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Volumes of tubes about Kähler submanifolds expressed in terms of Chern classes

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1. Introduction.

Let P be a topologically embedded Kähler submanifold of compact closure in a complete Kähler manifold M. Denote by $V_P^M(r)$ the volume of a tube of radius r about P. I shall give inequalities for $V_P^M(r)$ in terms of the Chern classes of P and M that depend on the sectional curvature of M. These inequalities are generalizations of Weyl's formula [**WY**] for the volumes of tubes about submanifolds of Euclidean space.

Let F be the Kähler form of P and denote by $\gamma_c(R^P-R^M)$ the c^{th} Chern form of R^P-R^M, where R^P and R^M are the curvature operators of P and M. Also let K^M denote the sectional curvature of M, and let $n=\dim_C M$, $q=\dim_C P$.

THEOREM 1.1. *Suppose $r>0$ is not larger than the distance from P to its nearest focal point.*

(i) *If $K^M\geqq 0$ then*

$$V_P^M(r)\leqq\sum_{c=0}^{q}\frac{(\gamma_c(R^P-R^M)\wedge F^{q-c})[P]}{(n-q+c)!\,(q-c)!}(\pi r^2)^{n-q+c}\leqq\frac{(\pi r^2)^{n-q}}{(n-q)!}\operatorname{vol}(P). \tag{1.1}$$

(ii) *If $K^M\leqq 0$ then*

$$V_P^M(r)\geqq\sum_{c=0}^{q}\frac{(\gamma_c(R^P-R^M)\wedge F^{q-c})[P]}{(n-q+c)!\,(q-c)!}(\pi r^2)^{n-q+c}. \tag{1.2}$$

(iii) *If M has nonnegative holomorphic bisectional curvature, then*

$$V_P^M(r)\leqq\frac{(\pi r^2)^{n-q}}{(n-q)!}\operatorname{vol}(P). \tag{1.3}$$

COROLLARY 1.2. *If $P\subset \boldsymbol{C}^n$ is a Kähler submanifold and $r>0$ is not greater than the distance from P to its nearest focal point, then*

$$V_P^{\boldsymbol{C}^n}(r)=\sum_{c=0}^{q}\frac{(\gamma_c(R^P)\wedge F^{q-c})[P]}{(n-q+c)!\,(q-c)!}(\pi r^2)^{n-q+c}. \tag{1.4}$$

Corollary 1.2 is a consequence either of theorem 1.1, or Weyl's tube formula together with an algebraic lemma (Lemma 2.3) about powers of curvature tensor fields of Kähler manifolds. Griffiths [**GS**] observed the form of the right side of (1.4), but he did not determine the value of the coefficients.

The rest of the paper will be devoted to generalizing Corollary 1.2 to complex submanifolds of Kähler manifolds of nonzero curvature. In Section 3 I shall prove

THEOREM 1.3. *Let P^{2q} be a topologically embedded complex submanifold (of real dimension $2q$) with compact closure in a space $M(\lambda)$ of constant holomorphic sectional curvature 4λ. Then the volume $V_P^{M(\lambda)}(r)$ of a tube of radius r about P in $M(\lambda)$ is completely expressible in terms of the Kähler form F and the Chern forms $\gamma_1, \cdots, \gamma_q$ of P. More precisely when $\lambda>0$*

$$V_P^{M(\lambda)}(r)=\sum_{d=0}^{q} C_d(\lambda, F, \gamma_1, \cdots, \gamma_d)(\sin\sqrt{\lambda}\, r)^{2(n-d)} \tag{1.5}$$

where

$$C_d(\lambda, F, \gamma_1, \cdots, \gamma_d)=\frac{(-1)^{d+q}}{(n-d)d!}\left(\frac{\pi}{\lambda}\right)^{n-q}\sum_{a=0}^{q-d} D_{da}\left(-\frac{\pi}{\lambda}\right)^{a}(q-a+1)!\,(\gamma_a\wedge F^{q-a})[P]$$

and $D_{da}=\sum_{b=0}^{q-d-a}\dfrac{1}{b!\,(q-a-b+1)!\,(q-a-b-d)!\,(n-q+a+b-1)!}$. *When $\lambda<0$ one puts* $\sinh\sqrt{|\lambda|}\, r$ *in place of* $\sin\sqrt{\lambda}\, r$ *in* (1.5). (*Formula* (1.5) *reduces to* (1.4) *when* $\lambda=0$).

For theorem 1.3 the Chern forms of $R^P-R^{M(\lambda)}$ must be calculated in terms of those of R^P. This computation is carried out in Section 3. Katz [**KA**] has performed similar calculations for complex hypersurfaces. It should also be remarked that for the case $\lambda>0$ Flaherty [**FL**] and Wolf [**WO**] have obtained tube formulas in which the tube coefficients are shown to be metric invariants. The formulas in these papers are found by transferring Weyl's formulas from Euclidean space to complex projective space via the natural projection. The main point of Theorem 1.3 is to establish the topological character of the tube coefficients for submanifolds (both in complex projective space and also in complex hyperbolic space).

In fact

COROLLARY 1.4. *Fix λ and f. Let P be any Kähler manifold for which*

$$[\gamma_c]=f(c)\left[\frac{\lambda}{\pi}F\right]^c \tag{1.6}$$

for $c=1, \cdots, q$. Then $V_P^{M(\lambda)}(r)$ depends only on the volume of P and is otherwise independent of the Kähler metric (compatible with the given complex structure).

