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# High energy resolvent estimates, II, higher order elliptic operators

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## § 1. Introduction.

This is a continuation of [1] and is concerned with elliptic partial differential operators on $\boldsymbol{R}^n$ whose coefficients are nearly constants at infinity. For such an operator $A(X, D_x)$ with real principal symbol $a(x, \xi)$ we shall show in this paper that if every classical orbit under the Hamiltonian $a(x, \xi)$ is not trapped, then the resolvent of $A(X, D_x)$ admits, as operators from a weighted $L_2$-space to its dual space, boundary values on the upper or lower bank of the reals which are bounded and uniformly Hölder continuous at infinity; and that the nontrapping condition for orbits is necessary to the uniform estimate for the resolvent.

In connection with time-decay for solutions of Schrödinger-type equations uniform estimates at high energy for resolvents of elliptic operators have been investigated (see [2], [4], [5], and references there). But such estimates were given only for operators whose leading and the next coefficients are constant. The purpose of this paper is to give high energy resolvent estimates for elliptic differential operators with variable leading coefficients. The results here will be used in [3] to study the asymptotic behavior as $t \rightarrow \infty$ of solutions for Schrödinger-type equations.

Now we prepare some notations in order to state our main results. We write $D_{x_j} = -i\partial/\partial x_j$, $D_x = (D_{x_1}, \cdots, D_{x_n})$, and $\langle x \rangle = (1+|x|^2)^{1/2}$. For a real number $\sigma$ and $s$, $H^{\sigma,s}$ denotes the weighted Sobolev space with the norm

$$\|f\|_{\sigma,s} \equiv \|f\|_{H^{\sigma,s}} = \|\langle x \rangle^s \langle D_x \rangle^\sigma f\|_{L_2(R^n)}\,. \tag{1.1}$$

We write $H^\sigma = H^{\sigma,0}$ and $L_2^s = H^{0,s}$. $B(\sigma, s\,;\, \tau, t)$ stands for the Banach space of all bounded linear operators from $H^{\sigma,s}$ to $H^{\tau,t}$. We write $B(s, t) = B(0, s\,;\, 0, t)$. For a positive number $h$ and a function $f$ on $\boldsymbol{C}$ to a Banach space, we put

$$\Delta_h^1 f(x) = f(x+h) - f(x)\,, \qquad \Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)\,.$$

For a real number $r$, $[r]$ denotes the largest number which is not larger than $r$.

We consider the elliptic partial differential operator

(1.2) $$A(X, D_x)=\sum_{|\alpha|\leqq m} a_\alpha(X)D_x^\alpha=p(D_x)+\sum_{|\alpha|\leqq m} q_\alpha(X)D_x^\alpha$$

of order $m$ satisfying the following conditions (A. I) and (A. II).

(A. I) (i) There exists a positive constant $c_0$ such that the principal symbol $a(x, \xi)$ of $A(X, D_x)$ satisfies $a(x, \xi)\geqq c_0|\xi|^m$ for all $(x, \xi)\in \boldsymbol{R}^n$. (ii) $p(\xi)$ is an elliptic polynomial of degree $m$ with real coefficients.

(A. II) There exists $\rho>1$ such that

(1.3) $$|\partial_x^\beta q_\alpha(x)|\leqq\begin{cases} C_\beta\langle x\rangle^{-\rho-1-|\beta|} & \text{for } |\alpha|=m \\ C_\beta\langle x\rangle^{-\rho-|\beta|} & \text{for } |\alpha|\leqq m-1\end{cases}$$

for any multi-index $\beta$, where the $C_\beta$ are constants independent of $x$.

It follows from (A. I) and (A. II) that the operator $A$ on $L_2(\boldsymbol{R}^n)$ defined by

(1.4) $$Au=A(X, D_x)u\,,\qquad u\in D(A)=H^m$$

is a closed operator whose resolvent $R(z)=(z-A)^{-1}$ exists for $z$ with $|\mathrm{Im}\,z|\gg 1$ or $-\mathrm{Re}\,z\gg 1$. Our problem is whether the resolvent $R(z)$ on $\{z;\ \pm\mathrm{Im}\,z\gg 1$ and $\mathrm{Re}\,z\gg 1\}$ can be extended to the set $\boldsymbol{N}_\pm=\{z;\ \pm\mathrm{Im}\,z\geqq 0$ and $\mathrm{Re}\,z>N\}$ for some $N>0$. In solving the problem the following condition (A. III) concerning the solution $\{x(t, y, \xi), \eta(t, y, \xi)\}$ of the Hamilton equation

(1.5) $$\frac{dx}{dt}=-\nabla_\eta a(x, \eta),\quad \frac{d\eta}{dt}=\nabla_x a(x, \eta),\quad \{x(0), \eta(0)\}=\{y, \xi\}$$

plays a crucial role.

