f(t)=\sum_{m\in Z^{n+q}} a^m(t'')\exp 2\pi\sqrt{-1}\langle m, t'\rangle.$$

We put

$$f^m(t):=a^m(t'')\exp 2\pi\sqrt{-1}\langle m, t'\rangle .$$

Using (1.1), we have

$$\frac{\partial}{\partial \bar{z}_i} f^m(t)=\Big\{\pi\sum_{k=1}^{q}\gamma_{ki}(\sum_{j=1}^{n} m_j v_{jk}-m_{n+k})a^m(t'') + \sqrt{-1}\sum_{k=q+1}^{n}\gamma_{ki}\Big(\pi m_k a^m(t'')+\frac{1}{2}\frac{\partial a^m(t'')}{\partial t_{n+k}}\Big)\Big\} \times\exp 2\pi\sqrt{-1}\langle m, t'\rangle, \quad 1\leqq i\leqq n . \tag{1.3}$$

Thus f is holomorphic on C^n/Γ if and only if $a^m(t'')$ satisfy

$$(\sum_{j=1}^{n} m_j v_{ji}-m_{n+i})a^m(t'')=0 \qquad \text{for} \quad 1\leqq i\leqq q \tag{1.4}$$

and

$$\pi m_i a^m(t'')+\frac{1}{2}\frac{\partial a^m(t'')}{\partial t_{n+i}}=0 \qquad \text{for} \quad q+1\leqq i\leqq n \tag{1.5}$$

for every $m\in Z^{n+q}$. We put

$$K_{m,i}:=\sum_{j=1}^{n} m_j v_{ji}-m_{n+i} \qquad \text{for} \quad 1\leqq i\leqq q$$

and

$$K_m:=\mathrm{Max}\{|K_{m,i}| \ ; \ 1\leqq i\leqq q\},$$

where $m={}^t(m_1, \cdots, m_{n+q})\in Z^{n+q}$.

Using (1.4) and (1.5), we prove the following proposition (cf. [4] and [8]).

PROPOSITION 1.3. *Let G be a connected complex abelian Lie group of dimension n. Then G is an (H, C)-group if and only if there exists a discrete subgroup Γ generated by $\{e_1, \cdots, e_n, v_1, \cdots, v_q \ ; \ 1\leqq q\leqq n\}$ such that G is isomorphic to C^n/Γ as a complex Lie group and $K_m>0$ for any $m\in Z^{n+q}-\{0\}$.*

PROOF. Suppose G is an (H, C)-group. Since G is abelian and admits no nonconstant holomorphic functions, we may assume that G is isomorphic to C^n/Γ for a discrete subgroup Γ generated by $\{e_1, \cdots, e_n, v_1, \cdots, v_q \ ; \ 1\leqq q\leqq n\}$. Suppose that there exists $m_0={}^t(m_1^0, \cdots, m_{n+q}^0)\in Z^{n+q}-\{0\}$ such that $K_{m_0}=0$. We put

$$f(t):=\exp 2\pi(\sum_{k=q+1}^{n} m_k^0 t_{n+k})\exp 2\pi\sqrt{-1}\langle m_0, t'\rangle .$$

Then we have $\bar{\partial}f(t)=0$. Thus $f(t)$ is a nonconstant holomorphic function on C^n/Γ. Conversely we suppose $G\cong C^n/\Gamma$ and $K_m>0$ for any $m\in Z^{n+q}-\{0\}$. Let

$$f(t)=\sum_{m\in Z^{n+q}} f^m(t'')\exp 2\pi\sqrt{-1}\langle m, t'\rangle$$

be a holomorphic function on C^n/Γ. By (1.4) and (1.5) we have $a^m(t'')\equiv 0$ for any $m\in Z^{n+q}-\{0\}$ and $\dfrac{\partial a^m(t'')}{\partial t_{n+i}}\equiv 0$. This means f is constant on C^n/Γ.

2. Formal solutions for the $\bar{\partial}$-problem on (H, C)-groups.

Throughout this section we assume G is an (H, C)-group of complex dimension n. Then we may assume $G=C^n/\Gamma$, where Γ is the discrete subgroup generated by $\{e_1, \cdots, e_n, v_1, \cdots, v_q\}$ and $K_m>0$ for any $m\in Z^{n+q}-\{0\}$. Let $\mathcal{A}$ be the sheaf of germs of (complex valued) real analytic functions on C^n/Γ and $\mathcal{A}^{p,q}$ the sheaf of germs of real analytic (p, q)-forms on C^n/Γ. We denote by $Z_{\bar{\partial}}(C^n/\Gamma, \mathcal{A}^{p,q})$ the space of $\bar{\partial}$-closed real analytic (p, q)-forms on C^n/Γ and by $B_{\bar{\partial}}(C^n/\Gamma, \mathcal{A}^{p,q})$ the space of $\bar{\partial}$-exact real analytic (p, q)-forms on C^n/Γ. Using the vanishing theorem ([**1**] and [**5**]): $H^p(C^n/\Gamma, \mathcal{A})=0$, $p\geqq 1$ and the resolution (for instance see [**11**])

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{A}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{0,n} \xrightarrow{\bar{\partial}} 0,$$

we have

$$H^p(C^n/\Gamma, \mathcal{O})=\frac{Z_{\bar{\partial}}(C^n/\Gamma, \mathcal{A}^{0,p})}{B_{\bar{\partial}}(C^n/\Gamma, \mathcal{A}^{0,p})}.$$

To calculate the cohomology groups $H^p(G, \mathcal{O})$, we shall study $\bar{\partial}$-exact forms and $\bar{\partial}$-closed forms on G.

