J. Math. Soc. Japan
Vol. 36, No. 1, 1984

On the structure of polarized manifolds with total deficiency one, III

By Takao FUJITA

(Received Jan. 8, 1983)

Introduction.

In this article we will study polarized manifolds (M, L) with $d(M, L)=\Delta(M, L)=1$, as a continuation of [**F4**]. But the arguments are completely independent of part II of it, and little knowledge of part I is required here. Moreover we consider here positive characteristic cases too, with the help of [**F5**].

In § 13, the first section of this part III, we study the structure of the rational mapping defined by $|L|$. It follows that $g=g(M, L)\geqq 1$. In § 14, assuming $\mathrm{char}(\mathfrak{K})\neq 2$ for the ground field $\mathfrak{K}$ from this time on throughout in this paper, we establish a precise structure theorem for (M, L) with $g=1$. When $g\geqq 2$, in general, we do not have so precise a result as in the case $g=1$. So we consider the case in which any curve $C=D_1\cap\cdots\cap D_{n-1}$ obtained by taking general members $D_1, \cdots, D_{n-1}$ of $|L|$ successively is a hyperelliptic curve. Such (M, L) will be said to be *sectionally hyperelliptic* (note that this is always the case when $g=2$). In § 15, they are classified into three types $(-)$, (∞) and $(+)$. Precise structures of them are described in § 16, § 17 and § 18 respectively. In particular, it turns out that $n=\dim M=2$ in case of type $(+)$, $n\leqq g+1$ in case of type (∞), and (M, L) is a weighted hypersurface of degree $4g+2$ in $\boldsymbol{P}(2g+1, 2, 1, \cdots, 1)$ in case of type $(-)$. In any case M is simply connected if $\mathfrak{K}=\boldsymbol{C}$. Moreover, all the (M, L) of the same type ($(-)$, (∞) or $(+)$) and with the same n and g form a single deformation family. It is easy to calculate the number of moduli of it.

Thus, when $\mathrm{char}(\mathfrak{K})\neq 2$, the classification theory of polarized manifolds (M, L) with $\Delta(M, L)=1$ is complete except the case $d(M, L)=1$, $g(M, L)\geqq 3$ and (M, L) is not sectionally hyperelliptic. In particular, all the Del Pezzo manifolds are completely classified.

This work was almost completed when the author was a Miller Fellow at the University of California, Berkeley. He would like to express his hearty thanks to people there, especially to Professors R. Hartshorne and A. Ogus.

§ 13. General case.

The notation in this part III is the same as in [**F5**], and is almost the same as in [**F4**] except that the ground field $\mathfrak{K}$ may be of any characteristic. We will study the structure of polarized manifolds (M, L) such that $d(M, L)=\Delta(M, L)=1$, $n=\dim M\geqq 2$ and $g(M, L)=g$.

(13.1) Since $h^0(M, L)=n$, $\rho_{|L|}$ is a rational mapping to $\boldsymbol{P}^{n-1}=P$. So $X=\mathrm{Bs}|L|\neq\emptyset$. On the other hand, $\dim X\leqq 0$ by [**F5**; (2.1)]. Therefore, if $D_1, D_2, \cdots, D_n$ are general members of $|L|$ and if we let $V_i=\bigcap_{j>i}^{n} D_j$, then $\dim V_i=i$ and $\{V_i\}$ give a ladder of (M, L). Of course $X=V_0$, which is a simple point on M, because $D_1\cdots D_n=d(M, L)=1$. Therefore D_j's meet transversally at the point X. In particular, V_i is non-singular at X. Hence, if $\mathrm{char}(\mathfrak{K})=0$, we can take $\{D_j\}$ so that V_i's are non-singular by Bertini's theorem.

(13.2) By virtue of [**F5**; (4.16)], we infer that $g\geqq 1$. So $\{V_i\}$ must be a regular ladder of (M, L) by [**F5**; (3.6)].

(13.3) Let $\pi: M^*\to M$ be the blowing-up with center X and let E be the exceptional divisor lying over X. Let H_j be the proper transform of D_j on M^*. Then, $H_1\cap\cdots\cap H_n=\emptyset$ by (13.1). So $\mathrm{Bs}|\pi^*L-E|=\emptyset$. Thus we get a morphism $f=\rho_{|\pi^*L-E|}: M^*\to P\cong\boldsymbol{P}_{\mathfrak{k}}^{n-1}$. Clearly E is a section of f, and M^* is identified with the graph of $\rho_{|L|}$.

(13.4) For every point x on P, the fiber $C_x=f^{-1}(x)$ is an irreducible reduced curve. Indeed, the mapping $C_x\to\pi(C_x)$ is a finite morphism since $C_x\cap E$ is a point. So L is ample on C_x. On the other hand, $L-E=H_{\xi}=0$ in $\mathrm{Pic}(C_x)$. Hence the restriction of E to C_x is an ample divisor. So C_x is an irreducible reduced curve, because $EC_x=1$.

Consequently f is flat.

(13.5) We easily see that V_1 is isomorphic to $H_2\cap\cdots\cap H_n$, which is a fiber of f. We have $h^1(V_1, \mathcal{O})=g(V_1, L)=g$. Since f is flat, every fiber of f is a curve of arithmetic genus g.

(13.6) Combining the preceding observations, we obtain:

THEOREM. *Let (M, L) be a polarized manifold with $n=\dim M$, $d(M, L)=\Delta(M, L)=1$. Then $X=\mathrm{Bs}|L|$ consists of one simple point. Let $\pi: M^*\to M$ be the blowing-up of M with center X and let E be the exceptional divisor over X. Then $\mathrm{Bs}|\pi^*L-E|=\emptyset$. This linear system defines a flat morphism f from M^* onto $P\cong\boldsymbol{P}_{\mathfrak{k}}^{n-1}$. E is a section of f, and every fiber of f is an irreducible reduced curve of arithmetic genus $g=g(M, L)\geqq 1$.*

REMARK. If $\mathrm{char}(\mathfrak{K})=0$, any general fiber of f is smooth by Bertini's theorem.

(13.7) Conversely, suppose that there is a flat morphism $f: N^*\to P\cong\boldsymbol{P}_{\mathfrak{k}}^{n-1}$

which satisfies the following conditions:

a) Every fiber of f is an irreducible reduced curve of arithmetic genus $g \geqq 1$.

b) There exists a section E of f, such that E can be blown down to a smooth point on another manifold N.

Then we get a polarized manifold (N, L) with $d(N, L)=\Delta(N, L)=1$ and $g(N, L)=g$ in the following way.

By the condition b), we have $E \cong P$ and $[E]_E=-H$, where H is the hyperplane section bundle $\mathcal{O}(1)$. Set $L^*=f^*H_\xi+E \in \mathrm{Pic}(N^*)$. Then $L^*_E=0$. So L^* is the pull-back of a line bundle on N, which is denoted by L. We have $L^n-1=(L^*-E)^n=0$ and $h^0(N, L)=h^0(N^*, L^*)=h^0(N^*, f^*H)=n$ because $g \geqq 1$ implies that E is a fixed part of $|L^*|$. So $d(N, L)=\Delta(N, L)=1$. It is easy to see that (N, L) has a ladder $\{V_j\}$ such that V_1 is isomorphic to a fiber of f. Therefore $g(N, L)=g(V_1)=g$. Now, it is enough to show the ampleness of L.