(A. III) For any $R>0$ there exists $T>0$ such that $|x(t, y, \xi)|>R$ for all $t>T$ if $|y|\leqq R$ and $|\xi|=1$.

REMARK. When $A(x, \xi)$ is real-valued, (A. III) is equivalent to the condition that every classical orbit with $|\xi|\gg 1$ under the Hamiltonian $A(x, \xi)$ is not trapped (cf. Proposition 2.5 below).

Our main results are as follows.

THEOREM 1. *Assume* (A. I)$\sim$(A. III). *Let* $0\leqq\theta<1$ *and* $k$ *be a nonnegative integer such that* $0<k+\theta<\rho-1$. *Then there exists a positive constant* $N$ *such that the resolvent* $R(z)$ *on* $\{z;\ \pm\mathrm{Im}\,z\gg 1$ *and* $\mathrm{Re}\,z\gg 1\}$ *can be extended to a* $k$*-times continuously differentiable function on the set* $\boldsymbol{N}_\pm=\{z;\ \pm\mathrm{Im}\,z\geqq 0$ *and* $\mathrm{Re}\,z>N\}$ *to* $B(s, -s)$, *where* $s>k+\theta+3/2$, *which is holomorphic in the interior of* $\boldsymbol{N}_\pm$ *and satisfies*

(1.6) $$\|R^{(j)}(z)\|_{B(0,s;\,\sigma,-s)}\leqq C|z|^{-\{(m-1)(j+1)-\sigma\}/m},$$
$$z\in\boldsymbol{N}_\pm,\quad j=0, \cdots, k,\quad 0\leqq\sigma\leqq m(j+1)\,,$$

(1.7) $$h^{-\theta}\|\Delta_h^1 R^{(k)}(z)\|_{B(0,s;\,\sigma,-s)}\leqq C|z|^{-\{(m-1)(k+\theta+1)-\sigma\}/m},$$
$$0<h<1,\quad z \text{ and } z+h\in\boldsymbol{N}_\pm,\quad 0\leqq\sigma\leqq m(k+1)\,,$$

*where C is a constant depending only on s. Furthermore, for any* $r>k+\theta+1$ *there exists a constant* $C_r$ *such that for all* $f\in L_2^r$

$$\int\Big[\langle t\rangle^{k+\theta-l}|t|^{l}\Big\|\Big(\frac{d}{dt}\Big)^{j}\int e^{it\lambda}\chi(\lambda)\lambda^{\delta/m}R(\lambda\pm i0)f\,d\lambda\Big\|_{\sigma,-r}\Big]^2 dt\leqq C_r\|f\|_r^2, \tag{1.8}$$

$$0\leqq\sigma+mj\leqq m(k+1),\qquad \delta\leqq(m-1)(k+\theta+1/2)-(\sigma+mj),$$

$$l=\max(0,\ (\sigma+mj+\delta)/(m-1)-1/2),$$

*where* $\chi(\lambda)$ *is a* $C^\infty$*-function such that* $\chi(\lambda)=1$ *for* $\lambda>N+1$ *and* $\operatorname{Supp}\chi\subset(N,\infty)$.

*Assume further that* $\rho>1+1/2(m-1)$. *Let* $1/2(m-1)<\gamma<\rho-1$, $s>\gamma+1$, $0\leqq\sigma+mj\leqq(m-1)\gamma-1/2$, *and* $l>(\sigma+mj+1/2)/(m-1)$. *Then*

$$\Big\|\Big(\frac{d}{dt}\Big)^{j}\int e^{it\lambda}\chi(\lambda)R(\lambda\pm i0)d\lambda\Big\|_{B(0,s;\ \sigma,-s)}\leqq C\langle t\rangle^{-\gamma+l}|t|^{-l},\quad t\neq0. \tag{1.9}$$

THEOREM 2. *Let* $A(X, D_x)$ *satisfy* (A. I) *and* (A. II). *Then the estimate* (1.8) *implies* (A. III).

A consequence of Theorem 1 is worth mentioning.

THEOREM 3. *Assume* (A. I)$\sim$(A. III). *Then the spectrum of A is contained in* $\{z\in\boldsymbol{C};\ |z|\leqq N\}\cup(N,\infty)$ for some $N>0$.