Let ϕ be a real analytic $(0, p)$-form on G. Since $G=C^n/\Gamma$ has global 1-forms $dz_1, \cdots, dz_n, d\bar{z}_1, \cdots, d\bar{z}_n$ for the natural coordinate $z={}^t(z_1, \cdots, z_n)$ in C^n, we can write $\phi=\dfrac{1}{p\,!}\sum_{1\leq i_1,\cdots,i_p\leq n}\phi_{i_1\cdots i_p}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p}$, where $\phi_{i_1\cdots i_p}$ are real analytic functions on C^n/Γ and skew-symmetric in all indices. We expand $\phi_{i_1\cdots i_p}$ as in (1.2):

$$\phi_{i_1\cdots i_p}(t)=\sum_{m\in Z^{n+q}}a^m_{i_1\cdots i_p}(t'')\exp 2\pi\sqrt{-1}\langle m, t'\rangle.$$

We put

$$\phi^m_{i_1\cdots i_p}(t):=a^m_{i_1\cdots i_p}(t'')\exp 2\pi\sqrt{-1}\langle m, t'\rangle$$

and

$$\phi^m:=\frac{1}{p\,!}\sum_{1\leq i_1,\cdots,i_p\leq n}\phi^m_{i_1\cdots i_p}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p}.$$

Then $\phi=\sum_{m\in Z^{n+q}}\phi^m$. Suppose $\phi=\sum_{m\in Z^{n+q}}\phi^m\in B_{\bar{\partial}}(G, \mathcal{A}^{0,p})$. There exists a real analytic $(0, p-1)$-form $\psi=\sum_{m\in Z^{n+q}}\psi^m$ such that $\phi=\bar{\partial}\psi$. Then we have $\phi^m=\bar{\partial}\psi^m$ for any $m\in Z^{n+q}$. We put

$$\psi^m=\frac{1}{(p-1)!}\sum_{1\leq i_1,\cdots,i_{p-1}\leq n}\psi^m_{i_1\cdots i_{p-1}}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_{p-1}}$$

and

$$\psi^m_{i_1\cdots i_{p-1}}(t)=b^m_{i_1\cdots i_{p-1}}(t'')\exp 2\pi\sqrt{-1}\langle m,\ t'\rangle .$$

The equation $\phi=\bar{\partial}\psi$ implies

$$\phi^m_{i_1\cdots i_p}=\sum_{k=1}^{p}(-1)^{k+1}\frac{\partial\psi^m_{i_1\cdots\hat{i}_k\cdots i_p}}{\partial\bar{z}_{i_k}}. \tag{2.1}$$

Combining (2.1) with (1.3), we have for any $m\in Z^{n+q}$

$$a^m_{i_1\cdots i_p}=\sum_{k=1}^{p}(-1)^{k+1}\Big\{\pi\sum_{l=1}^{q}\gamma_{li_k}K_{m,l}b^m_{i_1\cdots\hat{i}_k\cdots i_p} +\sqrt{-1}\sum_{l=q+1}^{n}\gamma_{li_k}\Big(\pi m_l b^m_{i_1\cdots\hat{i}_k\cdots i_p}+\frac{1}{2}\frac{\partial b^m_{i_1\cdots\hat{i}_k\cdots i_p}}{\partial t_{n+l}}\Big)\Big\}, \tag{2.2}$$

where $K_{m,l}=\sum_{j=1}^{n}m_jv_{jl}-m_{n+l}$.

LEMMA 2.1. *Let*

$$\phi=\frac{1}{p!}\sum_{1\leq i_1,\cdots,i_p\leq n}\phi_{i_1\cdots i_p}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p}$$

be a real analytic $\bar{\partial}$-exact $(0, p)$-form on C^n/Γ such that

(1) *$\phi_{i_1\cdots i_p}$ is constant for any $1\leqq i_1,\cdots,i_p\leqq n$*

(2) *if $\{i_1,\cdots,i_p\}\cap\{q+1,\cdots,n\}\neq\emptyset$, $\phi_{i_1\cdots i_p}\equiv 0$.*

Then $\phi=0$.

PROOF. The proof will be by induction on p. If $p=1$, by (2.2) we have a real analytic function $\psi(t'')$ on R^{n-q} with

$$\phi_i=\frac{\sqrt{-1}}{2}\sum_{k=q+1}^{n}\gamma_{ki}\frac{\partial\psi}{\partial t_{n+k}},\qquad 1\leqq i\leqq n .$$

Since $\gamma_{ki}=\delta_{ki}$, $q+1\leqq k$, $i\leqq n$ and $\phi_i=0$, $i\geqq q+1$, we obtain $\partial\psi/\partial t_{n+i}=0$, $i\geqq q+1$. This means ψ is constant on R^{n-q}. Thus $\phi_i=0$, $1\leqq i\leqq n$. Assume that the lemma holds for $p-1$, $p\geqq 2$. For

$$\phi=\frac{1}{p!}\sum_{1\leq i_1,\cdots,i_p\leq n}\phi_{i_1\cdots i_p}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p},$$

from (2.2) there exist real analytic functions $b_{i_1\cdots i_{p-1}}$ such that

$$\phi_{i_1\cdots i_p}=\frac{\sqrt{-1}}{2}\sum_{k=1}^{p}(-1)^{k+1}\sum_{l=q+1}^{n}\gamma_{li_k}\frac{\partial b_{i_1\cdots\hat{i}_k\cdots i_p}}{\partial t_{n+l}}.$$

