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Residues of complex analytic foliation singularities

By Tatsuo SUWA

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In [3], Baum and Bott defined the residues of complex analytic foliation singularities and proved a general residue formula using differential geometry based on the Bott vanishing theorem. Let M be a complex manifold. We define a foliation (of complete intersection type) on M to be a locally free subsheaf F of the cotangent sheaf Ω_M which satisfies the Frobenius integrability condition outside of the singular set (=the singular set of the coherent sheaf $\Omega_F=\Omega_M/F$). In this note, we express ((3.4) Theorem) a certain class of residues of F in terms of the Chern classes of F and the local Chern classes of the sheaf $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$, which appeared in the unfolding theory ([7]). As a corollary, the rationality of these residues is shown (cf. [3] p. 287 Rationality Conjecture). In a number of cases, the Riemann-Roch theorem for analytic embeddings (Atiyah-Hirzebruch [2]) can be used to compute the residues. The results of this paper were announced in [9].

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1. Residues.

We briefly review how the residues are defined in Baum-Bott [3]. Let M be an n-dimensional complex manifold. We denote by $\mathcal{O}_M$ (or simply by $\mathcal{O}$), Θ_M and Ω_M, respectively, the structure sheaf, the tangent sheaf and the cotangent sheaf of M. In [3] pp. 281-282, a foliation is defined to be a full integrable coherent subsheaf ξ of Θ_M. Let Q be the quotient sheaf Θ_M/ξ;

$$(1.1) \qquad 0 \longrightarrow \xi \longrightarrow \Theta_M \longrightarrow Q \longrightarrow 0 .$$

The singular set S of the foliation is defined by

$$(1.2) \qquad S=\{z\in M \mid Q_z \text{ is not a free } \mathcal{O}_z\text{-module}\},$$

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where for a sheaf $\mathcal{S}$ on M, $\mathcal{S}_z$ denotes the stalk of $\mathcal{S}$ over z. The sheaf ξ defines an ordinary foliation on $M-S$, whose codimension is denoted by q. Let Z be a connected component of S and assume that Z is compact. Take an open neighborhood U of Z in M such that Z is a deformation retract of U. Let $\sigma_1, \cdots, \sigma_n$ be the elementary symmetric functions in n variables $X_1, \cdots, X_n$. On $U-Z$, the sheaf Q is locally free and it admits a basic connection D_{-1}, which determines a closed $2i$-form $\sigma_i(K_{-1})$ on $U-Z$ for each i, $1 \leqq i \leqq n$. There exists a closed $2i$-form ω_i on U which coincides with $\sigma_i(K_{-1})$ outside of a compact set in U containing Z in its interior (cf. [3] p. 312 Proof of (0.23)).

If ϕ is a symmetric and homogeneous polynomial of degree d in $X_1, \cdots, X_n$, there is a polynomial $\tilde{\phi}$ in $\sigma_1, \cdots, \sigma_n$ with $\phi=\tilde{\phi}(\sigma_1, \cdots, \sigma_n)$. We set $\phi(Q)=(\sqrt{-1}/2\pi)^d\tilde{\phi}(\omega_1, \cdots, \omega_n)$, which is a closed $2d$-form on U. Note that in [3], the cohomology class of $\phi(Q)$ is denoted by $\phi(Q)$, however here the form itself is denoted by $\phi(Q)$. If $d>q$, then by the Bott vanishing theorem ([3](3.27)), $\phi(Q)$ has compact support and defines a cohomology class $[\phi(Q)]$ in $H_c^{2d}(U; \boldsymbol{C})$ (cohomology with compact support). We denote by L the composition of the two isomorphisms

$$(1.3) \qquad H_c^{2d}(U;\, \boldsymbol{C}) \xrightarrow[\sim]{D_U} H_{2n-2d}(U;\, \boldsymbol{C}) \xrightarrow[\sim]{i_*^{-1}} H_{2n-2d}(Z;\, \boldsymbol{C}),$$

where D_U denotes the Poincaré duality map and i is the embedding $Z \hookrightarrow U$. Then the residue is defined by

$$\mathrm{Res}_\phi(\xi, Z)=L([\phi(Q)]).$$

2. The sheaf $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$.

In [7](1.2), a (reduced) foliation is defined to be a full coherent subsheaf F of Ω_M satisfying the integrability condition. Let Ω_F be the quotient sheaf Ω_M/F;

$$(2.1) \qquad 0 \longrightarrow F \longrightarrow \Omega_M \longrightarrow \Omega_F \longrightarrow 0.$$