PROOF. Choose a positive constant $N$ such that Theorem 1 holds for $N$ and the spectrum of $A$ is contained in $\{z;\ |z|\leqq N\}\cup\{z;\ |\operatorname{Im}z|\leqq N$ and $\operatorname{Re}z>N\}$. Let $\zeta$ be $\operatorname{Im}\zeta\neq0$ and $\operatorname{Re}\zeta>N$. Suppose that $\zeta$ belongs to the spectrum of $A$. Then it is easily seen that $\zeta$ is a point spectrum and an eigenfunction $u\not\equiv0$ associated with $\zeta$ belongs to $H^{m,s}$ for any $s>0$. Thus Theorem 1 shows that $u=(\zeta-A)^{-1}(\zeta-A)u=0$. This is a contradiction. Q. E. D.

Theorem 1 is useful also in studying asymptotic behaviors of $e^{-itA}$ as $t\to\pm\infty$.

THEOREM 4. *Let A be a second order self-adjoint elliptic differential operator on* $\boldsymbol{R}^3$ *satisfying* (A. I), (A. II) *for some* $\rho>5$, *and* (A. III). *Let* $3/2<\gamma<\rho/2-1$ *and* $s>2\gamma-1/2$. *Then* $e^{-itA}$ *has the following expansion in* $B(s,-s)$ *as* $t\to\pm\infty$, *which can be differentiated* $[\gamma/2-1/4]$*-times in* $t$:

$$e^{-itA}=\sum_j e^{-it\lambda_j}P_j+\sum_{k=0}^{[\gamma-1/2]}e^{-it\mu}t^{-(k+1/2)}C_k+o(t^{-\gamma}), \tag{1.10}$$

*where the* $\lambda_j$ *are eigenvalues of A with the associated eigenprojection* $P_j$, $\mu=\min p(\xi)$, *and the* $C_k$ *are operators of finite rank.*

For the proof and more information, see [3]. The differentiability property in $t$ is easily seen from the proof of [Theorems 1.2 and 8.11, 3], although it is not discussed there.

The remainder of this paper is organized as follows. Theorems 1 and 2 are proved in Section 2 by applying the results in [1] for a first order pseudo-differential operator like $A^{1/m}$. In Section 3 an improvement of the estimates (1.6)

~(1.9) will be given for operators with constant leading coefficients. In Section 4 and the latter half of Section 3 Theorem 1 will be extended to differential operators with singular lower order coefficients.

## § 2. Proof of Theorems 1 and 2.

In this section Theorems 1 and 2 are proved by making use of corresponding theorems for a first order pseudo-differential operator like $A^{1/m}$.

Choose a positive number $K$ such that $\mathrm{Re}\,A(x, \xi)+K\geqq 1$, set

$$b(x, \xi)=(A(x, \xi)+K)^{1/m} + \frac{1}{2m}\left(\frac{1}{m}-1\right)(A(x, \xi)+K)^{1/m-2}\sum_{j=1}^{n}\partial_{\xi_j}A(x, \xi)D_{x_j}A(x, \xi), \tag{2.1}$$

and put $B=b(X, D_x)$. Then we have

PROPOSITION 2.1. *Assume* (A. I) *and* (A. II). *Then the following statements* (i) *and* (ii) *hold.*

(i) *The estimates* (1.6) *and* (1.7) *hold if and only if those estimates hold with* $R(z)=(z-B)^{-1}$, $m=1$, $\sigma=0$.

(ii) *The estimate* (1.8) *holds if and only if that estimate holds with* $R(\lambda\pm i0)=(\lambda\pm i0-B)^{-1}$, $m=1$, $\sigma=0$, $l=0$.

For the proof we prepare lemmas. Since they can be shown easily by usual calculations for pseudo-differential operators, we omit the proof of the lemmas. (For pseudo-differential operators, see [6] for example.)

LEMMA 2.2. *Let* $c(x, \xi)=(p(\xi)+\sum_{|\alpha|=m}q_\alpha(x)\xi^\alpha+K)^{1/m}-(p(\xi)+K)^{1/m}$ *and* $d(x, \xi)=b(x, \xi)-(p(\xi)+\sum_{|\alpha|=m}q_\alpha(x)\xi^\alpha+K)^{1/m}$. *Then*

$$b(x, \xi)=(p(\xi)+K)^{1/m}+c(x, \xi)+d(x, \xi), \tag{2.2}$$

$$|D_x^\alpha\partial_\xi^\beta c(x, \xi)|\langle x\rangle\langle\xi\rangle^{-1}+|D_x^\alpha\partial_\xi^\beta d(x, \xi)|\leqq C_{\alpha\beta}\langle x\rangle^{-\rho-|\alpha|}\langle\xi\rangle^{-|\beta|} \tag{2.3}$$

*for any multi-indices* $\alpha$ *and* $\beta$.