Then

$$\sum_{i=1}^{n}\beta_{is}\phi_{ii_1\cdots i_{p-1}}=\frac{-\sqrt{-1}}{2}\sum_{k=1}^{p-1}(-1)^{k+1}\sum_{l=q+1}^{n}\gamma_{li_k}\Big(\sum_{i=1}^{n}\beta_{is}\frac{\partial b_{ii_1\cdots\hat{i}_k\cdots i_{p-1}}}{\partial t_{n+l}}\Big)$$

for $1\leqq s\leqq q$. Thus

$$\frac{1}{(p-1)!}\sum_{1\leq i_1,\cdots,i_{p-1}\leq n}\Big(\sum_{i=1}^{n}\beta_{is}\phi_{ii_1\cdots i_{p-1}}\Big)d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_{p-1}}$$

are $\bar{\partial}$-exact $(0, p-1)$-forms for $1\leqq s\leqq q$. The induction hypothesis shows that $\sum_{i=1}^{n}\beta_{is}\phi_{ii_1\cdots i_{p-1}}=0$ for $1\leqq s\leqq q$. Then $0=\sum_{s=1}^{q}\sum_{i=1}^{n}\beta_{is}\gamma_{sk}\phi_{ii_1\cdots i_{p-1}}=\phi_{ki_1\cdots i_{p-1}}+\sum_{i=q+1}^{n}\sum_{s=1}^{q}\beta_{is}\gamma_{sk}\phi_{ii_1\cdots i_{p-1}}$ for $1\leqq k\leqq q$. Since $\phi_{ii_1\cdots i_{p-1}}=0$ for $i\geqq q+1$, $\phi_{ki_1\cdots i_{p-1}}=0$ for $1\leqq k\leqq q$. Hence $\phi=0$.

Now suppose $\phi=\sum_{m\in Z^{n+q}}\phi^m\in Z_{\bar{\partial}}(C^n/\Gamma, \mathcal{A}^{0,p})$. We have

$$\sum_{k=1}^{p+1}(-1)^{k+1}\frac{\partial\phi^m_{i_1\cdots\hat{i}_k\cdots i_{p+1}}}{\partial\bar{z}_{i_k}}=0$$

for any $m\in Z^{n+q}$. From (1.3) we obtain

$$\begin{aligned}\text{(2.3)}\qquad &\sum_{k=1}^{p+1}(-1)^{k+1}\Big\{\pi\sum_{l=1}^{q}\gamma_{li_k}K_{m,l}a^m_{i_1\cdots\hat{i}_k\cdots i_{p+1}}\\ &\quad+\sqrt{-1}\sum_{l=q+1}^{n}\gamma_{li_k}\Big(\pi m_l a^m_{i_1\cdots\hat{i}_k\cdots i_{p+1}}+\frac{1}{2}\frac{\partial a^m_{i_1\cdots\hat{i}_k\cdots i_{p+1}}}{\partial t_{n+l}}\Big)\Big\}\\ &=0.\end{aligned}$$

In (2.3) we take $(i, i_1, \cdots, i_p)$ instead of $(i_1, \cdots, i_{p+1})$. Then we have

$$\begin{aligned}&\pi\sum_{l=1}^{q}\gamma_{li}K_{m,l}a^m_{i_1\cdots i_p}+\sqrt{-1}\sum_{l=q+1}^{n}\gamma_{li}\Big(\pi m_l a^m_{i_1\cdots i_p}+\frac{1}{2}\frac{\partial a^m_{i_1\cdots i_p}}{\partial t_{n+l}}\Big)\\ &=\sum_{k=1}^{p}(-1)^{k+1}\Big\{\pi\sum_{l=1}^{q}\gamma_{li_k}K_{m,l}a^m_{ii_1\cdots\hat{i}_k\cdots i_p}+\sqrt{-1}\sum_{l=q+1}^{n}\Big(\pi m_l a^m_{ii_1\cdots\hat{i}_k\cdots i_p}\\ &\quad+\frac{1}{2}\frac{\partial a^m_{ii_1\cdots\hat{i}_k\cdots i_p}}{\partial t_{n+l}}\Big)\Big\}.\end{aligned}$$

Multiplying the above by β_{is} and adding from $i=1$ to n, we obtain

$$\begin{aligned}&\sum_{i=1}^{n}\Big\{\pi\sum_{l=1}^{q}\beta_{is}\gamma_{li}K_{m,l}a^m_{i_1\cdots i_p}+\sqrt{-1}\sum_{l=q+1}^{n}\beta_{is}\gamma_{li}\Big(\pi m_l a^m_{i_1\cdots i_p}+\frac{1}{2}\frac{\partial a^m_{i_1\cdots i_p}}{\partial t_{n+l}}\Big)\Big\}\\ &=\sum_{i=1}^{n}\sum_{k=1}^{p}(-1)^{k+1}\Big\{\pi\sum_{l=1}^{q}\beta_{is}\gamma_{li_k}K_{m,l}a^m_{ii_1\cdots\hat{i}_k\cdots i_p}+\sqrt{-1}\sum_{l=q+1}^{n}\beta_{is}\gamma_{li_k}\\ &\quad\times\Big(\pi m_l a^m_{ii_1\cdots\hat{i}_k\cdots i_p}+\frac{1}{2}\frac{\partial a^m_{ii_1\cdots\hat{i}_k\cdots i_p}}{\partial t_{n+l}}\Big)\Big\}.\end{aligned}$$