In particular

COROLLARY 1.5. *Let P be a complete intersection in $\boldsymbol{C}P^n(\lambda)$. Then $V_P^{\boldsymbol{C}P^n(\lambda)}(r)$ depends only on the degrees of the polynomials defining P.*

Finally in Section 4 a comparison theorem for the case $K^M \geqq \lambda$ that partially combines Theorems 1.2 and 1.5 will be given.

2. An identity for the c^{th} power of the curvature tensor field of a Kähler manifold.

In [**GR4**, Section 7] tensor fields having the same symmetries as the curvature tensor field were considered. Such tensor fields will be called curvature-like. One can form the c^{th} power R^c of a curvature-like tensor R and also the complete contraction $C^{2c}(R^c)$.

Now consider the case of an almost Hermitian manifold P of real dimension $2q$. Let J be the almost complex structure.

DEFINITION. A curvature-like tensor field R on an almost Hermitian manifold P is said to be *Kählerian* provided

$$R(wx)(yz)=R(Jw\,Jx)(yz) \tag{2.1}$$

for all tangent vectors w, x, y, z to P.

For a Kähler curvature-like tensor field R not only is there the contraction

$$C^{2c}(R^c)=\sum_{a_1\cdots a_{2c}=1}^{2q} R^c(e_{a_1}\cdots e_{a_{2c}})(e_{a_1}\cdots e_{a_{2c}}) \tag{2.2}$$

(where $\{e_1\cdots e_{2q}\}$ is any orthonormal basis of a tangent space P_p); one also has the contraction

$$2\sum_{\substack{a_1\cdots a_c=1\\ b_1\cdots b_c=1}}^{2q} R^c(e_{a_1}e^*_{a_1}\cdots e_{a_c}e^*_{a_c})(e_{b_1}e^*_{b_1}\cdots e_{b_c}e^*_{b_c}) \tag{2.3}$$

where $e_i^*=Je_i$. When $c=1$ the Kähler identity (2.1) implies that (2.2) and (2.3) coincide, the common value being the scalar curvature of R. In order to express the tube coefficients in terms of the Chern forms it will be necessary to know a relation between (2.2) and (2.3) for arbitrary c.

LEMMA 2.1 *Let R be a Kähler curvature-like tensor field on an almost Hermitian manifold P. Then (writing a_i for e_{a_i})*

$$\begin{aligned}\sum_{\substack{a_1\cdots a_c=1\\ b_1\cdots b_c=1}}^{2q} & R^c(a_1a_1^*\cdots a_ca_c^*)(b_1b_1^*\cdots b_cb_c^*) \\ &=\frac{(2^c c\,!\,)^2}{(2c)\,!}\sum_{a_1\cdots a_{2c}=1}^{2q} R^c(a_1\cdots a_{2c})(a_1\cdots a_{2c})=\frac{(2^c c\,!\,)^2}{(2c)\,!}C^{2c}(R^c),\end{aligned} \tag{2.4}$$

where $\{e_1 \cdots e_{2q}\}$ *is an arbitrary orthonormal basis of a tangent space* P_p *to* P.

PROOF. Put

(2.5) $$b(i, j)=\sum_{\substack{a_1\cdots a_i=1 \\ b_1\cdots b_i=1}}^{2q} R^c(a_1a_1^* \cdots a_ia_i^*d_{2i+1} \cdots d_{2c})(b_1b_1^* \cdots b_ib_i^*d_{2i+1} \cdots d_{2c}).$$

From the Kähler identity (2.1) follows the generalized Kähler identity

(2.6) $$R^c(x_1^* \cdots x_{2c}^*)(y_1 \cdots y_{2c})=R^c(x_1 \cdots x_{2c})(y_1 \cdots y_{2c})$$

for tangent vectors $x_1, \cdots, x_{2c}, y_1, \cdots, y_{2c} \in P_p$. Then from (2.5) and (2.6) one has

(2.7) $$b(i, j)=\sum R^c(a_1a_1^* \cdots a_ia_i^*d_{2i+1}^* \cdots d_{2c}^*)(b_1b_1^* \cdots b_ib_i^*d_{2i+1} \cdots d_{2c}).$$

There is also a generalized Bianchi identity [**TH3**]:

(2.8) $$\sum_{k=1}^{2c+1}(-1)^k R^c(x_1 \cdots \hat{x}_k \cdots x_{2c+1})(x_ky_2 \cdots y_{2c})=0$$

for $x_1, \cdots, x_{2c+1}, y_2, \cdots, y_{2c} \in P_p$. (See also [**GR1, 2**], [**TH1, 2**].) From (2.7) and (2.8) it follows after some calculation that

(2.9) $$b(i, j)=\left(\frac{2i}{2j+1}\right)b(i-1, j+1).$$

Repeated use of (2.9) yields $(2i)!\,b(i, 0)=(2^i i!)^2 b(0, i)$, which is just (2.4).

The rest of this section is devoted to showing how to express the tube coefficients $k_q(R)$ for a Kähler curvature-like tensor field R in terms of the Chern forms of R.