LEMMA 2.3. (i) *The symbol* $v(x, \xi)$ *of a pseudo-differential operator* $V=A+K-B^m$ *satisfies, for any multi-indices* $\alpha$ *and* $\beta$,

$$|D_x^\alpha\partial_\xi^\beta v(x, \xi)|\leqq C_{\alpha\beta}\langle x\rangle^{-\rho-2-|\alpha|}\langle\xi\rangle^{m-2-|\beta|}. \tag{2.4}$$

(ii) $V\in B(m-2+\sigma, s; \sigma, s+\rho+2)$ *for all* $s$, $\sigma\in\boldsymbol{R}^1$.

LEMMA 2.4. *There exists a positive constant* $N$ *such that for all* $z$ *with* $\mathrm{Re}\,z>N$ *and* $0\leqq\sigma\leqq m-1$

$$\left\|\left(\sum_{l=0}^{m-1}z^{l/m}B^{m-1-l}\right)^{-1}\right\|_{B(r, s; r+\sigma, s)}\leqq C_{rs}|z|^{-(m-1-\sigma)/m}, \qquad r, s\in\boldsymbol{R}^1. \tag{2.5}$$

PROOF OF PROPOSITION 2.1. Assume that the estimates (1.6) and (1.7) hold with $R(z)$ and $m$ replaced by $(z-B)^{-1}$ and 1, respectively. We have that for all $z\in N_{\pm}$ and $j=0, 1, \cdots, k$,

$$(2.6)\qquad \left(\frac{d}{dz}\right)^{j}(z-B^{m})^{-1}=(\sum_{l=0}^{m-1}z^{l/m}B^{m-1-l})^{-j-1}\left(\frac{d}{d\zeta}\right)^{j}(\zeta-B)^{-1}\Big|_{\zeta=z^{1/m}}.$$

This together with Lemma 2.4 shows that (1.6) for $\sigma\leqq(m-1)(j+1)$ holds with $R(z)$ replaced by $(z-B^{m})^{-1}$. In order to prove (1.6) for $(m-1)(j+1)<\sigma\leqq m(j+1)$, we have only to use the equality

$$(2.7)\quad (z-B^{m})^{-j-1}=(iM-B^{m})^{-j-1}[1+(iM-z)\sum_{l=0}^{j}(iM-B^{m})^{j-l}(z-B^{m})^{l}](z-B^{m})^{-j-1}$$

for $z\in N_{\pm}$, where $M$ is a sufficiently large constant. Since for some positive constants $C_1$ and $C_2$

$$C_1h\lambda^{1/m-1}\leqq|(\lambda+h)^{1/m}-\lambda^{1/m}|\leqq C_2h\lambda^{1/m-1},\qquad \lambda>2,\qquad 0<h<1,$$

(1.7) also holds with $R(z)$ replaced by $(z-B^{m})^{-1}$. By Lemma 2.3,

$$(2.8)\qquad \|V(z+K-B^{m})^{-1}\|_{B(s,\rho+2-s)}\leqq C_s|z|^{-1/m},\qquad 3/2<s<\rho+1/2.$$

Thus,

$$(2.9)\qquad (z-A)^{-1}=\sum_{j=0}^{\infty}(z+K-B^{m})^{-1}[V(z+K-B^{m})^{-1}]^{j}$$

in $B(r, -r)$ for $3/2<r<(\rho+2)/2$. Combining (2.8) and (2.9) we get (1.6) and (1.7) from the estimates for $(z-B^{m})^{-1}$. The converse can be shown similarly. This completes the proof of (i).

In the same way we can show (ii) by using Parseval's equality. Q. E. D.