The two definitions are equivalent if we set ([7](1.5)) $\xi=F^a=\{\theta\in\Theta_M \mid \omega(\theta)=0, \forall\omega\in F\}$ or $F=\xi^a=\{\omega\in\Omega_M \mid \omega(\theta)=0, \forall\theta\in\xi\}$. Note that F^a is identical with the dual sheaf $\mathcal{H}om_{\mathcal{O}}(\Omega_F, \mathcal{O})$ of Ω_F. The singular set $S(F)$ of F is defined by

$$(2.2) \qquad S(F)=\{z\in M \mid \Omega_{F,z} \text{ is not a free } \mathcal{O}_z\text{-module}\}$$

and is identical with S in (1.2). By taking the duals of (2.1), we obtain the exact sequence

$$(2.3) \qquad 0 \to \underset{\substack{\| \\ F^a}}{\mathcal{H}om_{\mathcal{O}}(\Omega_F, \mathcal{O})} \to \underset{\substack{\| \\ \Theta_M}}{\mathcal{H}om_{\mathcal{O}}(\Omega_M, \mathcal{O})} \to \underset{\substack{\| \\ F^*}}{\mathcal{H}om_{\mathcal{O}}(F, \mathcal{O})} \to \mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O}) \to 0.$$

By (2.2), the support of the sheaf $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$ is in S. Comparing (1.1) and (2.3), we get the exact sequence

$$0 \longrightarrow Q \longrightarrow F^* \longrightarrow \mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O}) \longrightarrow 0 . \tag{2.4}$$

From now on we consider only foliations of complete intersection type ([7](1.10)), i.e., we assume that F is a locally free $\mathcal{O}$-module (of rank q). We do not distinguish locally free sheaves from holomorphic vector bundles. Thus (2.4) can be viewed as a "decomposition" of the sheaf Q into the vector bundle part F^* and the singular part $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$.

3. Residues and the local Chern classes of $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$.

Let F be a codim q foliation (of complete intersection type) and let Z be a compact connected component of the singular set S. In this section, analytic objects on M are restricted to the open set U considered in section 1. Since $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$ is a coherent sheaf on U with support in Z, there is the associated "Grothendieck element" $\gamma_Z(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O}))$, which we simply denote by $\mathcal{E}$, in $K^0(U, U-Z)$ ([1] § 4, [4] Ch. I, cf. also [5]). The Chern character gives a mapping

$$\mathrm{ch}: K^0(U, U-Z) \longrightarrow H^*(U, U-Z\,;\, \boldsymbol{Q}) .$$

Since Z is a deformation retract of U, there is a canonical isomorphism

$$H^*(U, U-Z\,;\, \boldsymbol{Q}) \xrightarrow{\sim} H^*_c(U\,;\, \boldsymbol{Q}) .$$

Also there is a canonical homomorphism

$$\kappa: H^*_c(U\,;\, \boldsymbol{Q}) \longrightarrow H^*(U\,;\, \boldsymbol{Q}) . \tag{3.1}$$

Thus $\mathrm{ch}(\mathcal{E})$ determines the local Chern classes $c_1(\mathcal{E}), \cdots, c_n(\mathcal{E})$ in $H^*(U, U-Z\,;\, \boldsymbol{Q}) = H^*_c(U\,;\, \boldsymbol{Q})$ such that $1+\kappa(c_1(\mathcal{E}))+\cdots+\kappa(c_n(\mathcal{E}))$ is the total Chern class of the coherent sheaf $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$ on U. For each integer k with $1 \leqq k \leqq n$, we set

$$d_k(\mathcal{E}) = \sum_{r=1}^{k} (-1)^r \sum_{\substack{j_1+\cdots+j_r=k \\ j_\nu>0}} c_{j_1}(\mathcal{E}) \cdots c_{j_r}(\mathcal{E}) . \tag{3.2}$$