Similarly as in [**F3**; (5.7)], it suffices to prove that L_Z is strictly effective for any subvariety Z of N^* which is not contained in E. Clearly $L_Z=H_Z+E_Z$ is effective. If it is not strict, we must have $H_Z=E_Z=0$. The former implies that Z is (contained in) a fiber of f, which is impossible if $E_Z=0$ because E is a section of f. Thus L_Z is strictly effective, as required.

REMARK. Similar construction is possible without the assumption $g \geqq 1$. If $g=0$, we have $h^0(N, L)>n$ and $\Delta \leqq 0$. Hence $(N, L) \cong (\boldsymbol{P}^n, \mathcal{O}(1))$.

§ 14. The case $g=1$.

(14.1) Let the notation be as in (13.6). In this section we make a detailed analysis of the case $g=1$, assuming $\mathrm{char}(\mathfrak{K}) \neq 2$. As we saw in [**F5**; (5.7)], (M, L) is a Del Pezzo manifold, while we do not need this result in the sequel.

(14.2) Let $\mathcal{D}=\mathcal{O}_{M^*}(2L)$ and set $\mathcal{F}=f_*\mathcal{D}$. Since every fiber C_x of f over $x \in P$ is an irreducible reduced curve of arithmetic genus one, the restriction $\mathcal{D}_x$ of $\mathcal{D}$ to C_x is generated by global sections and $h^0(C_x, \mathcal{D}_x)=2$. Therefore $\mathcal{F}$ is locally free of rank two, and the natural homomorphism $f^*\mathcal{F} \to \mathcal{D}$ is surjective. This gives a morphism $\rho: M^* \to \boldsymbol{P}(\mathcal{F})$ such that $\rho^*\mathcal{O}(1)=\mathcal{D}$. $V=\boldsymbol{P}(\mathcal{F})$ is a $\boldsymbol{P}^1$-bundle over P. ρ is a finite morphism of degree two, because so is $\rho_x: C_x \to V_x \cong \boldsymbol{P}^1$ for every $x \in P$. By virtue of [**F4**; (2.3)] and [**F6**; (2.6)], we infer that ρ makes M^* a double covering of V with branch locus B, which is a non-singular divisor on V. So $M^* \cong R_B(V)$ in the notation of [**F4**] and [**F6**].

(14.3) E is a component of the ramification divisor R of ρ. Indeed, for every $x \in P$, $E_x=E \cap C_x$ is a ramification point of ρ_x, because ρ_x is the rational mapping defined by $|2L_x|$ and $L_x=E_x$ in $\mathrm{Pic}(C_x)$. Hence $S=\rho(E)$ is a component of B. Note that S is a section of $p: V \to P$, because E is a section of f.

(14.4) Let H_ζ be the tautological line bundle $\mathcal{O}(1)$ on $V=\boldsymbol{P}(\mathcal{F})$. Then $[H_\zeta]_S$

$=0$ since $2L_E=0$, $\rho^*H_\zeta=2L$ and $E\cong S$. So, taking p_* of $0\to\mathcal{O}_V[H_\zeta-S]\to\mathcal{O}_V[H_\zeta]\to\mathcal{O}_S[H_\zeta]\to 0$, we get an exact sequence $0\to\mathcal{O}_P[tH_\xi]\to\mathcal{F}\to\mathcal{O}_P\to 0$ on P, where t is an integer. The normal bundle of S in V is $H_\zeta-tH_\xi=\mathcal{O}_S(-t)$. On the other hand, we have $[B]_S\cong[2R]_E=[2E]_E=\mathcal{O}(-2)$. Thus we get $t=2$. Hence the above exact sequence on P splits for any $n\geqq 2$. In particular $\mathcal{F}\cong 2H_\xi\oplus[0]$ and $S\in|H_\zeta-2H_\xi|$.

(14.5) Set $B_2=B-S$ and $[B_2]=aH_\zeta+bH_\xi$ in $\mathrm{Pic}(V)$. $B_2\cap S=\emptyset$ because B is non-singular. This implies $b=0$. We have also $a=B_2V_x=3$, because B_x is the branch locus of ρ_x. Thus we have $B_2\in|3H_\zeta|$. Moreover, we easily see $H^1(V, -3H_\zeta)=0$, which implies that B_2 is connected.

(14.6) The preceding arguments altogether prove the following

THEOREM. *Let things be as in* (13.6) *and suppose in addition that* $g=1$ *and* $\mathrm{char}(\mathfrak{K})\neq 2$. *Then* M^* *is a double branched covering of* $V=\boldsymbol{P}(2H_\xi\oplus[0])$. *Let* H_ζ *be the tautological line bundle on* V. *Then the branch locus* B *of* $\rho: M^*\to V$ *consists of two connected components* B_1 *and* B_2, *where* B_1 *is the unique member* S *of* $|H_\zeta-2H_\xi|$ *and* B_2 *is a non-singular member of* $|3H_\zeta|$. *Furthermore,* S *is the image of* E *via the morphism* ρ, *and is a section of* p: $V\to P\cong\boldsymbol{P}_\xi^{n-1}$.

(14.7) Sectionally hyperelliptic polarized manifolds of type $(-)$ defined in the next section turn out to have similar structures as above. So, further investigations of such (M, L) will be done in § 16.

§ 15. Sectionally hyperelliptic cases.

(15.1) Let things be as in (13.6) and assume further that $g\geqq 2$. Let ω be the canonical dualizing sheaf of M^* and set $\mathcal{F}_t=f_*(\omega^{\otimes t})$ for each positive integer t. The restriction of ω to every fiber C_x of f over $x\in P$ is the dualizing sheaf of C_x. Hence $h^0(C_x, \omega_x^{\otimes t})$ is independent of x. This implies that $\mathcal{F}_t$ is a locally free sheaf on P. Furthermore, since ω_x is generated by global sections, the natural homomorphism $f^*\mathcal{F}_1\to\omega$ is surjective. This induces a morphism $\rho: M^*\to\boldsymbol{P}(\mathcal{F}_1)$ such that the restriction ρ_x of ρ to C_x is the canonical mapping. Let V be the image of ρ, and we regard ρ as a mapping onto V.

(15.2) Since $g(C_x)=g\geqq 2$ and ρ_x is the canonical mapping, the following conditions are equivalent to each other (cf., e.g., [**F6**; (1.4)]).

a) $V_x\cong\boldsymbol{P}^1$ and ρ_x is a double covering.

b) ω_x is not very ample.

c) The natural mapping $\mathrm{S}^3(H^0(C_x, \omega_x))\to H^0(C_x, \omega_x^{\otimes 3})$ is not surjective.

d) $x\in\mathrm{Supp}(\mathrm{Coker}(\mathrm{S}^3\mathcal{F}_1\to\mathcal{F}_3))$.

By the condition d), the set of points on P satisfying these conditions is Zariski closed.

(15.3) DEFINITION. (M, L) is said to be *sectionally hyperelliptic* if any general fiber of f satisfies the conditions in (15.2). If so, every fiber C_x of f is a hyperelliptic curve (which may be singular).