Let F denote the Kähler form of the almost Hermitian manifold P, and let $\wedge^c(P)$ be the space of c-forms on P. Thus $F(xy)=\langle Jx, y\rangle$ for tangent vectors x, y to P. Also $F^c \in \wedge^{2c}(P)$.

LEMMA 2.2. *Let* $\phi \in \wedge^{2c}(P)$ *and let* $\{e_1 \cdots e_{2q}\}$ *be any orthonormal basis of* P_p *compatible with the orientation of* P. *Then*

(2.10) $$(\phi \wedge F^{q-c})(e_1 \cdots e_{2q}) = \frac{(q-c)!}{2^c c!} \sum_{a_1\cdots a_c=1}^{2q} \phi(e_{a_1}e_{a_1}^* \cdots e_{a_c}e_{a_c}^*).$$

We omit the proof which is straightforward. (Note that the definition of wedge product involving shuffle permutations is being used.)

According to [**GR3**] the c^{th} Chern form $\gamma_c(R) \in \wedge^{2c}(P)$ of a Kähler curvature-like tensor field R is given by

(2.11) $$(2\pi)^c \gamma_c(R)(x_1 \cdots x_{2c}) = \frac{1}{2^c(c!)^2} \sum_{a_1\cdots a_{2c}=1}^{2q} R^c(a_1a_1^* \cdots a_ca_c^*)(x_1 \cdots x_{2c})$$

for tangent vectors $x_1, \cdots, x_{2c}$ to P. Therefore

LEMMA 2.3. *Let R be a Kähler curvature-like tensor field on an almost Hermitian manifold P. For $p\in P$ let $\{e_1\cdots e_{2q}\}$ be any orthonormal basis of P_p compatible with the orientation of P. Then*

$$(2.12)\qquad (2\pi)^c(\gamma_c(R)\wedge F^{q-c})(e_1\cdots e_{2q})=\frac{(q-c)!}{c!(2c)!}C^{2c}(R^c).$$

Hence

$$(2.13)\qquad k_{2c}(R)=\frac{(2\pi)^c}{(q-c)!}(\gamma_c(R)\wedge F^{q-c})[P],$$

provided the integrals in (2.13) *exist.*

PROOF. From Lemmas 2.1 and 2.2 together with equation (2.11) one gets (2.12). Then (2.13) follows from (2.12) and the definition of $k_{2c}(R)$ ([**GR4**, formula (7.6)]).

PROOF OF COROLLARY 1.2. In (2.13) we take $R=R^P$. Thus from Weyl's tube formula [**GR4**, formula (1.1)] we obtain (1.4).

3. Tubes about submanifolds of spaces of constant holomorphic sectional curvature.

Before proving theorems about Chern classes some preliminary facts about complex differential forms will be needed.

LEMMA 3.1. *Let $\alpha_1, \cdots, \alpha_c$ be complex 1-forms. Then*

$$\det\begin{pmatrix}\alpha_1\wedge\bar\alpha_1 & \cdots & \alpha_1\wedge\bar\alpha_c\\ \vdots & & \vdots\\ \alpha_c\wedge\bar\alpha_1 & \cdots & \alpha_c\wedge\bar\alpha_c\end{pmatrix}=c!\,\alpha_1\wedge\bar\alpha_1\wedge\cdots\wedge\alpha_c\wedge\bar\alpha_c.$$

PROOF. Since $\varepsilon_\sigma\bar\alpha_{\sigma(1)}\wedge\cdots\wedge\bar\alpha_{\sigma(c)}=\bar\alpha_1\wedge\cdots\wedge\bar\alpha_c$ it follows that $\varepsilon_\sigma\alpha_1\wedge\bar\alpha_{\sigma(1)}\wedge\cdots\wedge\alpha_c\wedge\bar\alpha_{\sigma(c)}=\alpha_1\wedge\bar\alpha_1\wedge\cdots\wedge\alpha_c\wedge\bar\alpha_c$. Write $a_{ij}=\alpha_i\wedge\bar\alpha_j$. Then by definition of the determinant,

$$\begin{aligned}\det(a_{ij}) &= \sum_{\sigma\in\mathfrak{S}_c}\varepsilon_\sigma a_{1\sigma(1)}\cdots a_{c\sigma(c)}\\ &= \sum_{\sigma\in\mathfrak{S}_c}\varepsilon_\sigma\alpha_1\wedge\bar\alpha_{\sigma(1)}\wedge\cdots\wedge\alpha_c\wedge\bar\alpha_{\sigma(c)}\\ &= c!\,\alpha_1\wedge\bar\alpha_1\wedge\cdots\wedge\alpha_c\wedge\bar\alpha_c.\end{aligned}$$

Next let P be any almost Hermitian manifold with almost complex structure J, metric $\langle\ ,\ \rangle$ and Kähler form F. Let $\{E_1\cdots E_qJE_1\cdots JE_q\}$ be a local orthonormal frame field preserved by J and let $\theta_1, \cdots, \theta_q, \theta_{1^*}, \cdots, \theta_{q^*}$ be the dual basis of 1-forms. For $a=1, \cdots, q$ write $\phi_a=\theta_a+\sqrt{-1}\,\theta_{a^*}$. Then it is easy to see that

$$\langle\ ,\ \rangle=\sum_{i=1}^{2q}\theta_i^2=2\sum_{a=1}^{q}\theta_a^2=\sum_{a=1}^{q}\phi_a\bar{\phi}_a \tag{3.1}$$

$$F=\frac{1}{2}\sum_{i=1}^{2q}\theta_i\wedge\theta_{i^*}=\sum_{a=1}^{q}\theta_a\wedge\theta_{a^*}=\frac{\sqrt{-1}}{2}\sum_{a=1}^{q}\phi_a\wedge\bar{\phi}_a \tag{3.2}$$