Let $c(F^*)=1+c_1(F^*)+\cdots+c_n(F^*)$ be the (rational) total Chern class in $H^*(U\,;\, \boldsymbol{Q})$ of F^*. Note that $c_i(F^*)=0$, $q+1 \leqq i \leqq n$, since F^* is a locally free sheaf of rank q. Also note that there is a canonical pairing

$$H^*(U\,;\, \boldsymbol{Q}) \times H^*_c(U\,;\, \boldsymbol{Q}) \longrightarrow H^*_c(U\,;\, \boldsymbol{Q}) .$$

(3.3) DEFINITION. For each integer j with $q<j \leqq n$, $c_j(F^*-\mathcal{E})$ denotes the element

$$c_q(F^*)d_{j-q}(\mathcal{E})+\cdots+c_1(F^*)d_{j-1}(\mathcal{E})+d_j(\mathcal{E})$$

in $H_c^{2j}(U;\boldsymbol{Q})$, and for each integer j with $1\leqq j\leqq q$, it denotes the element

$$c_j(F^*)+c_{j-1}(F^*)\kappa(d_1(\mathcal{E}))+\cdots+c_1(F^*)\kappa(d_{j-1}(\mathcal{E}))+\kappa(d_j(\mathcal{E}))$$

in $H^{2j}(U;\boldsymbol{Q})$.

(3.4) THEOREM. *Let F be a foliation (of complete intersection type) of codim q on M and let U and Z be as above. If $\phi=\sigma_{j_1}\cdots\sigma_{j_r}$ with $j_\nu>q$ for some ν, then*

$$\mathrm{Res}_\phi(F,Z)=L(c_{j_1}(F^*-\mathcal{E})\cdots c_{j_r}(F^*-\mathcal{E}))\,,$$

where $\mathrm{Res}_\phi(F,Z)=\mathrm{Res}_\phi(F^a,Z)$ *and L is the composition of two isomorphisms in* (1.3).

PROOF. Let D_{-1} be a basic connection for Q on $U-Z$. Since $Q=F^*$ on $U-Z$ and F^* is locally free on U, by [3] (4.41), the connection D_{-1} can be modified to obtain a connection $\tilde{D}_{-1}$ for F^* on U such that

$$\tilde{D}_{-1}=D_{-1}\quad\text{on}\quad U-\Sigma\,, \tag{3.5}$$

where Σ is a compact set in U containing Z in its interior. The connection $\tilde{D}_{-1}$ determines, for each i with $1\leqq i\leqq q$, a closed $2i$-form $\sigma_i(F^*)$ on U such that the class of $(\sqrt{-1}/2\pi)^i\sigma_i(F^*)$ in $H^*(U;\boldsymbol{C})$ is $c_i(F^*)$. The equation

$$\begin{aligned}(1+\sigma_1(Q)+\cdots+\sigma_n(Q))(1+\sigma_1(\mathcal{E})+\cdots+\sigma_n(\mathcal{E}))\\ =1+\sigma_1(F^*)+\cdots+\sigma_q(F^*)\end{aligned} \tag{3.6}$$

can be solved to find $\sigma_1(\mathcal{E}),\cdots,\sigma_n(\mathcal{E})$ such that, for each j, $1\leqq j\leqq n$, $\sigma_j(\mathcal{E})$ is a closed $2j$-form on U. By (3.5), $\sigma_j(Q)=\sigma_i(F)$, $1\leqq i\leqq q$ on $U-\Sigma$. Also by the Bott vanishing theorem, $\sigma_{q+1}(Q),\cdots,\sigma_n(Q)$ have compact support. Hence each $\sigma_j(\mathcal{E})$, $1\leqq j\leqq n$, has compact support. Moreover, the class of $(\sqrt{-1}/2\pi)^j\sigma_j(\mathcal{E})$ in $H_c^*(U;\boldsymbol{C})$ is $c_j(\mathcal{E})$. If we set, for $k=1,\cdots,n$,