From now on, we suppose (M, L) to be sectionally hyperelliptic.

(15.4) V_x is a Veronese curve of degree $g-1$ embedded in $\boldsymbol{P}(\mathcal{F}_1)_x \cong \boldsymbol{P}^{g-1}$ for every $x \in P$. Hence $p: V \to P$ is a $\boldsymbol{P}^1$-bundle. $\rho: M^* \to V$ is a finite morphism of degree two, since so is ρ_x for every x.

(15.5) Let S be the image $\rho(E)$ of E in V. Then S is a section of p. Set $[S]_S = \mathcal{O}(-e)$ for some integer e. Then the exact sequence $0 \to \mathcal{O}_V[eH_\xi] \to \mathcal{O}_V[S+eH_\xi] \to \mathcal{O}_S[S+eH_\xi] \to 0$ descends via p_* to the exact sequence $0 \to \mathcal{O}_P(e) \to \mathcal{E} \to \mathcal{O}_P \to 0$, where $\mathcal{E}$ is the locally free sheaf $p_*\mathcal{O}_V[S+eH_\xi]$ of rank two on P. It is easy to see $V \cong \boldsymbol{P}(\mathcal{E})$, and $S \in |H_\zeta - eH_\xi|$, where H_ζ is the tautological line bundle on $\boldsymbol{P}(\mathcal{E})$.

(15.6) From now on, we assume $\mathrm{char}(\mathfrak{K}) \neq 2$ throughout in this article. Then, by virtue of [**F6**; (2.6)] (or [**F4**; (2.3)] in case $\mathfrak{K} = \boldsymbol{C}$), we have $M^* \cong R_B(V)$ for some non-singular divisor B on V. Let i be the involution of M^* such that $V \cong M^*/i$. Then there are following three possibilities:

a) $i(E) = E$.

b) $i(E) \cap E = \emptyset$.

c) $i(E) \neq E$ and $i(E) \cap E \neq \emptyset$.

(15.7) Definition. A sectionally hyperelliptic polarized manifold (M, L) is said to be of type $(-)$ (resp. (∞), $(+)$) if the above condition a) (resp. b), c)) is satisfied.

§ 16. Type $(-)$.

We employ the same notation as in § 15 and (M, L) is assumed to be sectionally hyperelliptic of type $(-)$.

(16.1) The restriction of the involution i to E is the identity map because $f = f \circ i$ and E is a section of f. Hence E is a component of the ramification locus of ρ, and $S = \rho(E)$ is a component of the branch locus B of ρ. By the same argument as in (14.4), we infer $e=2$, $V \cong \boldsymbol{P}(2H_\xi \oplus [0])$ and $S \in |H_\zeta - 2H_\xi|$. Set $B = S + B_2$. Then, as in (14.5), we see $B_2 \in |(2g+1)H_\zeta|$ and B_2 is connected. Thus we obtain:

(16.2) Theorem. *Let (M, L) be a polarized manifold with $n = \dim M \geqq 2$, $d(M, L) = \Delta(M, L) = 1$, and let $f: M^* \to P \cong \boldsymbol{P}_\xi^{g-1}$ be as in* (13.6). *Suppose in addition that* $\mathrm{char}(\mathfrak{K}) \neq 2$ *and that (M, L) is sectionally hyperelliptic of type $(-)$ with $g(M, L) = g$. Then $M^* \cong R_B(V)$, where V is the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(2H_\xi \oplus [0])$ over P, $B = B_1 + B_2$ is a divisor on V, B_1 is the unique member of $|H_\zeta - 2H_\xi|$ with H_ζ being the tautological line bundle on V, and B_2 is a non-singular connected member of $|(2g+1)H_\zeta|$. Moreover, B_1 is the image of E via $\rho: M^* \to V$.*

(16.3) Because of the similarity of (14.6) and (16.2), (M, L) may be said to be of type $(-)$ when $g(M, L)=1$. So the results in this section are valid in case $g=1$ too.

(16.4) THEOREM. *Let things be as in* (16.2) *or* (14.6). *Then* $\mathrm{Bs}|2L|=\emptyset$ *and the image* $W=\rho_{|2L|}(M)$ *is a projective cone over a Veronese manifold* $(\boldsymbol{P}^{n-1}, 2H)$.

PROOF. We have $\rho^*H_\zeta=\rho^*(S+2H_\xi)=2E+2H_\xi=2L$ in $\mathrm{Pic}(M^*)$. Since $\mathrm{Bs}|H_\zeta|=\emptyset$ on V, we have $\mathrm{Bs}|2L|=\emptyset$ on M^*, which implies $\mathrm{Bs}|2L|=\emptyset$ on M. Moreover, because $H^0(M, 2L)\cong H^0(M^*, 2L)\cong H^0(V, H_\zeta)$, W is the image of V by the rational mapping defined by $|H_\zeta|$. This is nothing but the contraction of S to a normal point, and the resulting variety is the cone over $(\boldsymbol{P}_\xi^{n-1}, 2H_\xi)$.

(16.5) In the above situation, $M\to W$ is a finite morphism of degree two which is ramified over the image D of B_2 via the contraction $V\to W$ and over the vertex of W.

Conversely, given any such smooth divisor D on W, we can construct a polarized manifold (M, L) with $d(M, L)=\Delta(M, L)=1$ which is sectionally hyperelliptic of type $(-)$. Indeed, as in [**F6**; (4.5)], we lift D to a divisor B_2 on V and set $B=S+B_2$. Let $M^*=R_B(V)$ and let E be the component of the ramification locus lying over S. Then we can apply the method in (13.7).

(16.6) By the above observation we see that the results in [**F6**; (4.6)] apply in the present case too. In particular we have:

1) $K^M=(2g-n-1)L$.

2) $H^q(M, tL)=0$ for any $t\in\boldsymbol{Z}$, $0<q<n$.

3) For any general member Y of $|2L|$, Y is a double covering of $\boldsymbol{P}^{n-1}$ with branch locus being a smooth hypersurface of degree $4g+2$ and L_Y is the pull-back of the hyperplane bundle of $\boldsymbol{P}^{n-1}$.

4) $b_j(M)=b_j(\boldsymbol{P}^n)$ if $j<n$. Moreover, if $\mathfrak{K}=\boldsymbol{C}$, $H^{2i}(M;\boldsymbol{Z})$ is generated by $c_1(L)^i$ if $2i<n$.

5) $\mathrm{Pic}(M)$ is generated by L if $n\geqq 3$.

6) $\pi_1^{(p)}(M)=\{1\}$ if $p=\mathrm{char}(\mathfrak{K})>0$. When $\mathfrak{K}=\boldsymbol{C}$, M is topologically simply connected.

(16.7) THEOREM. *Let* (M, L) *be a polarized manifold with* $d(M, L)=\Delta(M, L)=1$. *Then the following conditions are equivalent to each other.*

a) (M, L) *is sectionally hyperelliptic of type* $(-)$.

b) $\mathrm{Bs}|2L|=\emptyset$.

c) $h^0(M, 2L)>n(n+1)/2$.

d) (M, L) *is a weighted hypersurface of degree* $4g+2$ *in the weighted projective space* $\boldsymbol{P}(2g+1, 2, 1, \cdots, 1)$.