We put

$$a^{m,s}_{i_1\cdots i_{p-1}}:=\sum_{i=1}^{n}\beta_{is}a^{m}_{ii_1\cdots i_{p-1}}.$$

Thus we have

$$\pi K_{m,s}a^{m}_{i_1\cdots i_p}=\sum_{k=1}^{p}(-1)^{k+1}\Big\{\pi\sum_{l=1}^{q}\gamma_{li_k}K_{m,l}a^{m,s}_{i_1\cdots\hat{i}_k\cdots i_p}+\sqrt{-1}\sum_{l=q+1}^{n}\gamma_{li_k}\Big(\pi m_l a^{m,s}_{i_1\cdots\hat{i}_k\cdots i_p}+\frac{1}{2}\frac{\partial a^{m,s}_{i_1\cdots\hat{i}_k\cdots i_p}}{\partial t_{n+l}}\Big)\Big\} \tag{2.4}$$

for $1\leqq s\leqq q$. Let $m\in Z^{n+q}-\{0\}$ and

$$s(m):=\min\{s\ ;\ |K_{m,s}|=K_m,\ 1\leqq s\leqq q\}.$$

Since $K_m=|K_{m,s(m)}|>0$ for $m\in Z^{n+q}-\{0\}$, we put

$$b^{m,s(m)}_{i_1\cdots i_{p-1}}:=a^{m,s(m)}_{i_1\cdots i_{p-1}}/\pi K_{m,s(m)}$$

and

$$\phi^{m,s(m)}:=\frac{1}{(p-1)!}\sum_{1\leq i_1,\cdots,i_{p-1}\leq n}b^{m,s(m)}_{i_1\cdots i_{p-1}}(t'')\times\exp 2\pi\sqrt{-1}\langle m,t'\rangle d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_{p-1}}. \tag{2.5}$$

Then, from (2.2) and (2.4), we have the following

LEMMA 2.2. *Let $\phi=\sum_{m\in Z^{n+q}}\phi^m$ be a real analytic $\bar{\partial}$-closed (0, p)-form on an (H, C)-group C^n/Γ. Take the (0, p−1)-form $\phi^{m,s(m)}$ defined by* (2.5) *for $m\in Z^{n+q}-\{0\}$. Then*

$$\phi^m=\bar{\partial}\phi^{m,s(m)}\qquad \text{for}\quad m\in Z^{n+q}-\{0\}.$$

In the case $m=0$ we get the following

LEMMA 2.3. *Let*

$$\phi^0=\frac{1}{p!}\sum_{1\leq i_1,\cdots,i_p\leq n}a^0_{i_1\cdots i_p}(t'')d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p}$$

be a real analytic $\bar{\partial}$-closed (0, p)-form on an (H, C)-group C^n/Γ. Then there exist a unique (0, p)-form

$$\chi=\frac{1}{p!}\sum_{1\leq i_1,\cdots,i_p\leq n}c_{i_1\cdots i_p}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p}$$

and a (0, p−1)-form

$$\psi=\frac{1}{(p-1)!}\sum_{1\leq i_1,\cdots,i_{p-1}\leq n}b_{i_1\cdots i_{p-1}}(t'')d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_{p-1}}$$

on C^n/Γ such that (1) *$\phi^0=\chi+\bar{\partial}\psi$,* (2) *for any $1\leqq i_1,\cdots,i_p\leqq n$, $c_{i_1\cdots i_p}$ is constant and* (3) *if $\{i_1,\cdots,i_p\}\cap\{q+1,\cdots,n\}\neq\emptyset$, $c_{i_1\cdots i_p}=0$.*

PROOF. The uniqueness of χ immediately follows by Lemma 2.1. We shall show the existence of χ and ψ. We set

$$l:=\max\{i_k\ ;\ 1\leqq i_k\leqq n,\ 1\leqq k\leqq p \text{ and } a^0_{i_1\cdots i_p}\neq 0\}.$$

The proof will be by induction on l, $l\geqq p$ with a fixed $p\geqq 1$. In the case $l=p$, we have $\phi^0=a^0_{12\cdots p}(t'')d\bar{z}_1\wedge\cdots\wedge d\bar{z}_p$. Since $\bar{\partial}\phi^0=0$, $\partial a^0_{12\cdots p}/\partial\bar{z}_r=0$ for $r>p$. If $p\leqq q$, by (1.3) $\partial a^0_{12\cdots p}/\partial\bar{z}_r=(\sqrt{-1}/2)\partial a^0_{12\cdots p}/\partial t_{n+r}=0$ for $q+1\leqq r\leqq n$. This means $a^0_{12\cdots p}$ is constant. If $p\geqq q+1$, we put

$$d_{12\cdots p-1}:=-2\sqrt{-1}\int_0^{t_{n+p}}a^0_{12\cdots p}(t_{n+q+1},\ \cdots,\ t_{n+p-1},\ \tau,\ t_{n+p+1},\ \cdots,\ t_{2n})d\tau\,.$$

Then

$$\bar{\partial}((-1)^{p-1}d_{12\cdots p-1}d\bar{z}_1\wedge\cdots\wedge d\bar{z}_{p-1})=a^0_{12\cdots p}d\bar{z}_1\wedge\cdots\wedge d\bar{z}_p=\phi^0\,.$$