Denote by $\{q(t, y, \xi), p(t, y, \xi)\}$ the solution of the Hamilton equation

$$(2.10)\quad \frac{dq}{dt}=-\nabla_{\xi}F(q, p),\quad \frac{dp}{dt}=\nabla_{x}F(q, p),\quad \{q(0, y, \xi), p(0, y, \xi)\}=\{y, \xi\},$$

where $F(x, \xi)=(p(\xi)+\sum_{|\alpha|=m}q_{\alpha}(x)\xi^{\alpha}+K)^{1/m}$. Then we have

PROPOSITION 2.5. *Assume* (A. I) *and* (A. II). *Then the condition* (A. III) *holds if and only if there exists a positive constant* $R_0$ *such that for any* $R>0$ *one can choose* $T>0$ *with*

$$(2.11)\qquad |q(t, y, \xi)|>R,\quad t>T,\quad |y|\leqq R,\quad |\xi|\geqq R_0.$$

PROOF. Denote by $\{x, \eta\}(t, y, \xi)$ and $\{Q, P\}(t, y, \xi)$ the solution of (2.10) with $F(x, \xi)=a(x, \xi)$ and $F(x, \xi)=a(x, \xi)^{1/m}$, respectively. Then simple calculations show that

$$(2.12)\qquad \{x, \eta\}(t, y, \xi)=\{Q, P\}(ma(y, \xi)^{(m-1)/m}t, y, \xi).$$

By the homogeneity of $a(x, \xi)$,

$$\{x, \eta\}(t, y, \lambda\xi)=\{x, \lambda\eta\}(\lambda^{m-1}t, y, \xi), \qquad \lambda\neq 0. \tag{2.13}$$

Combining (2.12) and (2.13) we get

$$\begin{cases} Q(t, y, \xi)=x(t, y, m^{-1/(m-1)}a(y, \xi)^{-1/m}\xi) \\ P(t, y, \xi)=m^{1/(m-1)}a(y, \xi)^{1/m}\eta(t, y, m^{-1/(m-1)}a(y, \xi)^{-1/m}\xi). \end{cases} \tag{2.14}$$

Using the ellipticity of $A(x, \xi)$ we obtain that there exists a positive constant $c$ such that $|p(t, y, \lambda\xi)|\geqq c\lambda$ for all $t>0$, $y\in \boldsymbol{R}^n$, $|\xi|=1$, and $\lambda\gg 1$. Thus there exists a positive constant $C$ such that for all $t>0$, $y\in \boldsymbol{R}^n$, $|\xi|=1$, and $\lambda\gg 1$

$$|q(t, y, \lambda\xi)-Q(t, y, \xi)|+|p(t, y, \lambda\xi)/\lambda-P(t, y, \xi)|\leqq e^{C(t+1)}/\lambda. \tag{2.15}$$

Since $q(t, y, \xi)$ and $Q(t, y, \xi)$ are nearly straight lines outside a sufficiently large ball (cf. Lemmas 2.1 and 2.2 in [**1**]), (2.14) and (2.15) show the proposition.

Q. E. D.

Now we can complete the proof of Theorems 1 and 2.

PROOF OF THEOREMS 1 AND 2. The estimates (1.6)~(1.8) and Theorem 2 follow from Propositions 2.1 and 2.5, Lemma 2.2, and [Theorems 1 and 2, **1**]. The estimate (1.9) can be shown by using (1.8) for $\delta/m>1/2$. Q. E. D.

## § 3. The constant leading coefficient case.

In this section an improvement of the estimates (1.6)~(1.9) is given for operators with constant leading coefficients.

Let $A$ be the closed operator in $L_2(\boldsymbol{R}^n)$ defined by an elliptic differential operator

$$A(X, D_x)=p(D_x)+\sum_{|\alpha|\leqq m-1} q_\alpha(X)D_x^\alpha \tag{3.1}$$

of order $m$ satisfying (A. I) and (A. II). Then we have

THEOREM 3.1. *Let* $0<s-1/2=k+\theta<\rho-1$ ($k$ : *a nonnegative integer*, $0<\theta\leqq 1$). *Then there exists a positive constant* $N$ *such that the resolvent* $R(z)=(z-A)^{-1}$ *on* $\{z;\ \pm\mathrm{Im}\,z\gg 1,\ \mathrm{Re}\,z\gg 1\}$ *can be extended to a k-times continuously differentiable function on the set* $\boldsymbol{N}_\pm=\{z;\ \pm\mathrm{Im}\,z\geqq 0$ *and* $\mathrm{Re}\,z>N\}$ *to* $B(s, -s)$, *which is holomorphic in the interior of* $\boldsymbol{N}_\pm$ *and satisfies the following estimates:*

(i) *For all* $z\in \boldsymbol{N}_\pm$, $j=0, \cdots, k$, $0\leqq\sigma\leqq m(j+1)$,

$$\|R^{(j)}(z)\|_{B(0,s;\sigma,-s)}\leqq C_s|z|^{-\{(m-1)(j+1)-\sigma\}/m}. \tag{3.2}$$