REMARK. The condition d) implies also $d(M, L)=\Delta(M, L)=1$.

PROOF. We use the notation in (13.6). We have a) $\Rightarrow$ d) by [**F6**; (4.6.7)] and (16.2). d) $\Rightarrow$ c) is obvious. To prove c) $\Rightarrow$ b), assume $\mathrm{Bs}\,|2L|\neq\emptyset$. Then E must be a fixed component of $\pi^*|2L|$. So $H^0(M^*, 2L-E)\cong H^0(M^*, 2L)$. The restriction of $2L-E$ to each fiber C_x of f is E_x. Clearly E_x is a fixed part of $|E_x|$ because $g\geqq 1$. Hence E is a fixed component of $|2L-E|$, so $H^0(M^*, 2L-E)\cong H^0(M^*, 2L-2E)\cong H^0(M^*, 2H_\xi)\cong H^0(P, 2H_\xi)$ because $f_*\mathcal{O}_{M^*}=\mathcal{O}_P$. Thus we infer $h^0(M, 2L)=h^0(M^*, 2L)=h^0(P, 2H)=n(n+1)/2$, contradicting c).

To show b) $\Rightarrow$ a), let x be any point on P. The restriction of $2L$ to C_x is $2E_x$. Hence $\mathrm{Bs}\,|2E_x|=\emptyset$. This linear system defines a morphism onto $\boldsymbol{P}^1$ of degree two. Consequently C_x is hyperelliptic. Moreover, E_x is a ramification point of this morphism. Hence (M, L) is sectionally hyperelliptic and E is a component of the ramification locus of $M^*\to V$. Thus we obtain a).

(16.8) Now, similarly as in [**F6**; §7], we will study deformations of (M, L).

THEOREM. *Let $(\mathcal{M}, T, \pi, \mathcal{L})$ be a deformation family of prepolarized manifolds, that means, $\pi: \mathcal{M}\to T$ is a proper smooth morphism between manifolds $\mathcal{M}$, T which may not be complete, and $\mathcal{L}$ is a line bundle on $\mathcal{M}$. Suppose that there is a point o on T such that (M_o, L_o), the fiber $M_o=\pi^{-1}(o)$ together with the restriction L_o of $\mathcal{L}$ to M_o, is a sectionally hyperelliptic polarized manifold of type $(-)$ with $d(M_o, L_o)=\Delta(M_o, L_o)=1$ and $n=\dim M_o\geqq 2$. Then there exists a Zariski open neighborhood U of o such that (M_t, L_t) is a sectionally hyperelliptic polarized manifold of the same type $(-)$ for every $t\in U$.*

PROOF. By [**EGA**; Chap. III, (4.7.1)], we find a neighborhood U_1 of o such that L_t is ample on M_t for every $t\in U_1$. By (16.6; 2) and by the upper-semicontinuity theorem, there is a neighborhood U_2 of o such that $H^1(M_t, L_t)=H^1(M_t, 2L_t)=0$ for every t on U_2. Then, as is well known (cf. [**EGA**; Chap. III] or [**H**; Chap. III, §12]), $h^0(M_t, L_t)$ and $h^0(M_t, 2L_t)$ are constant functions of $t\in U_2$. Thus, for every point t on $U=U_1\cap U_2$, the criterion (16.7; c) applies.

(16.9) QUESTION. Let (M, L) be a polarized manifold of the type (16.7). Then, does any small deformation of M carry a family of line bundles so that it becomes a deformation family of (M, L)?

When $n=\dim M\geqq 3$, we have $H^2(M, \mathcal{O})=0$ by (16.6; 2) and there is no obstruction for extending L as a family of line bundles. When $n=2$ and $\mathfrak{R}=\boldsymbol{C}$, let $\{\lambda_t\in H^2(M_t; \boldsymbol{Z})\}$ be the locally constant family of cohomology classes which extends $c_1(L)$. Since $K^M=(2g-3)L$ by (16.6; 1), we have $c_1(K_t)=(2g-3)\lambda_t$ for every t, where K_t is the canonical line bundle of any small deformation M_t of M. This implies that the image of λ_t in $H^2(M_t, \mathcal{O})$ vanishes. Hence $\lambda_t=c_1(L_t)$ for some $L_t\in\mathrm{Pic}(M_t)$. Such a line bundle L_t is unique since $h^1(M_t, \mathcal{O})=h^1(M, \mathcal{O})=0$. So $\{L_t\}$ form a family of line bundles.

Let us consider this problem from another viewpoint. By the observation (16.5), we see that polarized manifolds of the type (16.7) are parametrized by smooth members of $|(2g+1)H_\zeta|$. Our question is equivalent to asking whether this family is complete in the sense of Kodaira-Spencer [**KS**] as a deformation family of the complex manifold M. So, a positive answer follows from their criterion and the result below.

(16.10) THEOREM. *Let U be the open subset of $H^0(V, (2g+1)H_\zeta)$ corresponding to non-singular members of $|(2g+1)H_\zeta|$ and let $\{(M_u, L_u)\}$ be the induced family of polarized manifolds of type $(-)$ parametrized by U. Then, at any point u on U, the characteristic mapping $T_u^U(\cong H^0(V, (2g+1)H_\zeta))\to H^1(M_u, \Theta_u)$ is surjective if $n=\dim M_u\geqq 2$, where Θ_u is the sheaf of vector fields on M_u.*

In the following proof, we omit the subscript u for the sake of brevity of notation, for most objects lie over u. Thus we use the notation in (16.2). Note first that the characteristic mapping factors through $H^1(M^*, \Theta^*)$, where Θ^* is the sheaf of vector fields on M^*.

(16.11) Let τ^+ (resp. τ^-) be the eigen space belonging to the eigenvalue 1 (resp. -1) of the action on $H^1(M^*, \Theta^*)$ of the involution i of M^* covering $M^*\to V$. Then, as in [**F6**; (7.9)], $\tau^-\cong H^1(V, T^V[-F])$ for $F=(g+1)H_\zeta-H_\xi$, and there is an exact sequence $H^0(V, T^V)\to H^0(B, [B])\to\tau^+\to H^1(V, T^V)\to H^1(B, [B])$.

(16.12) The relative tangent bundle of $p: V\to P\cong \boldsymbol{P}_\xi^{n-1}$ is $2H_\zeta-2H_\xi$. Therefore we have the following exact sequences:

(1) $0\to[2H_\zeta-2H_\xi]\to T^V\to T_V^P\to 0$, and

(2) $0\to\mathcal{O}_V\to H^0(P, H_\xi)^\vee\otimes[H_\xi]\to T_V^P\to 0$.

From (2) we get an exact sequence $H^1(V, [-F])\to H^0(P, H_\xi)^\vee\otimes H^1(V, H_\xi-F)\to H^1(V, T_V^P[-F])\to H^2(V, -F)\to H^0(P, H_\xi)^\vee\otimes H^2(V, H_\xi-F)$. It is easy to see $h^1(V, -F)=0$, $h^1(V, H_\xi-F)=1$ and further $h^2(V, -F)=0$ unless $n=2$. When $n=2$, the last mapping is the dual of the surjective mapping $H^0(P, H_\xi)\otimes H^0(V, K^V+F-H_\xi)\to H^0(V, K^V+F)$. So, in any case, we obtain $h^1(V, T_V^P[-F])=n$. On the other hand, we see $H^1(V, (1-g)H_\zeta-H_\xi)=0$ by Serre duality. Now, in view of (1), we infer $\dim\tau^-=h^1(V, T^V[-F])\leqq h^1(V, T_V^P[-F])=n$.