Thus the lemma is proved for $l=p$. Now assume the lemma is proved for $l-1$, $l>p$. We write

$$\phi^0=\frac{1}{p\,!}\sum_{1\leq i_1,\cdots,i_p\leq l-1}a^0_{i_1\cdots i_p}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p}$$
$$+\frac{1}{(p-1)\,!}\sum_{1\leq i_2,\cdots,i_p\leq l-1}a^0_{li_2\cdots i_p}d\bar{z}_l\wedge d\bar{z}_{i_2}\wedge\cdots\wedge d\bar{z}_{i_p}\,.$$

Since $\bar{\partial}\phi^0=0$,

$$(*)\qquad \frac{\partial a^0_{li_2\cdots i_p}}{\partial\bar{z}_r}=0,\qquad \text{for}\quad l+1\leqq r\leqq n\,.$$

If $l\leqq q$,

$$\frac{\partial a^0_{li_2\cdots i_p}}{\partial\bar{z}_r}=\frac{\sqrt{-1}}{2}\frac{\partial a^0_{li_2\cdots i_p}}{\partial t_{n+r}}=0\qquad \text{for}\quad q+1\leqq r\leqq n\,.$$

This means $a^0_{li_2\cdots i_p}$ are constant. And thus

$$\frac{1}{p\,!}\sum_{1\leq i_1,\cdots,i_p\leq l-1}a^0_{i_1\cdots i_p}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p}$$
$$=\phi^0-\frac{1}{(p-1)\,!}\sum_{1\leq i_2,\cdots,i_p\leq l-1}a^0_{li_2\cdots i_p}d\bar{z}_l\wedge d\bar{z}_{i_2}\wedge\cdots\wedge d\bar{z}_{i_p}$$

is a $\bar{\partial}$-closed $(0,\,p)$-form satisfying the induction hypothesis; then the lemma is proved for $l\leqq q$. If $l>q$, we put

$$e_{i_2\cdots i_p}(t''):=\frac{2}{\sqrt{-1}}\int_0^{t_{n+l}}a^0_{li_2\cdots i_p}(t_{n+q+1},\ \cdots,\ t_{n+l-1},\ \tau,\ t_{n+l+1},\ \cdots,\ t_{2n})d\tau$$

and

$$\omega:=\frac{1}{(p-1)\,!}\sum_{1\leq i_2,\cdots,i_p\leq l-1}e_{i_2\cdots i_p}(t'')d\bar{z}_{i_2}\wedge\cdots\wedge d\bar{z}_{i_p}\,.$$

We have by (*) $\partial e_{i_2\cdots i_p}/\partial\bar{z}_r=0$ for $l+1\leqq r\leqq n$, and

$$\frac{\partial e_{i_2\cdots i_p}}{\partial\bar{z}_l}=\frac{\sqrt{-1}}{2}\frac{\partial e_{i_2\cdots i_p}}{\partial t_{n+l}}=a^0_{li_2\cdots i_p}.$$

Thus the form $\phi^0-\bar{\partial}\omega$ is $\bar{\partial}$-closed and satisfies the induction hypothesis. Then the lemma is proved for all l.

Summarizing Lemmas 2.1, 2.2 and 2.3, we have the following

PROPOSITION 2.4. *Let*

$$\phi=\frac{1}{p!}\sum_{1\leq i_1,\cdots,i_p\leq n}\sum_{m\in Z^{n+q}}a^m_{i_1\cdots i_p}(t'')\exp 2\pi\sqrt{-1}\langle m,\ t'\rangle d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p}$$

be a $\bar{\partial}$-closed real analytic $(0, p)$-form on an (H, C)-group C^n/Γ. Put

$$\psi^{m,s(m)}:=\frac{1}{(p-1)!}\sum_{1\leq i_1,\cdots,i_{p-1}\leq n}\Big(\sum_{i=1}^{n}\beta_{is(m)}a^m_{ii_1\cdots i_{p-1}}/\pi K_{m,s(m)}\Big)$$
$$\exp 2\pi\sqrt{-1}\langle m,\ t'\rangle d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_{p-1}}$$

for $m\in Z^{n+q}-\{0\}$, where $s(m):=\min\{s\ ;\ |K_{m,s}|=K_m,\ 1\leqq s\leqq q\}$. Then there exist a unique $(0, p)$-form

$$\chi=\frac{1}{p!}\sum_{1\leq i_1,\cdots,i_p\leq q}c_{i_1\cdots i_p}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p}$$

and a real analytic $(0, p-1)$-form

$$\psi^0=\frac{1}{(p-1)!}\sum_{1\leq i_1,\cdots,i_{p-1}\leq n}b^0_{i_1\cdots i_{p-1}}(t'')d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_{p-1}}$$

on C^n/Γ such that

(1) $\phi=\chi+\bar{\partial}\psi^0+\sum_{m\in Z^{n+q}-\{0\}}\bar{\partial}\psi^{m,s(m)}$

(2) $c_{i_1\cdots i_p}$ *is constant for any* $1\leqq i_1,\cdots,i_p\leqq q$.

REMARK. Proposition 2.4 implies that for any $\phi\in Z_{\bar{\partial}}(C^n/\Gamma,\ \mathcal{A}^{0,p})$, $p\geqq 1$ the $\bar{\partial}$-equation

$$\phi\equiv\bar{\partial}\psi\quad \mathrm{mod}\{\sum_{1\leq i_1,\cdots,i_p\leq q}c_{i_1\cdots i_p}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p}\ ;\ c_{i_1\cdots i_p}\in C\}$$

has usually a formal solution ψ.