(where $\theta_{(a+q)^*}=-\theta_a$). Furthermore an easy calculation using Lemma 3.1 shows

COROLLARY 3.2.

$$\sum_{a_1\cdots a_c=1}^{q}\det\begin{pmatrix}\phi_{a_1}\wedge\bar{\phi}_{a_1}\cdots\phi_{a_1}\wedge\bar{\phi}_{a_c}\\ \vdots \qquad\qquad \vdots\\ \phi_{a_c}\wedge\bar{\phi}_{a_1}\cdots\phi_{a_c}\wedge\bar{\phi}_{a_c}\end{pmatrix}=c\,!\,(-2\sqrt{-1}\,F)^c\,.$$

The Cartan structure equations will be written down in a convenient form. Let ∇ be the Riemannian connection of P. For $X, Y\in\mathfrak{X}(P)$ put $\omega_{ij}(X)=\langle\nabla_X E_i, E_j\rangle$, $\omega_{ij^*}(X)=\langle\nabla_X E_i, JE_j\rangle$, $\Omega_{ij}(XY)=\langle R_{XY}E_i, E_j\rangle$ and $\Omega_{ij^*}(XY)=\langle R_{XY}E_i, JE_j\rangle$. Then (as is well-known) we have the *real structure equations*

$$d\theta_i=\sum_{j=1}^{2q}\omega_{ij}\wedge\theta_j\,, \tag{3.3}$$

$$d\omega_{ij}=\sum_{k=1}^{2q}\omega_{ik}\wedge\omega_{kj}-\Omega_{ij}\,,\qquad i, j=1,\cdots,2q\,. \tag{3.4}$$

Also put $\psi_{ab}=\omega_{ab}-\sqrt{-1}\,\omega_{ab^*}$ and $\Xi_{ab}=\Omega_{ab}-\sqrt{-1}\,\Omega_{ab^*}$ for $a, b=1,\cdots,q$. When P is a Kähler manifold,

$$\omega_{i^*j^*}=\omega_{ij}\quad\text{and}\quad\Omega_{i^*j^*}=\Omega_{ij}\,,\qquad i, j=1,\cdots,2q\,. \tag{3.5}$$

Therefore from (3.3), (3.4), (3.5) follow the *complex structure equations*

$$d\phi_a=\sum_{b=1}^{q}\psi_{ab}\wedge\phi_b\,, \tag{3.6}$$

$$d\psi_{ab}=\sum_{c=1}^{q}\psi_{ac}\wedge\psi_{cb}-\Xi_{ab}\,,\qquad a, b=1,\cdots,q\,, \tag{3.7}$$

for a Kähler manifold P. Using matrix notation (3.3), (3.4), (3.6) and (3.7) can be written as

$$\begin{cases} d\theta=\omega\wedge\theta \qquad d\omega=\omega\wedge\omega-\Omega\,,\\ d\phi=\psi\wedge\phi \qquad d\psi=\psi\wedge\psi-\Xi\,. \end{cases} \tag{3.8}$$

From (3.8) one obtains the following form of the Bianchi identities

$$\Omega\wedge\theta=\Xi\wedge\phi=0\,, \tag{3.9}$$

$$d\Omega-\omega\wedge\Omega+\Omega\wedge\omega=d\Xi-\psi\wedge\Xi+\Xi\wedge\psi=0\,. \tag{3.10}$$

In the sequel only (3.9) will be needed.

Next let $\tilde{\Xi}_{ab}=\Xi_{ab}-\lambda(\phi_a\wedge\bar{\phi}_b-2\sqrt{-1}\,\delta_{ab}F)$ and let $\tilde{\Xi}$ be the corresponding matrix. Note that the $\lambda(\phi_a\wedge\bar{\phi}_b-2\sqrt{-1}\,\delta_{ab}F)$ are just the complex curvature forms of a space $M(\lambda)$ of constant holomorphic curvature 4λ. Although $\tilde{\Xi}$ is defined abstractly, when P is a complex submanifold of $M(\lambda)$ the matrix $\tilde{\Xi}$ can be interpreted as the difference between the matrices of complex curvature forms of P and $M(\lambda)$.

The problem solved now will be the computation of Chern forms $\tilde{\gamma}_a$ of $\tilde{\Xi}$ in terms of the Chern forms γ_a of Ξ. By definition (see for example [**GR3**])

(3.11) $$1+\tilde{\gamma}_1+\cdots+\tilde{\gamma}_q=\tilde{\gamma}=\det\Big(\delta_{ab}+\frac{\sqrt{-1}}{2\pi}\tilde{\Xi}_{ab}\Big),$$

and similarly for the γ_a's.