(ii) *For all* $z\in \boldsymbol{N}_\pm$, $0<h<1$, $0\leqq\sigma\leqq m(k+1)$,

$$h^{-\theta}\|\Delta_h^1 R^{(k)}(z)\|_{B(0,s;\sigma,-s)}\leqq C_s|z|^{-\{(m-1)(k+\theta+1)-\sigma\}/m} \quad \text{when} \quad \theta<1, \tag{3.3}$$

(3.3′) $\quad h^{-1}\|\Delta_h^2 R^{(k)}(z)\|_{B(0,s;\sigma,-s)}\leqq C_s|z|^{-\{(m-1)(k+2)-\sigma\}/m}$ *when* $\theta=1$.

(iii) *For all* $T>0$, $0\leqq\sigma+mj\leqq m(k+1)$, $\delta\leqq(m-1)s-(\sigma+mj)$, $l=\max(0, (\sigma+mj+\delta)/(m-1)-1/2)$,

$$\int_{\Omega_T}\left[\langle t\rangle^{s-1/2-l}|t|^l\left\|\left(\frac{d}{dt}\right)^j\int e^{it\lambda}\chi(\lambda)\lambda^{\delta/m}R(\lambda\pm i0)f\,d\lambda\right\|_{\sigma,-s}\right]^2 dt \leqq C_s\|f\|_s^2, \tag{3.4}$$

*where* $\Omega_T=\{t\in\boldsymbol{R}^1;\ T\leqq|t|\leqq 2T\}$ *and* $\chi(\lambda)$ *is a* $C^\infty$*-function on* $\boldsymbol{R}^1$ *such that* $\chi(\lambda)=1$ *for* $\lambda\geqq N+1$ *and* $\operatorname{Supp}\chi\subset(N,\infty)$.

*Assume further that* $\rho>1+1/2(m-1)$. *Let* $1/2(m-1)<\gamma<\rho-1$, $s=\gamma+1/2$, $0\leqq\sigma+mj<(m-1)\gamma-1/2$, $l>(\sigma+mj+1/2)/(m-1)$. *Then the estimate* (1.9) *holds.*

PROOF. Using [Theorem 6.1, **1**] instead of [Theorem 1, **1**], the first half of Theorem 3.1 is shown in the same way as Theorem 1. The second half is deduced from (3.4). Q. E. D.

Now, let us extend Theorem 3.1 to operators whose lower order coefficients may be singular. Let $A$ be the operator

$$A=p(D_x)+\sum_{|\alpha|=m-1}q_\alpha(X)D_x^\alpha+V \tag{3.5}$$

in $L_2(\boldsymbol{R}^n)$ with domain $H^m$ satisfying the following conditions:

(B. I) $p(\xi)$ is a real-valued elliptic polynomial of degree $m$.

(B. II) There exist $\rho>1$ and $C>0$ such that for $|\beta|\leqq 3n+11$

$$|\partial_x^\beta q_\alpha(x)|\leqq C\langle x\rangle^{-\rho-|\beta|},\qquad x\in\boldsymbol{R}^n. \tag{3.6}$$

(B. III) $V$ is a closed operator in $L_2(\boldsymbol{R}^n)$ such that for some $m'<m-1$

$$V\in B(m', s;\, 0, s+\rho),\qquad s\in\boldsymbol{R}^1. \tag{3.7}$$

THEOREM 3.2. *Assume* (B. I)~(B. III). *Then the conclusion of Theorem* 3.1 *holds also for the operator A defined by* (3.5) *under the restriction that* $\sigma\leqq m$.

PROOF. By [Remark 4.3, **1**], the theorem holds with $R(z)$ replaced by $(z-A_1)^{-1}$, where $A_1=p(D_x)+\sum q_\alpha(X)D_x^\alpha$. Thus we get the theorem by using the Neuman series

$$R(z)=\sum_{j=0}^{\infty}[(z-A_1)^{-1}V]^j(z-A_1)^{-1}. \tag{3.8}$$

Q. E. D.

THEOREM 3.3. *Let* $A=p(D_x)+V$, *where* $p(\xi)$ *and* $V$ *satisfy* (B. I) *and* (B. III) *with* $\rho>2$, *respectively. Let* $1<s<\rho-1$, $0\leqq\sigma<m-1$, $0\leqq\sigma+mj\leqq(m-1)s$, *and* $l=(\sigma+mj)/(m-1)$. *Then*

$$\left\|\left(\frac{d}{dt}\right)^j\int e^{it\lambda}\chi(\lambda)R(\lambda\pm i0)d\lambda\right\|_{B(0,s;\sigma,-s)}\leqq C\langle t\rangle^{-s+l}|t|^{-l},\qquad t\neq 0, \tag{3.9}$$