(16.13) We claim that the mapping $H^0(B, [B])\to\tau^+$ in (16.11) is surjective. When $n\geqq 3$, we obtain $H^1(V, T^V)=0$ by similar arguments as in (16.12). So the assertion is clear. Any way, it suffices to show that $H^1(V, T^V)\to H^1(S, [S])$ is injective, for the latter is a direct sum component of $H^1(B, [B])$. Using the exact sequences (1) and (2), we infer $H^1(V, T^V[-S])=0$ since $[S]=H_\zeta-2H_\xi$. This implies that the natural mapping $H^1(V, T^V)\to H^1(S, T_S^V)$ is injective. On the other hand, since S is a section of $p: V\to P$, we see that $T_S^V\to T_S^P\cong T^S$ gives a splitting of the exact sequence $0\to T^S\to T_S^V\to[S]_S\to 0$. So $H^1(S, T_S^V)\cong H^1(S, [S])$, because $H^1(S, T^S)=0$. Combining these facts we prove our claim.

(16.14) Let us consider a general situation where M^* is the blowing-up of a manifold M with center being a submanifold C in M. Let E be the exceptional divisor lying over C. Then $E\cong\boldsymbol{P}(N^\vee)$ for the conormal bundle $N^\vee$ of C in M and the tautological line bundle $\mathcal{O}_E(1)$ is the restriction of $[-E]\in\mathrm{Pic}(M^*)$.

$\mathcal{O}_E(-1)$ is naturally a subbundle of N_E, the pull-back of the normal bundle N of C. Let Q be the quotient bundle $N_E/\mathcal{O}_E(-1)$.

We have a natural homomorphism $\theta: \Theta^* \to \pi^*\Theta$, where Θ^* (resp. Θ) denotes the sheaf of vector fields on M^* (resp. M). It is easy to see that θ is injective and $\mathcal{C}=\mathrm{Coker}(\theta)$ is supported on E. Moreover $\mathcal{C}\cong\mathcal{O}_E[Q]$. Using Leray spectral sequence for $E\to C$, we infer $H^q(\mathcal{C})\cong H^q(C, N)$ for any $q\in\boldsymbol{Z}$. Thus we get a long exact sequence $0\to H^0(M^*, \Theta^*)\to H^0(M, \Theta)\to H^0(C, N)\to H^1(M^*, \Theta^*)\to H^1(M, \Theta)\to H^1(C, N)\to\cdots$.

(16.15) In our particular case, $C=X=\mathrm{Bs}|L|$ is a point on M, which is an isolated fixed point of the involution of M induced by that of M^* over V. Therefore $h^0(C, N)=n$ and the image of $\delta: H^0(C, N)\to H^1(M^*, \Theta^*)\cong\tau^+\oplus\tau^-$ lies in τ^-. Any vector field on M, as an infinitesimal automorphism of M, preserves the line bundle L because $H^1(M, \mathcal{O}_M)=0$. Hence it does not move $C=\mathrm{Bs}|L|$. So $H^0(M, \Theta)\to H^0(C, N)$ is a zero map. This implies that δ is injective. By (16.12) we infer that $\mathrm{Im}(\delta)=\tau^-$. This implies that $\tau^+\to H^1(M, \Theta)$ is bijective.

On the other hand, $H^0(V, [B])\to H^0(B, [B])$ is surjective because $H^1(V, \mathcal{O}_V)=0$. So, combining with (16.13), we infer that $H^0(V, [B])\to H^1(M, \Theta)$ is surjective, proving (16.10).

§ 17. Type (∞).

We employ the same notation as in § 15 and (M, L) is assumed to be sectionally hyperelliptic of type (∞).

(17.1) In the case of type (∞) we have $i(E)\cap E=\emptyset$ and $\rho^{-1}(S)=E\cup i(E)$. So $S\cap B=\emptyset$, hence $e=1$. In particular $V\cong\boldsymbol{P}(H_\xi\oplus\mathcal{O}_P)$ and the rational mapping σ defined by $|H_\zeta|$ makes V a blowing-up of $\boldsymbol{P}_\zeta^n$ with center being a point y to which S is contracted.

(17.2) Since $B\cap S=\emptyset$ and $BV_x=2g+2$ for any $x\in P$, we infer $B\in|(2g+2)H_\zeta|$. So $B'=\sigma(B)$ is a hypersurface on $\boldsymbol{P}_\zeta^n$ of degree $2g+2$ and $B'\cong B$. Let M' be the double covering $R_{B'}(\boldsymbol{P}_\zeta^n)$ of $\boldsymbol{P}_\zeta^n$ and let y_1, y_2 be points on M' lying over y. Then M^* is the blowing-up of M' at these points and the exceptional divisors over them are E and $i(E)$. So M is the blowing-up of M' at one of y_1 and y_2. The choice between these points does not affect the isomorphism class of M, because they are interchangeable by the involution of M'. Since $L=H_\xi+E=H_\zeta-i(E)$, (M, L) is determined by the pair (B', y).

(17.3) For any line l_x on $\boldsymbol{P}_\zeta^n$ passing through y, there exists a point on $l_x\cap B'$ at which they meet in odd order.

Indeed, otherwise, the fiber C_x of $f: M^*\to P$ over the point $x\in P$ corresponding to l_x would be isomorphic to $R_{B'\cap l_x}(l_x)$ and hence not irreducible.

(17.4) Conversely, for any pair (B', y) as in (17.2) satisfying the condition (17.3), we can construct a sectionally hyperelliptic polarized manifold (M, L) of type (∞) by reversing the preceding process. The condition (17.3) is necessary to prove the ampleness of L.

(17.5) Given any (n, g), the set of pairs (B', y) satisfying the condition (17.3) forms a Zariski open subset of the space parametrizing all the pairs as in (17.2). However, this open set may be empty. In fact we have the following

LEMMA. *Let B be a smooth hypersurface of degree $2g+2$ in $\boldsymbol{P}_\zeta^n$ and let y be a point on $\boldsymbol{P}_\zeta^n$ off B. Then, if $n \geqq g+2$, there exists a line l passing y such that the intersection multiplicity of l and B at every point on $l \cap B$ is even.*

Such a line l will be said to be evenly in contact with B. A proof of this lemma will be given below, ending in (17.12).

(17.6) Given a vector bundle E on a space T, let $E^\vee$ (resp. $\mathrm{S}^k E$) denote the dual bundle (resp. k-th symmetric product) of E. For every $t \in T$, the fiber $(\mathrm{S}^k E)_t$ is canonically identified with the space of homogeneous polynomial functions of degree k on $(E^\vee)_t$. Hence, taking m-powers at each point t of T, we get a mapping $\mathrm{S}^k E \to \mathrm{S}^{mk} E$ for each positive integer m. Of course, usually, this is not a homomorphism of vector bundles. Any way, this induces a morphism $\mu_m : \boldsymbol{P}((\mathrm{S}^k E)^\vee) \to \boldsymbol{P}((\mathrm{S}^{mk} E)^\vee)$. Denoting by H_k and H_{mk} the tautological line bundles on these spaces, we have $\mu_m^* H_{mk} = m H_k$ by definition of μ_m.