LEMMA 3.3. *We have*

(3.12) $$\tilde{\gamma}_c=\sum_{a=0}^{c}\binom{q-a+1}{c-a}\Big(-\frac{\lambda}{\pi}F\Big)^{c-a}\wedge\gamma_a\,,\quad \textit{or more concisely}$$

(3.13) $$\tilde{\gamma}=\sum_{a=0}^{q}\Big(1-\frac{\lambda}{\pi}F\Big)^{q-a+1}\wedge\gamma_a\,.$$

PROOF. From (3.11) follows

(3.14) $$\tilde{\gamma}=\det\Big(\Big(1-\frac{\lambda}{\pi}F\Big)\delta_{ab}+\frac{\sqrt{-1}}{2\pi}(\Xi_{ab}-\lambda\phi_a\wedge\bar{\phi}_b)\Big).$$

One expands the right hand side of (3.14) and gets

(3.15) $$\tilde{\gamma}=\sum_{c=0}^{q}\Big(1-\frac{\lambda}{\pi}F\Big)^{q-c}\Big(\frac{\sqrt{-1}}{2\pi}\Big)^{c}\tilde{\Phi}_c\quad\text{where}$$

(3.16) $$\tilde{\Phi}_c=\frac{1}{c!}\sum_{a_1\cdots a_c=1}^{q}\det\begin{pmatrix}\Xi_{a_1a_1}-\lambda\phi_{a_1}\wedge\bar{\phi}_{a_1}\cdots\Xi_{a_1a_c}-\lambda\phi_{a_1}\wedge\bar{\phi}_{a_c}\\ \vdots\\ \Xi_{a_ca_1}-\lambda\phi_{a_c}\wedge\bar{\phi}_{a_1}\cdots\Xi_{a_ca_c}-\lambda\phi_{a_c}\wedge\bar{\phi}_{a_c}\end{pmatrix}.$$

In turn one expands the right hand side of (3.16) in powers of λ and obtains

(3.17) $$\begin{aligned}\tilde{\Phi}_c=\frac{1}{c!}\sum_{a_1\cdots a_c=1}^{q}\sum_{\sigma\in\mathfrak{S}_c}\varepsilon_\sigma\Big\{&\Xi_{a_1\sigma(a_1)}\wedge\cdots\wedge\Xi_{a_c\sigma(a_c)}\\ &-\lambda\sum_{b=1}^{c}\Xi_{a_1\sigma(a_1)}\wedge\cdots\wedge\phi_{a_b}\wedge\bar{\phi}_{\sigma(a_b)}\wedge\cdots\wedge\Xi_{a_c\sigma(a_c)}\\ &+\cdots+(-\lambda)^c\phi_{a_1}\wedge\bar{\phi}_{\sigma(a_1)}\wedge\cdots\wedge\phi_{a_c}\wedge\bar{\phi}_{\sigma(a_c)}\Big\}.\end{aligned}$$

Using the first Bianchi identity (3.9) on (3.17) one finds

(3.18) $$\begin{aligned}\tilde{\Phi}_c=\frac{1}{c!}\sum_{a_1\cdots a_c=1}^{q}\Bigg\{&\det\begin{pmatrix}\Xi_{a_1a_1}\cdots\Xi_{a_1a_c}\\ \vdots\qquad\vdots\\ \Xi_{a_ca_1}\cdots\Xi_{a_ca_c}\end{pmatrix}\\ &-c\lambda\phi_{a_c}\wedge\bar{\phi}_{a_c}\det\begin{pmatrix}\Xi_{a_1a_1}\cdots\Xi_{a_1a_{c-1}}\\ \vdots\qquad\vdots\\ \Xi_{a_{c-1}a_1}\cdots\Xi_{a_{c-1}a_{c-1}}\end{pmatrix}\end{aligned}$$

$$+\cdots+(-\lambda)^c \det\begin{pmatrix}\phi_{a_1}\wedge\bar\phi_{a_1}\cdots\phi_{a_1}\wedge\bar\phi_{a_c}\\ \vdots \qquad \vdots\\ \phi_{a_c}\wedge\bar\phi_{a_1}\cdots\phi_{a_c}\wedge\bar\phi_{a_c}\end{pmatrix}\Bigg\}.$$

From (3.18) and Corollary 3.2 it follows that

$$\tilde\psi_c=\sum_{b=0}^{c}(2\lambda\sqrt{-1}\,F)^{c-b}\psi_b\,, \tag{3.19}$$

where ψ_b is defined by (3.16) with $\lambda=0$. From (3.15) and the definition of the Chern classes of Ξ one gets

$$\gamma_c=\left(\frac{\sqrt{-1}}{2\pi}\right)^c\psi_c\,. \tag{3.20}$$

Hence using (3.20) one can rewrite (3.19) as

$$\tilde\psi_c=(2\lambda\sqrt{-1}\,)^c\sum_{b=0}^{c}\left(-\frac{\pi}{\lambda}\right)^b F^{c-b}\wedge\gamma_b\,. \tag{3.21}$$

Substituting (3.21) into (3.15) and rearranging terms it follows that

$$\tilde\gamma=\sum_{b=0}^{q}\sum_{c=b}^{q}\left(1-\frac{\lambda}{\pi}F\right)^{q-c}\left(-\frac{\lambda}{\pi}F\right)^{c-b}\gamma_b\,. \tag{3.22}$$