*where* $\chi(\lambda)$ *is a* $C^\infty$*-function on* $\boldsymbol{R}^1$ *such that* $\chi(\lambda)=1$ *for* $|\lambda|\geqq N+1$ *and* $\mathrm{Supp}\chi\subset\{\lambda;\ |\lambda|>N\}$.

For the proof we need a lemma.

LEMMA 3.4. *Let* $R_0(z)=(z-p(D_x))^{-1}$, $s\geqq 0$, $\sigma\geqq 0$, $0\leqq\sigma+mj\leqq(m-1)s$, *and* $l=(\sigma+mj)/(m-1)$. *Then* (3.9) *holds with* $R(\lambda\pm i0)$ *replaced by* $R_0(\lambda\pm i0)$, *where the constant* $C$ *can be chosen independent of* $s$, $\sigma$, $j$ *when* $s$ *runs over a compact set.*

PROOF. Choosing another $N$ if necessary, we may assume that $|\nabla_\xi p(\xi)|\geqq c|\xi|^{m-1}$ on $\mathrm{Supp}\chi(p(\xi))$, where $c$ is some positive constant. Making use of the identity $(it)^{-1}(\nabla_\xi p(\xi)/|\nabla_\xi p(\xi)|^2)\cdot\nabla_\xi e^{itp(\xi)}=e^{itp(\xi)}$ for $t\neq 0$ and $\xi\in\mathrm{Supp}\chi(p(\xi))$, we obtain by integration by parts that for any nonnegative integer $j$

$$\|\chi(p(D_x))e^{itp(D_x)}\|_{B(0,j;\,(m-1)j,\,-j)}\leqq C_j t^{-j},\qquad t\neq 0\,. \tag{3.10}$$

The estimate (3.10) with $j$ replaced by any nonnegative number $s$ is derived from the above inequality by using the interpolation theorem for weighted Sobolev spaces (for which, see [J. Math. Soc. Japan, **31** (1979), p. 477]). Then elementary calculations show the lemma. Q. E. D.

PROOF OF THEOREM 3.3. Choose a $C^\infty$-function $\psi$ on $\boldsymbol{R}^1$ such that $\psi(\lambda)=1$ on $\mathrm{Supp}\,\chi$ and $\mathrm{Supp}\,\psi\subset\{\lambda;\ |\lambda|>N\}$. We have

$$\int e^{it\lambda}\chi(\lambda)R(\lambda\pm i0)d\lambda=\int e^{it\lambda}\chi(\lambda)\sum_{j=0}^{\infty}R_0(\lambda\pm i0)[V\psi(\lambda)R_0(\lambda\pm i0)]^j d\lambda\,. \tag{3.11}$$

We write $W(t)=\int e^{it\lambda}\psi(\lambda)R_0(\lambda\pm i0)d\lambda$. Choose $\sigma'$ such that $\max(m',\sigma)<\sigma'<m-1$. By Lemma 3.4, there exists a constant $C$ such that for any $1<r\leqq\rho$

$$\|VW(t)\|_{B(r,\,\rho-r)}\leqq CN^{-\min(1,\,\sigma'-m')}\langle t\rangle^{-r+\sigma'/(m-1)}|t|^{-\sigma'/(m-1)}\,.$$

Thus, rechoosing $N$ if necessary we obtain from (3.11) that

$$\int e^{it\lambda}\chi(\lambda)R(\lambda\pm i0)d\lambda=\sum_{j=0}^{\infty}\Big(\int e^{it\lambda}\chi(\lambda)R_0(\lambda\pm i0)d\lambda\Big)(*VW(t))^j \tag{3.12}$$

in $B(0,s;\sigma,-s)$, where $*$ denotes the convolution. From this the theorem follows. Q. E. D.

## §4. A generalization.

In this section we extend Theorem 1 to operators with singular lower order coefficients.