(17.7) To prove the lemma (17.5), let V be the blowing-up of $\boldsymbol{P}_\zeta^n$ with center y and let S be the exceptional divisor over y. Then $(V, H_\zeta) \cong (\boldsymbol{P}(E), \mathcal{O}(1))$ for the vector bundle $E = H_\xi \oplus \mathcal{O}$ on $\boldsymbol{P}_\xi^{n-1}$. S is the unique member of $|H_\zeta - H_\xi|$. The fibers of $p : V \to \boldsymbol{P}_\xi^{n-1} = P$ are in one to one correspondence with the lines on $\boldsymbol{P}_\zeta^n$ passing through y. B is defined by a section of $H^0(\boldsymbol{P}_\zeta^n, \mathcal{O}(2g+2)) \cong H^0(V, (2g+2)H_\zeta) \cong H^0(P, \mathrm{S}^{2g+2} E)$. This section of $\mathrm{S}^{2g+2} E$ does not vanish at any point x on P because B does not contain the fiber V_x over x. Hence this defines a subbundle of $\mathrm{S}^{2g+2} E$ isomorphic to $\mathcal{O}_P$. Correspondingly, we have a section b of the bundle $\boldsymbol{P}((\mathrm{S}^{2g+2} E)^\vee)$ over P.

On the other hand, as we saw in (17.6), there is a natural morphism μ: $\boldsymbol{P}((\mathrm{S}^{g+1} E)^\vee) = G \to \boldsymbol{P}((\mathrm{S}^{2g+2} E)^\vee)$ defined by square. By definition of μ, the line l_x corresponding to $x \in P$ is evenly in contact with B if and only if $b(x) \in \mu(G)$. Therefore, we should show $b(P) \cap \mu(G) \neq \emptyset$.

(17.8) First we consider the case $n = g+2$. Then $\dim b(P) = n-1 = \operatorname{codim} \mu(G)$. We will calculate the intersection number $I = b(P)\mu(G)$ in the Chow ring of $\boldsymbol{P}((\mathrm{S}^{2g+2} E)^\vee)$ and show $I > 0$.

(17.9) The section b defines a subbundle N of $\mathrm{S}^{2g+2} E \cong \bigoplus_{j=0}^{2g+2} \mathcal{O}_P(j)$. The direct sum component $\mathcal{O}_P$ corresponds to the quotient sheaf $p_*(\mathcal{O}_S[(2g+2)H_\zeta])$ of $p_*(\mathcal{O}_V[(2g+2)H_\zeta]) \cong \mathcal{O}_P[\mathrm{S}^{2g+2} E]$. Since $B \cap S = \emptyset$, we infer that N maps bijectively onto this quotient bundle. Hence $\mathrm{S}^{2g+2} E / N \cong \bigoplus_{j=1}^{2g+2} \mathcal{O}_P(j)$. For each $j = 1, \cdots, 2g+2$, the quotient $\mathcal{O}_P(j)$ of $S^{2g+2}E$ defines a divisor $D_j \in |H_\sigma + jH_\xi|$ on $\boldsymbol{P}((S^{2g+2}E)^\vee)$, where H_σ denotes the tautological line bundle. Clearly $b(P)=D_1\cap\cdots\cap D_{2g+2}$. So $I=(H_\sigma+H_\xi)\cdots(H_\sigma+(2g+2)H_\xi)\mu(G)=(2H_\tau+H_\xi)\cdots(2H_\tau+(2g+2)H_\xi)_G$, where H_τ is the tautological line bundle on $G=\boldsymbol{P}((S^{g+1}E)^\vee)$, because $\mu^*H_\sigma=2H_\tau$ by (17.6).

(17.10) For the convenience of calculation, we introduce the following notation. Let $\boldsymbol{P}^1\subset\boldsymbol{P}^2\subset\cdots\subset\boldsymbol{P}^n\subset\boldsymbol{P}^{n+1}\subset\cdots$ be an infinite sequence of linear embeddings and let $P^\infty=\bigcup_{n\geq 1}\boldsymbol{P}^n$. There is a line bundle H on P^∞ such that its restriction to each $\boldsymbol{P}^n$ is $\mathcal{O}(1)$. The Chow ring of P^∞ is defined to be the projective limit of $\mathrm{Ch}(\boldsymbol{P}^n)$, which turns out to be the ring $\boldsymbol{Z}[[h]]$ of formal power series in $h=c_1(H)$ with integral coefficients. For $\varphi\in\boldsymbol{Z}[[h]]$, we denote the coefficient of h^d by $[\varphi]_d$.

Now, for any totally decomposable vector bundle E on P^∞, the total Chern class $c(E)$ of E is well-defined by the following axioms: $c(E_1\oplus E_2)=c(E_1)c(E_2)$ and $c(tH)=1+th$. Similarly total Segre classes are defined by the axioms $s(E_1\oplus E_2)=s(E_1)s(E_2)$ and $s(tH)=\sum_{j=0}^{\infty}(th)^j$. So $c(E)s(E^\vee)=1$ for any E. We denote $[c(E)]_d$ and $[s(E)]_d$ by $c_d(E)$ and $s_d(E)$ respectively.

(17.11) Under the above notation, it is easy to see $I=\sum_{j=0}^{2g+2}2^j c_{2g+2-j}(H\oplus 2H\oplus\cdots\oplus(2g+2)H)H_\tau^j H_\xi^{2g+2-j}\{G\}$. On the other hand, since $n-1=g+1$ and $(S^{g+1}E)^\vee\cong\bigoplus_{j=0}^{g+1}[-jH_\xi]$, we have $H_\tau^{g+1+r}H_\xi^{g+1-r}=s_r(\bigoplus_{j=0}^{g+1}[-jH])$ if $r\geqq 0$, and $=0$ if $r<0$. Therefore $I=\sum_{r=0}^{g+1}2^{g+1+r}s_r(\bigoplus_{j=0}^{g+1}[-jH])c_{g+1-r}(\bigoplus_{j=1}^{2g+2}[jH])=2^{g+1}[s(\bigoplus_{j=1}^{g+1}[-2jH])c(\bigoplus_{j=1}^{g+1}[(2jH])c(\bigoplus_{j=1}^{g+1}[(2j-1)H])]_{g+1}=2^{g+1}c_{g+1}(\bigoplus_{t=0}^{g}[(2t+1)H])=2^{g+1}\prod_{t=0}^{g}(2t+1)>0$. Thus we prove the assertion in case $n=g+2$.

(17.12) In general, we can calculate the intersection number $I=b(\boldsymbol{P}^{n-1})\mu(G)H_\xi^{n-g-2}$ and show that $I>0$, which implies $b(P)\cap\mu(G)\neq\emptyset$ as required. Alternately, taking a general hyperplane section passing through y, we can prove the lemma (17.5) by induction on n. Details are left to the reader.

(17.13) REMARK. When $n\leqq g+1$, it is not difficult to see that any general pair (B', y) as in (17.2) satisfies the condition (17.3).