Moreover

$$\begin{aligned}&\sum_{c=b}^{q}\left(1-\frac{\lambda}{\pi}F\right)^{q-c}\left(-\frac{\lambda}{\pi}F\right)^{c-b}\\ &\qquad=\frac{\left(1-\frac{\lambda}{\pi}F\right)^{q-b+1}-\left(-\frac{\lambda}{\pi}F\right)^{q-b+1}}{\left(1-\frac{\lambda}{\pi}F\right)-\left(-\frac{\lambda}{\pi}F\right)}\\ &\qquad=\left(1-\frac{\lambda}{\pi}F\right)^{q-b+1}-\left(-\frac{\lambda}{\pi}F\right)^{q-b+1}.\end{aligned} \tag{3.23}$$

Note that $F^{q-b+1}\wedge\gamma_b=0$ for all b because it is a $q+1$ form on a q-dimensional manifold. Therefore from (3.22) and (3.23) one obtains (3.13). This completes the proof.

PROOF OF THEOREM 1.3. Let $A_P^M(r)$ be the $(2n-1)$-dimensional volume of the boundary of the tube of radius r about P. Then

$$\frac{d}{dr}V_P^M(r)=A_P^M(r)\,. \tag{3.24}$$

(See [**GR4**, Lemma 7.2].) Furthermore

$$\begin{aligned}&A_P^{M(\lambda)}(r)\\ &\quad=\frac{2\pi^{n-q}}{\Gamma(n-q)}(\cos\sqrt{\lambda}\,r)^{2n}\sum_{c=0}^{q}\left(\frac{\tan\sqrt{\lambda}\,r}{\sqrt{\lambda}}\right)^{2(n-q+c)-1}\frac{k_{2c}(R^P-R^{M(\lambda)})}{2^c(n-q)\cdots(n-q+c-1)},\end{aligned} \tag{3.25}$$

where $R^{M(\lambda)}$ denotes the restriction of the curvature tensor of $M(\lambda)$ to P. Formula (3.25) is proved in [**GV**, Corollary 7.5]; alternatively it follows from slight simplifications of calculations in the next section. From Corollary 2.3 one has

$$k_{2c}(R^P-R^{M(\lambda)})=\frac{(2\pi)^c}{(q-c)!}(\gamma_c(R^P-R^{M(\lambda)})\wedge F^{q-c})[P]. \tag{3.26}$$

Also by Lemma 3.3

$$\gamma_c(R^P-R^{M(\lambda)})=\sum_{b=0}^{c}\binom{q-b+1}{c-b}\left(-\frac{\lambda}{\pi}F\right)^{c-b}\wedge\gamma_b(R^P). \tag{3.27}$$

From (3.24)-(3.27) one gets (1.5).

The proof of Corollary 1.4 is obvious. Also Corollary 1.5 follows from Corollary 1.4 and standard facts about the Chern classes of complete intersections. For example, see Schwarzenberger's appendix to Hirzebruch's book [**HI**, p. 159].

4. Comparison theorems for the volumes of tubes in Kähler manifolds of nonnegative or nonpositive curvature.

PROOF OF THEOREM 1.1. In [**GR4**] it is shown that if P is a $2q$-dimensional topological embedded submanifold with compact closure in a complete Riemannian manifold M of dimension $2n$, then $K^M\geqq 0$ implies

$$V_P^M(r)\leqq\frac{(\pi r^2)^{n-q}}{(n-q)!}\sum_{c=0}^{q}\frac{k_{2c}(R^P-R^M)r^{2c}}{2^c(n-q+1)\cdots(n-q+c)}. \tag{4.1}$$

Furthermore if P is a minimal variety of M then the right hand side of (4.1) is not greater than $\{(\pi r^2)^{n-q}/(n-q)!\}\operatorname{vol}(P)$. Now assume that P is a Kähler submanifold of M. In Lemma 2.3 take $R=R^P-R^M$ and substitute into (4.1). The result is (1.1), because P is a minimal variety of $\overline{M}$. This proves (i); (ii) is proved in a similar way.

To prove (iii) the arguments of [**GR4**] must be generalized. Let γ be a unit speed geodesic in M normal to P with $\gamma(0)=p\in P$. Choose a holomorphic basis $\{e_1e_1^*\cdots e_ne_n^*\}$ of the tangent space M_p so that $e_1e_1^*\cdots e_qe_q^*$ are tangent to P, and $e_{q+1}=\gamma'(0)$. Also it may be assumed that $e_1e_1^*\cdots e_qe_q^*$ diagonalize the symmetric bilinear form $(x, y)\rightarrow T_{xyu}$, where $u=\gamma'(0)$ and T is the second fundamental form of P. Let $\kappa_1(0)$, $\kappa_1^*(0)$, $\cdots$, $\kappa_q(0)$, $\kappa_q^*(0)$ be the corresponding eigenvalues. Now extend $e_1e_1^*\cdots e_ne_n^*$ to orthonormal vector fields $F_1F_1^*$ $F_nF_n^*$ along γ so that at each point $F_{q+1}(t)=\gamma'(t)$ and the other $F_\alpha(t)$ diagonalize the second fundamental form of the hypersurface P_t. Here P_t is the tubular hypersurface at a distance t from P. Let $\kappa_1(t)$, $\cdots$, $\kappa_q(t)$, $\kappa_{(q+1)^*}(t)$, $\cdots$, $\kappa_{n^*}(t)$ be the principal curvatures of P_t. In [**GR4**] it is shown that (except where the $\kappa_\alpha(t)$ are nondifferentiable) the following differential equations are satisfied:

(4.2) $$\kappa'_\alpha(t)=\kappa_\alpha(t)^2+R^M_{\gamma'(t)F_\alpha(t)\gamma'(t)F_\alpha(t)}$$

for $\alpha=1,\cdots,n^*$, $\alpha\neq q+1$. Since M is a Kähler manifold it follows from (4.2) that

(4.3) $$\begin{cases}\kappa'_\alpha(t)+\kappa'^*_\alpha(t)=\kappa_\alpha(t)^2+\kappa^*_\alpha(t)^2+R^M_{\gamma'(t)J\gamma'(t)F_\alpha(t)JF_\alpha(t)}, & \alpha\neq q+1,\\ \kappa'_{(q+1)^*}(t)=\kappa_{(q+1)^*}(t)^2+R^M_{\gamma'(t)J\gamma'(t)\gamma'(t)J\gamma'(t)}. \end{cases}$$

Then (4.3) and the assumption that M has nonnegative holomorphic bisectional curvature imply that

(4.4) $$\begin{cases}\kappa'_\alpha(t)+\kappa'^*_\alpha(t)\geqq\kappa_\alpha(t)^2+\kappa^*_\alpha(t)^2\geqq\frac{1}{2}(\kappa_\alpha(t)+\kappa^*_\alpha(t))^2\geqq 0,\\ \kappa'_{(q+1)^*}(t)\geqq\kappa_{(q+1)^*}(t)^2. \end{cases}$$

Now because P is a complex submanifold of M it follows that

(4.5) $$\kappa_a(0)+\kappa_{a^*}(0)=0 \qquad \text{for} \quad a=1,\cdots,q.$$

Furthermore

(4.6) $$\kappa_i(0)=-\infty \qquad \text{for} \quad i=(q+1)^*,\cdots,n^*.$$

Then (4.4), (4.5) and (4.6) imply

(4.7) $$\begin{cases}\kappa_a(t)+\kappa_{a^*}(t)\geqq 0, & a=1,\cdots,q,\\ \kappa_i(t)+\kappa_{i^*}(t)\geqq -\frac{2}{t}, & i=q+2,\cdots,n,\\ \kappa_{(q+1)^*}(t)\geqq-\frac{1}{t}. \end{cases}$$

The second fundamental form of the real hypersurface P_t at the point $\gamma(t)$ will be denoted by $S(t)$. Then $\operatorname{tr}S(t)=\kappa_1(t)+\cdots+\kappa_{n^*}(t)$ and so from (4.7) it follows that

(4.8) $$\operatorname{tr}S(t)\geqq-(2n-2q-1)/t.$$

By a continuity argument given in [**GR4**] this inequality holds even at points where a $\kappa_\alpha(t)$ is not differentiable. Also one has [**GR4**, Lemma 6.1]

(4.9) $$\frac{\theta'_u(t)}{\theta_u(t)}=-\Big(\frac{2n-2q-1}{t}+\operatorname{tr}S(t)\Big), \qquad \theta_u(0)=1,$$

where θ_u is the infinitesimal change of volume in the normal direction u. From (4.8) and (4.9) follows

(4.10) $$\theta_u(t)\leqq 1.$$

Now there is the general formula (see [**GR4**, Lemma 7.1])

$$(4.11)\qquad A_P^M(r)=r^{2n-2q-1}\int_P\int_{S^{2n-2q-1}(1)}\theta_u(r)\,du\,dP.$$

From (4.10) and (4.11) one obtains

$$(4.12)\qquad A_P^M(r)\leqq\frac{2\pi^{n-q}r^{2n-2q-1}}{\Gamma(n-q)}\mathrm{vol}(P).$$

Then (1.3) is obtained by integrating (4.12) from 0 to r.

In the rest of the section a comparison theorem that combines Theorem 1.3 with Theorem 1.1 (i) will be proved. For this it will be necessary to generalize some of the analytical results of [**GR4**]. The following notion will be needed.

DEFINITION. Let M be a Kähler manifold with sectional curvature K^M, and let $0\leqq\theta<2\pi$. The *θ-holomorphic sectional curvature* $K^M(\theta)$ is the restriction of the sectional curvature K^M to those 2-dimensional subspaces of tangent spaces that make an angle θ with the holomorphic 2-dimensional spaces.

THEOREM 4.1. *Let P be a $2q$-dimensional topologically embedded Kähler submanifold with compact closure in a complete Kähler manifold M. Assume that the holomorphic sectional curvature $K^M(0)$ satisfies $K^M(0)\geqq 4\lambda$ and that the antiholomorphic sectional curvature $K^M\left(\frac{\pi}{2}\right)$ satisfies $K^M\left(\frac{\pi}{2}\right)\geqq\lambda$. Then*

$$(4.13)\qquad V_P^M(r)\leqq\begin{cases}\sum_{d=0}^{q}C_d(\lambda,F,\gamma_1,\cdots,\gamma_d)(\sin\sqrt{\lambda}\,r)^{2(n-d)} & \text{for }\lambda>0,\\ \sum_{d=0}^{q}\frac{(\gamma_d\wedge F^{q-d})[P]}{(q-d)!\,(n-q+d)!}(\pi r^2)^{n-q+d} & \text{for }\lambda=0,\\ \sum_{d=0}^{q}C_d(\lambda,F,\gamma_1,\cdots,\gamma_d)(\sinh\sqrt{|\lambda|}\,r)^{2(n-d)} & \text{for }\lambda<0.\end{cases}$$