3. (H, C)-groups which have finite-dimensional cohomology groups.

We find a condition for an (H, C)-group G to have the finite-dimensional cohomology groups $H^p(G, \mathcal{O})$, $p>0$ as follows.

THEOREM 3.1. *Let C^n/Γ be an (H, C)-group where Γ is generated by $\{e_1,\cdots,e_n,v_1,\cdots,v_q\}$ and let $K_{m,s}:=\sum_{j=1}^{n}m_jv_{js}-m_{n+s}$, $1\leqq s\leqq q$ and $K_m:=\max\{|K_{m,s}|\ ;$*

$1 \leqq s \leqq q\}$ *for* $m \in Z^{n+q}$. *For any* $\varepsilon > 0$ *if there exists a positive number C such that*

$$\exp(-\varepsilon|m|) \leqq CK_m \qquad \text{for any} \quad m \in Z^{n+q} - \{0\},$$

then

$$\dim H^p(C^n/\Gamma, \mathcal{O}) = \begin{cases} \binom{q}{p} & q \geqq p \geqq 1 \\ 0 & p > q. \end{cases}$$

PROOF. We identify C^n/Γ with $T^{n+q} \times R^{n-q}$ as a real Lie group as in §2. We put

$$s(m) := \min\{s \ ; \ |K_{m,s}| = K_m, \ 1 \leqq s \leqq q\},$$

for $m \in Z^{n+q} - \{0\}$. We take

$$\phi = \frac{1}{p!} \sum_{1 \leq i_1, \cdots, i_p \leq n} \phi_{i_1 \cdots i_p} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} \in Z_{\bar{\partial}}(C^n/\Gamma, \mathcal{A}^{0,p})$$

with the Fourier expansions

$$\phi_{i_1 \cdots i_p} = \sum_{m \in Z^{n+q}} a^m_{i_1 \cdots i_p}(t'') \exp 2\pi\sqrt{-1}\langle m, t' \rangle .$$

By Lemma 1.2 there exist a neighbourhood V of R^{n-q} in C^{n-q} and holomorphic functions $a^m_{*i_1 \cdots i_p}$ in V for all $m \in Z^{n+q}$ such that $a^m_{*i_1 \cdots i_p}|_{R^{n-q}} = a^m_{i_1 \cdots i_p}$ and to every compact subset K of V we have positive numbers C and ε satisfying $\sup_K |a^m_{*i_1 \cdots i_p}| \leqq C \exp(-\varepsilon|m|)$ for any $m \in Z^{n+q}$ and $1 \leqq i_1, \cdots, i_p \leqq n$. Since $K_{m,s(m)} \neq 0$ for any $m \in Z^{n+q} - \{0\}$ by Proposition 1.3, we put

$$b^m_{*i_1 \cdots i_{p-1}} := \sum_{i=1}^{n} \beta_{is(m)} a^m_{*ii_1 \cdots i_{p-1}} / \pi K_{m,s(m)}$$

and $b^m_{i_1 \cdots i_{p-1}} := b^m_{*i_1 \cdots i_{p-1}}|_{R^{n-q}}$. From the assumption of the theorem there exists a positive number C' such that $\exp\left(-\frac{\varepsilon}{2}|m|\right) \leqq C'|K_{m,s(m)}|$ for any $m \in Z^{n+q} - \{0\}$. Thus we have

$$\sup_K |b^m_{*i_1 \cdots i_{p-1}}| \leqq C\,C'\left(\sum_{i=1}^{n} |\beta_{is}|\right) \exp\left(-\frac{\varepsilon}{2}|m|\right)$$

for any $m \in Z^{n+q} - \{0\}$. This means by Lemma 1.2

$$\psi_{i_1 \cdots i_{p-1}} = \sum_{m \in Z^{n+q} - \{0\}} b^m_{i_1 \cdots i_{p-1}}(t'') \exp 2\pi\sqrt{-1}\langle m, t' \rangle$$

is real analytic on C^n/Γ. By Proposition 2.4 we have a real analytic $(0, p-1)$-form

$$\psi^0 = \frac{1}{(p-1)!} \sum_{i_1, \cdots, i_{p-1}} b^0_{i_1 \cdots i_{p-1}}(t'') d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_{p-1}}$$

on C^n/Γ with

$$\phi-\bar{\partial}\Big(\psi^{0}+\frac{1}{(p-1)!}\sum_{i_1,\cdots,i_{p-1}}\psi_{i_1\cdots i_{p-1}}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_{p-1}}\Big)$$
$$=\sum_{1\leq i_1<i_2\cdots<i_p\leq q}c_{i_1i_2\cdots i_p}d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_p},\quad \text{where } c_{i_1i_2\cdots\cdots i_p}\in C.$$

Hence $\dim H^p(C^n/\Gamma, \mathcal{O})=\binom{q}{p}$ if $q\geqq p\geqq 1$ and $\dim H^p(C^n/\Gamma, \mathcal{O})=0$ if $p>q$.

It is well-known that $\dim H^p(T^n_C, \mathcal{O})=\binom{n}{p}$ for any complex torus T^n_C of dimension n. This fact is obtained by the theory of harmonic integrals (See [9]). By Theorem 3.1, we give another proof of the fact not using the theory of harmonic integrals.

COROLLARY 3.2. *Let T^n_C be a complex torus of dimension n. Then* $\dim H^p(T^n_C, \mathcal{O})=\binom{n}{p}$.