(17.14) Putting things together, we get the following

THEOREM. *Let (M, L) be a polarized manifold with $d(M, L)=\Delta(M, L)=1$. Suppose that* $\mathrm{char}(\mathfrak{K})\neq 2$ *and that (M, L) is sectionally hyperelliptic of type* (∞). *Then, there exist a non-singular hypersurface B of degree $2g(M, L)+2$ on $\boldsymbol{P}^n$ and a point y' on $M'=R_B(\boldsymbol{P}^n)$ off the ramification locus of $M'\to\boldsymbol{P}^n$ such that M is isomorphic to the blowing-up of M' with center y' and $L=H-E'$ for the exceptional divisor E' over y'. Furthermore, if y is the image of y' on $\boldsymbol{P}^n$, any line on $\boldsymbol{P}^n$ passing through y is not evenly in contact with B. In particular*

$n=\dim M\leqq g(M, L)+1$.

(17.15) COROLLARY. *Polarized manifolds of the above type form a single deformation family for any fixed* $n=\dim M$ *and* $g=g(M, L)$. *This family is complete in the sense of Kodaira-Spencer* [**KS**] *unless* $n=g=2$ *and* M' *is a* K3-*surface.*

This follows from the observation (16.14) and the result [**F6**; (7.7)]. From the results on M' in [**F6**; § 4], we obtain also the results below.

(17.16) COROLLARY. *Let* (M, L) *be as in* (17.14). *Then* $\pi_1^{\{p\}}(M)=\{1\}$ *for* $p=\mathrm{char}(\mathfrak{K})$ *and* $\pi_1(M)=\{1\}$ *if* $\mathfrak{K}=\boldsymbol{C}$.

(17.17) COROLLARY. *Let* j *be an integer such that* $0<j<2n$ *and* $j\neq n$. *Then* $b_j(M)=0$ *if* j *is odd and* $b_j(M)=2$ *if* j *is even. If* $\mathfrak{K}=\boldsymbol{C}$, $h^{p,q}(M)=0$ *unless* $p=q$ *or* $p+q=n$.

(17.18) COROLLARY. *If* $n\geqq 3$, $\mathrm{Pic}(M)$ *is generated by* L *and* E'.

(17.19) REMARK. The conclusion of (17.14) is valid also in case $n=2$ and $g=1$. In this case (M, L) is a Del Pezzo surface with $d=1$.

§ 18. Type $(+)$.

We employ the same notation as in § 15 and (M, L) is assumed to be sectionally hyperelliptic of type $(+)$.

(18.1) We have $\rho^{-1}(S)=E\cup i(E)$ and $S\cap B\neq\emptyset$. Hence B_S must be of the form $2Y$, Y being an effective divisor on $S\cong\boldsymbol{P}^{n-1}$. Set $\delta=\deg Y>0$.

(18.2) We claim $\delta-1=-e$. Indeed, the pull-back of Y by the morphism $E\rightarrow S$ is $i(E)_E$. Hence, restricting $[E+i(E)]=[S]$ to E and considering the degrees, we get the desired equality.

(18.3) LEMMA. $n=2$.

PROOF. We have natural exact sequences $0\rightarrow T^S\rightarrow T_S^V\rightarrow[S]\rightarrow 0$ on S and $0\rightarrow T^B\rightarrow T_B^V\rightarrow[B]\rightarrow 0$ on B. Restrict them to $Y=B\cap S$. Since $B_S=2Y$, the intersection of B and S along Y is not transversal. So, at each point y on Y, the subspaces T_y^S and T_y^B of T_y^V coincide with each other. This implies $[S]_Y\cong[B]_Y$. Hence, if $n>2$, we have $-e=\deg[S]_S=\deg[B]_S=2\delta$. Then $\delta-1=2\delta$ by (18.2), which is absurd because δ is positive. Thus we prove $n=2$.

(18.4) V is isomorphic to a Hirzebruch surface $\Sigma_k\cong\boldsymbol{P}(kH_\xi\oplus\mathcal{O})$ for some $k\geqq 0$. Let H_α be the tautological line bundle on Σ_k. Note that $H_\alpha^2=k$, and that, if $k>0$, $C^2=-k$ for the unique member C of $|H_\alpha-kH_\xi|$. Set $[B]=2aH_\alpha+2bH_\xi$ and $[S]=H_\alpha+\sigma H_\xi$. Then $-e=S^2=k+2\sigma$, $2\delta=BS=2ak+2b+2a\sigma$ and $2a=BH_\xi=2g+2$. So, from (18.2), we obtain

$$(\sharp)\qquad gk+(g-1)\sigma+b=1\,.$$

(18.5) Assume that $k>0$ and let C be the curve as above. Since $S^2=-e=\delta-1\geqq 0$, we see $S\neq C$, which implies $\sigma=SC\geqq 0$. We have also $2b=BC\geqq -k$ since B is smooth. So (♯) yields $2\geqq 2(gk+b)\geqq(2g-1)k\geqq 2g-1$. This contradicts the assumption $g\geqq 2$. Thus we infer $k=0$. Hence $V\cong \boldsymbol{P}_\alpha^1\times\boldsymbol{P}_\xi^1$.

(18.6) We claim $b>0$. Indeed, otherwise, B consists of $2a$ fibers of ρ_α: $V\to\boldsymbol{P}_\alpha^1$ since $b=0$. So $M^*\cong\boldsymbol{P}_\xi^1\times T$ for some hyperelliptic curve T of genus g. The mapping $M^*\to V\to\boldsymbol{P}_\alpha^1$ factors through T. Hence the image of E is a point because E is rational. So S maps to a point on $\boldsymbol{P}_\alpha^1$ via ρ_α. Then $S\subset B$ or $S\cap B=\emptyset$, contradicting the assumption.

(18.7) Now, by (♯) and (18.5), we infer $b=1$ and $\sigma=0$. In particular, S is a fiber of ρ_α and $e=0$. We also have $\delta=1$ and $B\in|(2g+2)H_\alpha+2H_\xi|$. So B is ample on V and hence connected. $\rho_\alpha(S)$ is a branch point of the double covering $B\to\boldsymbol{P}_\alpha^1$.

(18.8) Since every fiber of $f: M^*\to\boldsymbol{P}_\xi^1$ is irreducible and reduced, B satisfies the following condition:

(♯♯) For every fiber V_x of $V\to\boldsymbol{P}_\xi^1$, there exists a point on V_x at which B and V_x intersect with odd multiplicity.

(18.9) Conversely, suppose that we have a non-singular member B of $|(2g+2)H_\alpha+2H_\xi|$ on $V\cong\boldsymbol{P}_\alpha^1\times\boldsymbol{P}_\xi^1$, which satisfies the above condition (♯♯). Take a branch point z of the double covering $B\to\boldsymbol{P}_\alpha^1$ and let S be the fiber of $V\to\boldsymbol{P}_\alpha^1$ over z. Then $B_S=2Y$ for some effective divisor Y on S. Let $M^*=R_B(V)$ and let $\rho: M^*\to V$ be the natural morphism. Then $\rho^*S=S_1+S_2$ for some divisors S_1, S_2 on M^*. $S_1S_2=\deg Y=1$, $S_1^2=S_2^2=-1$ and both S_1 and S_2 are sections of the mapping $f: M^*\to\boldsymbol{P}_\xi^1$. The condition (♯♯) implies that every fiber of f is irreducible and reduced. So, we can apply (13.7) to obtain a polarized manifold (M, L) with $d(M, L)=\Delta(M, L)=1$ by contracting either S_1 or S_2 to a point. Clearly (M, L) is sectionally hyperelliptic of type $(+)$.