$$\tau_k(\mathcal{E})=\sum_{r=1}^{k}(-1)^r\sum_{\substack{j_1+\cdots+j_r=k\\ j_\nu>0}}\sigma_{j_1}(\mathcal{E})\cdots\sigma_{j_r}(\mathcal{E})\,,$$

then we have $(1+\tau_1(\mathcal{E})+\cdots+\tau_n(\mathcal{E}))(1+\sigma_1(\mathcal{E})+\cdots+\sigma_n(\mathcal{E}))=1$. From (3.6), we have

$$\sigma_j(Q)=\sum_{\substack{i+k=j\\ i,k\geqq 0}}\sigma_i(F^*)\tau_k(\mathcal{E})\,,\qquad j=1,\cdots,n\,, \tag{3.7}$$

where we set $\sigma_0(F^*)=\tau_0(\mathcal{E})=1$ and $\sigma_{q+1}(F^*)=\cdots=\sigma_n(F^*)=0$. Thus if $j>q$, each term in the right hand side of (3.7) has compact support and the class of $\left(\frac{\sqrt{-1}}{2\pi}\right)^j\sigma_j(Q)$ in $H_c^{2j}(U;\boldsymbol{Q})$ is $c_j(F^*-\mathcal{E})$ (see (3.3) Definition). Therefore, if $\phi=\sigma_{j_1}\cdots\sigma_{j_r}$ with $j_\nu>q$ for some ν, then $c_{j_1}(F^*-\mathcal{E})\cdots c_{j_r}(F^*-\mathcal{E})$ is in $H_c^{2j}(U;\boldsymbol{Q})$, $j=j_1+\cdots+j_r$, and is the class of $\phi(Q)$, Q.E.D.

(3.8) COROLLARY. *Let F and Z be as above and let ϕ be a symmetric and homogeneous polynomial of degree d in $X_1, \cdots, X_n$. If each monomial in the expression $\phi=\tilde{\phi}(\sigma_1, \cdots, \sigma_n)$ contains σ_j with $j>q$, then* $\mathrm{Res}_{\phi}(F, Z)$ *is rational, i.e. it is in* $H_{2n-2d}(Z\,;\,\boldsymbol{Q})$ (cf. [3] p. 287 Rationality Conjecture).

Suppose now that Z is non-singular and that there is a holomorphic vector bundle E on Z such that $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})=i_!\mathcal{O}_Z(E)$ (=the sheaf $\mathcal{O}_Z(E)$ of germs of holomorphic sections of E extended by zero on $U-Z$), where i is the embedding $Z\hookrightarrow U$. Then (the finer version of) the Riemann-Roch theorem for analytic embeddings (Atiyah-Hirzebruch [2] Theorem (3.1), see also the proof of Theorem (3.3)) gives the local Chern classes of $\mathcal{E}=\gamma_Z(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O}))$;

$$\mathrm{ch}(\mathcal{E})=i_*(\mathrm{td}(N)^{-1}\mathrm{ch}(E)) \tag{3.9}$$

or

$$c_1(\mathcal{E})+\cdots+c_n(\mathcal{E})=i_*\left(\frac{c(\lambda_{-1}(N^*))*c(E)-1}{c_r(N)}\right), \tag{3.10}$$

where N is the normal bundle of Z in U, $r=\mathrm{rank}\,N=\mathrm{codim}\,Z$ in U, $\lambda_{-1}(N^*) =\sum_{i=0}^{r}(-1)^i\lambda^i(N^*)$ ($\lambda^i(N^*)=i$-th exterior power of N^*), $c(\lambda_{-1}(N^*))*c(E)$ is the total Chern class of the tensor product $\lambda_{-1}(N^*)\otimes E$ and i_* is the Thom-Gysin homomorphism

$$i_*: H^*(Z\,;\,\boldsymbol{Q})\longrightarrow H^*(U, U-Z\,;\,\boldsymbol{Q})=H^*_c(U\,;\,\boldsymbol{Q}). \tag{3.11}$$

By our assumption, Z is non-singular. Thus we have a commutative diagram

$$\begin{array}{ccc} H^p(Z\,;\,\boldsymbol{Q}) & \xrightarrow{\ i_*\ } & H_c^{p+2r}(U\,;\,\boldsymbol{Q}) \\ D_Z\downarrow\wr & \swarrow L & \wr\downarrow D_U \\ H_{2n-2r-p}(Z\,;\,\boldsymbol{Q}) & \underset{\sim}{\xrightarrow{\ i_*\ }} & H_{2n-2r-p}(U\,;\,\boldsymbol{Q}), \end{array}$$

where D_Z is the Poincaré duality map, and i_* in (3.11) is an isomorphism.