Let $A_1(X,D_x)$ be the elliptic differential operator of order $m$ whose symbol

$$A_1(x,\xi)=\sum_{|\alpha|\leqq m}a_\alpha(x)\xi^\alpha=p(\xi)+\sum_{|\alpha|\leqq m}q_\alpha(x)\xi^\alpha$$

satisfies the assumption (A. I), (A. III), and (A. II′): The functions $q_\alpha(x)$, $|\alpha|\leqq m$, satisfy (1.3) for $|\beta|\leqq 3n+11$. Denote by $A_1$ the closed operator in $L_2(\boldsymbol{R}^n)$ defined by: $A_1u=A_1(X, D_x)u$, $u\in D(A_1)=H^m$. Let $V$ be a closed operator in $L_2(\boldsymbol{R}^n)$ such that for some $m'<m-1$

$$V\in B(m', s; 0, s+\rho+\max(m'-m+3, 0)), \qquad s\in\boldsymbol{R}^1. \tag{4.1}$$

Put $A=A_1+V$. Then we have

THEOREM 4.1. *The conclusion of Theorem 1 holds also for the above operator $A$ under the restriction that $\sigma\leqq m$.*

For the proof we need a lemma for $R_1(z)=(z-A_1)^{-1}$.

LEMMA 4.2. (i) *Let $0<k+\theta<\rho-1$, $0\leqq\tau_j\leqq 1$ and $s_j>k+\theta+3/2-\tau_j$, $j=1, 2$. Then for any $0\leqq\sigma\leqq m$*

$$\|R_1^{(j)}(z)\|_{B(0, s_1; \sigma, -s_2)}\leqq C|z|^{-\{(m-1)(j+1)-\sigma-\tau_1-\tau_2\}/m}, \tag{4.2}$$

$$z\in\boldsymbol{N}_\pm, \quad j=0, \cdots, k,$$

$$h^{-\theta}\|\Delta_h^1 R_1^{(k)}(z)\|_{B(0, s_1; \sigma, -s_2)}\leqq C|z|^{-\{(m-1)(k+\theta+1)-\sigma-\tau_1-\tau_2\}/m}, \tag{4.3}$$

$$0<h<1, \quad z \text{ and } z+h\in\boldsymbol{N}_\pm.$$

(ii) *Let $0\leqq\kappa_j\leqq 1/2$ and $r_j>k+\theta+1-\kappa_j$, $j=1, 2$. Then for all $f\in L_2^{\tau_1}$*

$$\int\left[\langle t\rangle^{k+\theta-l}|t|^l\left\|\left(\frac{d}{dt}\right)^j\int e^{it\lambda}\chi(\lambda)\lambda^{\delta/m}R_1(\lambda\pm i0)f\,d\lambda\right\|_{\sigma, -r_2}\right]^2 dt \tag{4.4}$$

$$\leqq C\|f\|_{r_1}^2,$$

*where* $0\leqq\sigma\leqq m$, $0\leqq\sigma+mj\leqq m(k+1)$, $\delta\leqq(m-1)(k+\theta+1/2)-(\sigma+mj+2\kappa_1+2\kappa_2)$, $l=\max(0, (\sigma+mj+\delta)/(m-1))$.

PROOF. Even if (A. II′) is assumed instead of (A. II), Theorem 1 holds also for $A_1$ under the restriction that $\sigma\leqq m$ (cf. [Remark 4.3, **1**]). Thus the statement (i) for $\tau_1=\tau_2=0$ is valid. Since Theorem 3.1 holds with $R(z)$ and $\rho$ replaced by $R_0(z)$ and $\infty$, respectively, (i) for $\tau_1=1$ and $\tau_2=0$ is shown by using the resolvent equation

$$R_1(z)=R_0(z)+R_1(z)\{\sum_{|\alpha|\leqq m} q_\alpha(X)D_x^\alpha\}R_0(z).$$

Thus the interpolation method shows (i) for $0\leqq\tau_1\leqq 1$ and $\tau_2=0$. The lemma for the other cases can be shown similarly. Q. E. D.

PROOF OF THEOREM 4.1. Lemma 4.2 shows that for any sufficiently small $\varepsilon>0$

$$\|VR_1(z)\|_{B(0, 1/2+l+\varepsilon; 0, 1/2+l+\varepsilon)}\leqq C_\varepsilon|z|^{-\varepsilon},$$

where $l=\max(0, (m'+3-m)/2)$. Thus we obtain that for any $z$ with $\mathrm{Re}\, z\gg 1$ and $\pm\mathrm{Im}\, z\geqq 0$

$$R(z)=\sum_{j=0}^{\infty} R_1(z)[VR_1(z)]^j \quad \text{in} \quad B(s, -s), \quad s>3/2.$$

This yields the theorem. Q.E.D.

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