PROOF. Assume $\lambda>0$. Let $p\in P$ and let $u\in P_p^{\perp}$ be a unit vector. Denote by γ the unit speed geodesic in M with $\gamma(0)=p$ and $\gamma'(0)=u$. The second fundamental form of the hypersurface P_t at the point $\gamma(t)$ will be denoted by $S(t)$. Also let $R(t)x=R^M_{\gamma'(t)x}\gamma'(t)$. In [**GR4**, Corollary 4.2] it is shown that S satisfies the Riccati differential equation

$$(4.14)\qquad S'(t)=S(t)^2+R(t).$$

Suppose E is a unit parallel vector field along γ with $E(0)\in P_p^{\perp}$ and $\langle E(0),u\rangle=0$. Put $\cos\theta=\langle E,J\gamma'(t)\rangle$ and let $f(t)=\langle SE,E\rangle(t)$. Here θ is a constant and $f(0)=-\infty$. Using (4.14) and [**GR4**, Lemma 5.2] it follows that for $\theta=0,\frac{\pi}{2}$:

$$(4.15)\qquad \langle SE,E\rangle(t)\geqq\frac{\sqrt{\lambda(1+\cos^2\theta)}}{\tan(t\sqrt{\lambda(1+3\cos^2\theta)})}.$$

Next take a parallel unit vector field H along γ with $H(0)\in P_p$. Then $\langle H, \gamma'\rangle=\langle H, J\gamma'\rangle=0$. Put $g(t)=\langle SH, H\rangle(t)$. Then

$$g(0)=T_{hhu}, \tag{4.16}$$

where T is the second fundamental form of P in M and $H(0)=h$. Now (4.14) and (4.16) together with [**GR4**, Lemma 5.1] imply

$$\langle SH, H\rangle(t)\geqq\frac{T_{hhu}+\sqrt{\lambda}\tan\sqrt{\lambda}\,t}{1-\dfrac{\tan\sqrt{\lambda}\,t}{\sqrt{\lambda}}T_{hhu}}. \tag{4.17}$$

Let $\langle T_u x, y\rangle=T_{xyu}$. Then (4.15) and (4.17) imply

$$\operatorname{tr}S(t)\geqq\operatorname{tr}\left(\frac{(\sqrt{\lambda}\tan\sqrt{\lambda}\,t)I+T_u}{I-\dfrac{1}{\sqrt{\lambda}}(\tan\sqrt{\lambda}\,t)T_u}\right)-\frac{2(n-q-1)\sqrt{\lambda}}{\tan\sqrt{\lambda}\,t}-\frac{2\sqrt{\lambda}}{\tan 2\sqrt{\lambda}\,t}. \tag{4.18}$$

From (4.18) and (4.9) follows

$$\theta_u(t)\leqq(\cos\sqrt{\lambda}\,t)^{2q+1}\left(\frac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}\,t}\right)^{2n-2q-1}\det\left(I-\frac{\tan\sqrt{\lambda}\,t}{\sqrt{\lambda}}T_u\right). \tag{4.19}$$

From (4.19) one obtains

$$A_P^M(r)\leqq(\cos\sqrt{\lambda}\,r)^{2q+1}\left(\frac{\sin\sqrt{\lambda}\,r}{\sqrt{\lambda}}\right)^{2n-2q+1} \tag{4.20}$$

$$\int_P\int_{S^{2n-2q-1}(1)}\det\left(I-\frac{\tan\sqrt{\lambda}\,r}{\sqrt{\lambda}}T_u\right)du\,dP.$$

The right hand side of (4.20) can be integrated just as in the case of the Weyl tube formula. See [**GR4**, Theorem 1.3] for details. The result is the first part of (4.13). Similar proofs for the cases $\lambda\leqq 0$ yields the rest of (4.13).

Denote by $V_m^M(r)$ the volume of a geodesic ball of radius r in M. By taking P to be a point in a Kähler manifold M and integrating one gets

COROLLARY 4.2. *Let M be a complete Kähler manifold for which $K^M(0)\geqq 4\lambda$ and $K^M\left(\frac{\pi}{2}\right)\geqq\lambda$. Then if r is less than or equal to the distance from m to its nearest conjugate point,*

$$V_m^M(r)\leqq\begin{cases}\dfrac{\pi^n}{n!}\left(\dfrac{\sin\sqrt{\lambda}\,r}{\sqrt{\lambda}}\right)^{2n} & \text{for } \lambda>0,\\[2ex] \dfrac{(\pi r^2)^n}{n!} & \text{for } \lambda=0,\\[2ex] \dfrac{\pi^n}{n!}\left(\dfrac{\sinh\sqrt{|\lambda|}\,r}{\sqrt{|\lambda|}}\right)^{2n} & \text{for } \lambda<0.\end{cases} \tag{4.21}$$

Equation (4.21) *is analogous to an estimate of Bishop* [**BC**, p. 256] *for real manifolds, but simpler.*

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