In particular, if the singularity is isolated, we have

(3.12) PROPOSITION. *Let U be a polydisk about the origin 0 in $\boldsymbol{C}^n$ and let $F=(\omega)$ be a codim 1 foliation on U with an isolated singularity at 0. We denote the stalks $\mathcal{O}_{\boldsymbol{C}^n,0}$ and $\Omega_{F,0}$ simply by $\mathcal{O}$ and Ω_F, respectively. Then we have*

$$\mathrm{Res}_{\sigma_n}(F, \{0\})=(-1)^n(n-1)!\,\dim_{\boldsymbol{C}}\mathrm{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O}) \quad \text{in} \quad H_0(\{0\}\,;\,\boldsymbol{Q})=\boldsymbol{Q}.$$

PROOF. Since $H^{2j}_c(U\,;\,\boldsymbol{Q})=0$ for $j\neq n$, we have $c_j(\mathcal{E})=0$ for $1\leqq j\leqq n$. Also $c_i(F^*)=0$ for $i>0$. Hence by (3.4) Theorem and (3.2), we have

$$\mathrm{Res}_{\sigma_n}(F, \{0\})=L(d_n(\mathcal{E}))=-L(c_n(\mathcal{E})).$$

On the other hand, for a point z in U,

$$\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})_z = \begin{cases} \operatorname{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O}), & \text{if } z=0, \\ 0 \quad , & \text{if } z \neq 0. \end{cases}$$

We set $E=\operatorname{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$ and think of it as a vector bundle over $Z=\{0\}$ of rank $\mu=\dim_C E$. Then we have $\mathcal{E}xt^1_{\mathcal{O}}(\Omega_F, \mathcal{O})=i_! \mathcal{O}_Z(E)$. In (3.9), we have $\operatorname{td}(N)=1$ and $\operatorname{ch}(E)=\mu$ in $H^0(\{0\}\,;\, \boldsymbol{Q})=\boldsymbol{Q}$. Thus denoting by θ the image of 1 by the isomorphism $i_*: H^0(\{0\}\,;\, \boldsymbol{Q}) \to H^{2n}_c(U\,;\, \boldsymbol{Q})$, we have

$$\operatorname{ch}(\mathcal{E})=\mu\theta. \tag{3.13}$$

Writing formally $1+c_1(\mathcal{E})+\cdots+c_n(\mathcal{E})=\prod_{i=1}^{n}(1+\gamma_i)$, we have $\operatorname{ch}(\mathcal{E})=\sum_{i=1}^{n}(e^{\gamma_i}-1)$. From (3.13),

$$\gamma_1^j+\cdots+\gamma_n^j=\begin{cases} 0, & \text{if } 1 \leqq j \leqq n-1, \\ n!\,\mu\theta, & \text{if } j=n. \end{cases}$$

Thus we have $n\gamma_1\cdots\gamma_n+(-1)^n(\gamma_1^n+\cdots+\gamma_n^n)=0$. Hence $c_n(\mathcal{E})=\gamma_1\cdots\gamma_n=(-1)^{n+1}(n-1)!\,\mu\theta$. Q.E.D.

(3.14) REMARK. In the situation of (3.12), if we write $\omega=\sum_{i=1}^{n} f_i(z)dz_i$, then

$$\operatorname{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O})=\boldsymbol{C}\{z_1, \cdots, z_n\}/(f_1, \cdots, f_n),$$

where $\mathcal{O}=\boldsymbol{C}\{z_1, \cdots, z_n\}$ is the ring of convergent power series in $z_1, \cdots, z_n$ and $(f_1, \cdots, f_n)$ is the ideal generated by the germs of $f_1(z), \cdots, f_n(z)$ at 0 ([**7**] (4.5)). Especially, if $\omega=df$ for some f, then $f_i=\dfrac{\partial f}{\partial z_i}$. Thus (3.12) can be viewed as a formula for the "generalized" multiplicity (cf. [**6**]). For the significance of $\operatorname{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O})$, see also [**8**].