PROOF. We may regard T^n_C as an (H, C)-group C^n/Γ, where Γ is the subgroup generated by $\{e_1, \cdots, e_n, v_1, \cdots, v_n\}$. We put

$$S:=\{\sum_{i=1}^{n}m_i(v_{i1}, \cdots, v_{in})+\sum_{i=1}^{n}m_{n+i}{}^te_i\ ;\ \text{for any } m\in Z^{2n}-\{0\}\},$$

where ${}^te_i=(\delta_{i1}, \cdots, \delta_{in})$. Since $(v_{i1}, \cdots, v_{in})$, ${}^te_j$, $1\leqq i, j\leqq n$ are linearly independent over R, then $0\notin S$ and S is a discrete subset of C^n. Then

$$\rho:=\min\{\sqrt{|u_1|^2+\cdots+|u_n|^2}\ ;\ (u_1, \cdots, u_n)\in S\}>0.$$

Since $(K_{m,1}, \cdots, K_{m,n})=(\sum_{j=1}^{n}m_jv_{j1}-m_{n+1}, \cdots, \sum_{j=1}^{n}m_jv_{jn}-m_{2n})\in S$ for $m\in Z^{2n}-\{0\}$, $K_m=\max\{|K_{m,s}|\ ;\ 1\leqq s\leqq n\}\geqq\rho/\sqrt{n}>0$ for any $m\in Z^{2n}-\{0\}$. This shows that $\{K_m\}$ satisfies the assumption of Theorem 3.1.

Finally we give in the following an example of an (H, C)-group G with the infinite-dimensional cohomology group $H^1(G, \mathcal{O})$. To give the example we need to topologize $H^p(G, \mathcal{O})$. Let $\mathcal{A}(R)$ be the vector space of (complex valued) real analytic functions on R. We regard R as a closed real analytic submanifold of C under the natural inclusion. We take a compact subset K of R and an open and connected neighbourhood U_j of K in C for $1\leqq j\leqq\infty$ satisfying $U_{j+1}\Subset U_j$ and $\bigcap_j U_j=K$. Let $\mathcal{A}(K)$ be the vector space of real analytic functions in a neighbourhood of K in R. We denote by $\mathcal{H}(U_j)$ the space of bounded holomorphic functions on U_j for $j\geqq 1$. Put $\|f\|:=\sup\{|f(z)|\ ;\ z\in U_j\}$ for $f\in\mathcal{H}(U_j)$. This norm makes $\mathcal{H}(U_j)$ into a Banach space. By the inductive limit $\mathcal{A}(K)=\operatorname{ind\,lim}\mathcal{H}(U_j)$ we regard $\mathcal{A}(K)$ as a (D, F, S)-space. The restriction mapping: $\mathcal{A}(K_1)\to\mathcal{A}(K_2)$, $K_2\subset K_1$ induces the projective limit $\mathcal{A}(R)=\operatorname{proj\,lim}\mathcal{A}(K)$. It is known that the above locally convex topology on $\mathcal{A}(R)$ is complete and semi-Montel. Similarly we can make the vector space $H^0(G, \mathcal{A}^{p,q})$ of real analytic (p, q)-forms on an

(H, C)-group G into a complete and semi-Montel locally convex space. Thus the closed subspace $Z_{\bar{\partial}}(G, \mathcal{A}^{p,q})$ of $H^0(G, \mathcal{A}^{p,q})$ is also a complete and semi-Montel locally convex space.

EXAMPLE. Let $\alpha \in R$ and Γ the discrete subgroup of C^n generated by $e_1 = {}^t(\delta_{11}, \cdots, \delta_{n1})$, $e_2 = {}^t(\delta_{12}, \cdots, \delta_{n2})$, $\cdots$, $e_n = {}^t(\delta_{1n}, \cdots, \delta_{nn})$, $v_1 = \sqrt{-1}\, e_1, \cdots, v_{n-2} = \sqrt{-1}\, e_{n-2}$ and $v_{n-1} = \sqrt{-1}\, e_{n-1} + \alpha e_n$. By the definition of $K_{m,s}$ we have

$$K_{m,s} = \begin{cases} m_s\sqrt{-1} - m_{n+s} & 1 \leqq s \leqq n-2 \\ m_{n-1}\sqrt{-1} + m_n\alpha - m_{2n-1} & s = n-1. \end{cases}$$

By Proposition 1.3 C^n/Γ is an (H, C)-group if and only if α is irrational. Now suppose α is irrational and algebraic. Then by Liouville's theorem there exist $M>0$ and a positive integer l such that $\left|\alpha - \frac{p}{q}\right| > M/|q|^l$ for any $p, q \in Z$ with $q \neq 0$. Since $K_m = \max_{1 \leqq s \leqq n-1} |K_{m,s}| \geqq \min\{1, |m_n\alpha - m_{2n-1}|\} \geqq \min\{1, M/|m_n|^{l-1}\}$, $m_n \neq 0$, for any $\varepsilon > 0$ there exists $C>0$ such that $CK_m \geqq \exp(-\varepsilon|m|)$ for any $m \in Z^{2n-1} - \{0\}$. By Theorem 3.1 we have $\dim H^p(C^n/\Gamma, \mathcal{O}) = \binom{n-1}{p}$, $1 \leqq p \leqq n-1$. Next we choose a sequence $\{r_0, r_1, r_2, \cdots ;\ r_0 < r_1 < r_2 < \cdots\}$ of integers such that $r_0 := 0$ and

$$r_{k+1} \geqq r_k + 1 + (\log 10)^{-1} k^2 \sqrt{(10^{r_k})^2 + (\sum_{j=0}^{k} 10^{r_k - r_j})^2},$$