The isomorphism class of (M, L) is determined by the pair (B, z) and is independent of the choice between S_1 and S_2, because they are interchangeable by the involution i of M^*.

(18.10) It is not difficult to see that the above condition (♯♯) is satisfied by any general member of $|(2g+2)H_\alpha+2H_\xi|$. So, as in the case of type (∞), (M, L)'s of type $(+)$ form a single deformation family for each fixed $g=g(M, L)$. However, this family is not complete in the sense of Kodaira-Spencer [**KS**]. In fact, for a general small deformation (M_t, L_t) of (M, L), we have $h^0(M_t, L_t)<2$, whence $\Delta(M_t, L_t)>1$. Note that $h^1(M, L)>0$ in case of type $(+)$, unlike the cases of type $(-)$ and (∞).

(18.11) Combining the preceding arguments, we obtain the following.

THEOREM. *Let (M, L) be a polarized manifold with $d(M, L)=\Delta(M, L)=1$. Suppose that* $\mathrm{char}(\mathfrak{K})\neq 2$ *and that (M, L) is sectionally hyperelliptic of type $(+)$.*

Then $\dim M=2$ *and* $M^*\cong R_B(\boldsymbol{P}^1_\alpha\times\boldsymbol{P}^1_\xi)$ *for a non-singular member* B *of* $|(2g+2)H_\alpha+2H_\xi|$, *where* M^* *is as in* (13.6). *The image of* E *on* $V=\boldsymbol{P}^1_\alpha\times\boldsymbol{P}^1_\xi$ *is a fiber over a branch point of the double covering* $B\to\boldsymbol{P}^1_\alpha$. *All the polarized surfaces of this type with fixed* $g=g(M, L)$ *form a single deformation family.*

(18.12) COROLLARY. *M is a rational surface.*

Indeed, the mapping $M^*\to\boldsymbol{P}^1_\alpha$ gives a $\boldsymbol{P}^1$-ruling.

Appendix 1. Table of numerical invariants of (M, L) with $n=\dim M$, $g=g(M, L)$.

	type $(-)$	type (∞)	type $(+)$
range of n	any n	$n\leqq g+1$	$n=2$
$\kappa(M)=n$ if	$n<2g-1$	$n<g$	—
$=0$ if	$n=2g-1$	$n=g$	—
<0 if	$n>2g-1$	$n=g+1$	always
$\pi_1^{(p)}(M)$	$\{1\}$	$\{1\}$	$\{1\}$
$b_2(M)$ $(n\geqq 3)$	1	2	—
$p_g(M)$ $(n=2)$	$g(g-1)$	$g(g-1)/2$	0
$c_1(M)^2$ $(n=2)$	$(2g-3)^2$	$2g^2-8g+7$	$3-4g$

Appendix 2. Here we present a proof of the following

THEOREM. *Let M be a Kähler threefold whose cohomology ring $H^{\circ}(M; \boldsymbol{Z})$ is isomorphic to $H^{\circ}(\boldsymbol{P}^3; \boldsymbol{Z})$. Suppose that $c_1(M)$ is positive. Then M is analytically isomorphic to $\boldsymbol{P}^3$.*

PROOF. M is projective since $H^2(M, \mathcal{O}_M)=0$. So Pic(M) is generated by an ample line bundle H such that $H^3=1$. Set $K^M=kH$. k is negative by assumption and is even because $2g(M, H)-2=(K^M+2H)H^2=k+2$. If $k\leqq -4$, we can apply [**KO**]. So we should consider the case $k=-2$. Then (M, H) is a Del Pezzo manifold and our theorem (14.6) applies. Hence it suffices to show the following

LEMMA. *Let (M, L) be as in* (14.6). *Then the topological Euler number $e(M)$ of M is* -38.

PROOF. Given any manifold X, we denote the tangent bundle of X by T^X. We have two exact sequences $0\to[2H_\zeta-2H_\xi]\to T^V\to T^P_V\to 0$ and $0\to\mathcal{O}_V\to H_\xi\oplus H_\xi\oplus H_\xi\to T^P_V\to 0$ on V as in (16.12). We have also $0\to T^{B_2}\to T^V_{B_2}\to[3H_\zeta]_{B_2}\to 0$

on B_2. Using them we express the total Chern class $c(T^{B_2})$ in terms of $c_1(H_\zeta)$ and $c_1(H_\xi)$. Calculating intersection numbers we obtain $e(B_2)=c_2(B_2)=45$. Therefore $e(B)=e(B_1)+e(B_2)=48$, $e(V)=6$ and $e(M^*)=2e(V)-e(B)=-36$. On the other hand we have $e(M^*)=e(M)+e(E)-1=e(M)+2$. Hence $e(M)=-38$.

REMARK. By a similar method we can calculate the Euler numbers of polarized manifolds studied in this paper.

References

[F1] T. Fujita, On the structure of polarized varieties with Δ-genera zero, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **22** (1975), 103-115.

[F2] T. Fujita, Defining equations for certain types of polarized varieties, Complex Analysis and Algebraic Geometry, 165-173, Iwanami, Tokyo, 1977.

[F3] T. Fujita, On the hyperplane section principle of Lefschetz, J. Math. Soc. Japan, **32** (1980), 153-169.

[F4] T. Fujita, On the structure of polarized manifolds with total deficiency one, I and II, J. Math. Soc. Japan, **32** (1980), 709-725 and **33** (1981), 415-434.

[F5] T. Fujita, On polarized varieties of small Δ-genera, Tôhoku Math. J., **34** (1982), 319-341.

[F6] T. Fujita, On hyperelliptic polarized varieties, Tôhoku Math. J., **35** (1983), 1-44.

[F7] T. Fujita, Classification of projective varieties of Δ-genus one, Proc. Japan Acad., **58** (1982), 113-116.

[G] (referred also as [EGA]) A. Grothendieck, Eléments de géométrie algébrique, Publ. Math. I. H. E. S., **4**, **8**, **11**, **17**.

[H] R. Hartshorne, Algebraic Geometry, Graduate Texts in Math., **52**, Springer, 1977.

[KO] S. Kobayashi and T. Ochiai, Characterizations of complex projective spaces and hyperquadrics, J. Math. Kyoto Univ., **13** (1973), 31-47.

[KS] K. Kodaira and D. C. Spencer, A theorem of completeness for complex analytic fiber spaces, Acta Math., **100** (1958), 281-294.

[M] S. Mori, On a generalization of complete intersections, J. Math. Kyoto Univ., **15** (1975), 619-646.

Takao FUJITA
Department of Mathematics
College of General Education
University of Tokyo
Komaba, Meguro, Tokyo 153
Japan

Note added in proof.

Professor Y. Miyaoka points out to the author that the answer to the Question (16.9) is affirmative in positive characteristic cases too. His method uses etale cohomology.