Here is an example with non-isolated singular set.

(3.15) EXAMPLE. Let $\boldsymbol{P}^1=\boldsymbol{P}^1(\boldsymbol{C})$ be the projective line with homogeneous coordinates $(\zeta_0 : \zeta_1)$. It is covered by two coordinate neighborhoods U_0 and U_1 with coordinates $z_0=\zeta_1/\zeta_0$ and $z_1=\zeta_0/\zeta_1$, respectively. We denote by H the hyperplane bundle over $\boldsymbol{P}^1$. Letting l and m be two integers, consider the vector bundle N of rank 2 over $\boldsymbol{P}^1$ given by $N=H^l \oplus H^m$. Thus N can be expressed as a union $N=\boldsymbol{C}^2\times U_0 \cup \boldsymbol{C}^2 \times U_1$, where a point (x_0, y_0, z_0) in $\boldsymbol{C}^2\times U_0$ is identified with (x_1, y_1, z_1) in $\boldsymbol{C}^2\times U_1$ if and only if

$$x_0=z_1^{-l}x_1, \quad y_0=z_1^{-m}y_1 \quad \text{and} \quad z_0=z_1^{-1}. \tag{3.16}$$

We identify $\boldsymbol{P}^1$ with the zero section $x_i=y_i=0$, $i=0, 1$, in N. Let a and b be positive integers satisfying $l(a-1)=m(b-1)$. We set $r=l(a-1)=m(b-1)$. C: each $W_i=\boldsymbol{C}^2\times U_i$, $i=0, 1$, we consider two holomorphic 1-forms τ_i and ω_i given by

$$\tau_i = dz_i \quad \text{and} \quad \omega_i = y_i^b dx_i - x_i^a dy_i .$$

In the intersection $W_0 \cap W_1$, we have

$$\begin{pmatrix} \tau_0 \\ \omega_0 \end{pmatrix} = \begin{pmatrix} -z_1^{-2} & 0 \\ x_1 y_1 z_1^{-s-1}(m x_1^{a-1} - l y_1^{b-1}) & z_1^{-s} \end{pmatrix} \begin{pmatrix} \tau_1 \\ \omega_1 \end{pmatrix}, \tag{3.17}$$

where $s = r + l + m$. Thus we may consider the locally free sub-$\mathcal{O}_N$-module F of Ω_N generated by τ_i and ω_i on W_i. Clearly F satisfies the integrability condition and defines a codim 2 foliation on N with singular set the zero section $\boldsymbol{P}^1$. Now we find the sheaf $\mathcal{E}xt^1_{\mathcal{O}_N}(\Omega_F, \mathcal{O}_N)$. From (2.3), we have

$$\mathcal{E}xt^1_{\mathcal{O}_N}(\Omega_F, \mathcal{O}_N)|_{W_i} \cong \mathcal{O}^2_{W_i} / ((0, y_i^b), (0, x_i^a), (1, 0)),$$

where the denominator in the right hand side denotes the sub-$\mathcal{O}_{W_i}$-module of $\mathcal{O}^2_{W_i}$ generated by $(0, y_i^b)$, $(0, x_i^a)$ and $(1, 0)$. Hence we have

$$\mathcal{E}xt^1_{\mathcal{O}_N}(\Omega_F, \mathcal{O}_N)|_{W_i} \cong \mathcal{O}_{W_i} / (x_i^a, y_i^b), \tag{3.18}$$

where (x_i^a, y_i^b) is the ideal generated by the sections x_i^a and y_i^b. For an element h in $\mathcal{O}_{W_i}$, we denote by $[h]$ its class in $\mathcal{O}_{W_i}/(x_i^a, y_i^b)$. The right hand side of (3.18) is a free $\mathcal{O}_{U_i}$-module generated by $[x_i^\alpha y_i^\beta]$, $0 \leqq \alpha \leqq a-1$, $0 \leqq \beta \leqq b-1$. Moreover, by (3.16), we have