$k = 0, 1, 2, \cdots$. We put $\alpha := \sum_{j=0}^{\infty} 10^{-r_j}$,

$$m_i^{(k)} := \begin{cases} 0 & 1 \leqq i \leqq n-1 \text{ and } n+1 \leqq i \leqq 2n-2 \\ 10^{r_k} & i = n \\ \sum_{j=0}^{k} 10^{r_k - r_j} & i = 2n-1 \end{cases}$$

and $m^{(k)} := (m_1^{(k)}, \cdots, m_{2n-1}^{(k)})$ for $k = 0, 1, 2, \cdots$. Then

$$K_{m^{(k)}} = |m_n^{(k)}\alpha - m_{2n-1}^{(k)}| = |\sum_{j=k+1}^{\infty} 10^{r_k - r_j}| \leqq 10^{r_k - r_{k+1} + 1} \leqq \exp(-k^2|m^{(k)}|).$$

We put

$$\phi^m(t'') := \begin{cases} \exp(-|k|\,|m^{(k)}| - 2\pi m_n^{(k)} t_{2n})/K_{m^{(k)}} \\ \qquad\qquad \text{if } m = m^{(k)} \text{ for some } k \geqq 0 \\ 0 \qquad\qquad \text{if } m \notin \{m^{(k)} ;\ k = 0, 1, 2, \cdots\}. \end{cases}$$

From (1.3) we have

$$\bar{\partial}(\phi^m(t'') \exp 2\pi\sqrt{-1}\langle m, t'\rangle) = \pi \sum_{s=1}^{n-1} K_{m,s}\phi^m \exp 2\pi\sqrt{-1}\langle m, t'\rangle d\bar{z}_s.$$

Let

$$\psi_*^m(w) := \begin{cases} \exp(-|k||m^{(k)}| - 2\pi m_n^{(k)} w)/K_{m^{(k)}} \\ \qquad \text{if} \quad m = m^{(k)} \quad \text{for some} \quad k \geqq 0 \\ 0 \qquad \text{if} \quad m \notin \{m^{(k)} ; k = 0, 1, 2, \cdots\} \end{cases}$$

for $w \in C$. Then $\psi_*^m|_R = \psi^m$, where $t_{2n} = \mathrm{Re}\, w$. Since $|K_{m,s}| \leqq K_m$, we have

$$|\pi \sum_{s=1}^{n-1} K_{m^{(k)},s} \psi_*^m| \leqq \pi(n-1)\exp(-|k||m^{(k)}| + 2\pi m_n^{(k)} |t_{2n}|),$$

where $t_{2n} = \mathrm{Re}\, w$. By Lemma 1.2 we have the real analytic (0, 1)-form

$$\phi := \sum_{m \in Z^{2n-1}} \bar{\partial}(\psi^m \exp 2\pi\sqrt{-1}\langle m, t'\rangle)$$

on C^n/Γ. It is easy to check

$$\phi = \lim_{N \to \infty} \bar{\partial}(\sum_{|m| \leqq N} \psi^m \exp 2\pi\sqrt{-1}\langle m, t'\rangle) \in \overline{B_{\bar{\partial}}(C^n/\Gamma, \mathcal{A}^{0,1})},$$

where $\overline{B_{\bar{\partial}}(C^n/\Gamma, \mathcal{A}^{0,1})}$ is the closure of $B_{\bar{\partial}}(C^n/\Gamma, \mathcal{A}^{0,1})$ with respect to the locally convex topology of $H^0(C^n/\Gamma, \mathcal{A}^{0,1})$. Suppose $\phi \in B_{\bar{\partial}}(C^n/\Gamma, \mathcal{A}^{0,1})$. Then there exists a real analytic function

$$\lambda = \sum_{m \in Z^{2n-1}} \lambda^m(t'') \exp 2\pi\sqrt{-1}\langle m, t'\rangle$$

such that $\phi = \bar{\partial}\lambda$. This means

$$\bar{\partial}(\lambda^m \exp 2\pi\sqrt{-1}\langle m, t'\rangle) = \bar{\partial}(\psi^m \exp 2\pi\sqrt{-1}\langle m, t'\rangle).$$

Since $H^0(C^n/\Gamma, \mathcal{O}) = C$, there exists a constant $c \in C$ such that $\lambda^0 = c + \psi^0$ and $\lambda^m = \psi^m$ for any $m \in Z^{2n-1} - \{0\}$. Hence $\lim_{N \to \infty} \sum_{|m| < N} \lambda^m \exp 2\pi\sqrt{-1}\langle m, t'\rangle$ is not convergent on any real analytic function. It is a contradiction. Thus $B_{\bar{\partial}}(C^n/\Gamma, \mathcal{A}^{0,1})$ is not closed in $Z_{\bar{\partial}}(C^n/\Gamma, \mathcal{A}^{0,1})$ and $H^1(C^n/\Gamma, \mathcal{O})$ is infinite-dimensional.

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Supplementary notes. After this paper was submitted, the referee informed the authors that C. Vogt obtained similar results concerning $H^1(C^n/\Gamma, \mathcal{O})$ in the following paper: Line bundles on toroidal groups, J. Reine Angew. Math., **335** (1982), 197-215.