$$x_0^\alpha y_0^\beta = z_1^{-(\alpha l + \beta m)} x_1^\alpha y_1^\beta .$$

Hence we may write $\mathcal{E}xt^1_{\mathcal{O}_N}(\Omega_F, \mathcal{O}_N) = i_! \mathcal{O}_{\boldsymbol{P}^1}(E)$, where E is the vector bundle over $\boldsymbol{P}^1$ of rank ab given by

$$E = \bigoplus_{\substack{0 \leqq \alpha \leqq a-1 \\ 0 \leqq \beta \leqq b-1}} H^{\alpha l + \beta m} .$$

We have

$$\mathrm{ch}(E) = \sum_{\substack{0 \leqq \alpha \leqq a-1 \\ 0 \leqq \beta \leqq b-1}} (1+\eta)^{\alpha l + \beta m} = ab(1 + r\eta),$$

where η is the first Chern class of H and is a generator of $H^2(\boldsymbol{P}^1; \boldsymbol{Q}) \cong \boldsymbol{Q}$. On the other hand, from $N = H^l \oplus H^m$, we have

$$\mathrm{td}(N)^{-1} = 1 - \frac{l+m}{2} \eta .$$

Hence we have

$$\mathrm{td}(N)^{-1} \mathrm{ch}(E) = \mathrm{ab}\Big(1 + \Big(r - \frac{l+m}{2}\Big)\eta\Big) .$$

We have the Thom isomorphism $H^p(\boldsymbol{P}^1; \boldsymbol{Q}) \overset{i_*}{\simeq} H^{p+4}(N, N - \boldsymbol{P}^1; \boldsymbol{Q})$. Setting $\theta_2 = i_*(1) \in H^4(N, N - \boldsymbol{P}^1; \boldsymbol{Q})$ and $\theta_3 = i_*(\eta) \in H^6(N, N - \boldsymbol{P}^1; \boldsymbol{Q})$, we have from (3.9),

$$\mathrm{ch}(\mathcal{E})=ab\left(\theta_2+\left(r-\frac{l+m}{2}\right)\theta_3\right).$$

Thus we have

$$c_1(\mathcal{E})=0,\quad c_2(\mathcal{E})=-ab\theta_2 \quad\text{and}\quad c_3(\mathcal{E})=ab(2r-(l+m))\theta_3\,.$$

From (3.2), we have

$$d_1(\mathcal{E})=0,\quad d_2(\mathcal{E})=ab\theta_2 \quad\text{and}\quad d_3(\mathcal{E})=ab(l+m-2r)\theta_3\,.$$

Next we find $c(F^*)$. Since $H^*(N;\boldsymbol{Q})\overset{i^*}{\simeq}H^*(\boldsymbol{P}^1;\boldsymbol{Q})$, it suffices to find $c(i^*F^*)$. From (3.17), we have $i^*F^*=H^2\oplus H^s$. Thus $c(i^*F^*)=1+(s+2)\eta$. Therefore, $c(F^*)=1+(s+2)\sigma$, where σ denotes $i^{*-1}\eta$ and is a generator of $H^2(N;\boldsymbol{Q})\cong\boldsymbol{Q}$. We have

$$\begin{aligned}c_3(F^*-\mathcal{E})&=c_2(F^*)d_1(\mathcal{E})+c_1(F^*)d_2(\mathcal{E})+d_3(\mathcal{E})\\&=ab(s+2)\sigma\theta_2+ab(l+m-2r)\theta_3\\&=ab(2(l+m+1)-r)\theta_3\,.\end{aligned}$$

Therefore,

$$\mathrm{Res}_{\sigma_3}(F,\boldsymbol{P}^1)=ab(2(l+m+1)-r)\quad\text{in}\quad H_0(\boldsymbol{P}^1;\boldsymbol{Q})=\boldsymbol{Q}\,.$$

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Tatsuo SUWA
Department of Mathematics
Hokkaido University
Sapporo 060
Japan