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# How many primes decompose completely in an infinite unramified Galois extension of a global field?

Dedicated to Professor I. R. Šafarevič on his 60-th birthday

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**1.** Let $M/k$ be an infinite unramified Galois extension of a global field $k$. By investigating an analogue of $d\log\zeta(s)$ ($\zeta$: the zeta function) for the infinite extension field $M$, and its analytic continuation especially towards $s=1/2$, we obtain an upper bound for some "weighted cardinality" of the set of primes of $k$ that decompose almost completely in $M$. In the function field case, our upper bound is attained by those $M/k$ which correspond to torsion-free co-compact irreducible discrete subgroups $\Gamma$ of $PSL_2(\boldsymbol{R})\times PGL_2(F_{\mathfrak{p}})$ ($F_{\mathfrak{p}}$: a $\mathfrak{p}$-adic field) (called $\Gamma$-classfields in [**4**], and obtained by the reduction mod $\mathfrak{p}$ of towers of Shimura curves cf. also [**3**][**5**]). In a sense, our inequality may be viewed as playing the role of "the second norm index inequality for (non-abelian) classfield theory of $\Gamma$-type". In the number field case, we must assume the generalized Riemann hypothesis to obtain an equally good bound. But this conditional result will also be presented, with the hope that its comparison with the situation in the function field case might be suggestive for further study of infinite unramified extensions.

We state our main results in § 2, and give their proofs in §§ 3~12. The crucial part of the proof lies in the study of the limit of

$$[K:k]^{-1}d\log\zeta_K(s) \qquad \begin{array}{l} k\subset K\subset M \\ [K:k]<\infty \end{array}$$

as $K\to M$, especially in the domain $1/2<\mathrm{Re}(s)<1$ where the Dirichlet series expression for $d\log\zeta_K(s)$ is no longer valid. This is done by careful examination of the property of their inverse Mellin transforms. In the number field case, the "effective analysis" for $d\log\zeta_K(s)$ initiated by Stark [**15**] and continued by Odlyzko, Lagarias, Serre, Poitou, $\cdots$ is crucial. In § 13, we indicate what modifications are necessary if we allow some tame ramifications in $M/k$. In § 14, we give some examples of Golod-Šafarevič type. The final section § 15 is for various remarks and discussions.

Relations with other works will be mentioned briefly at the end of § 2. The author wishes to express his acknowledgement to Professor J-P. Serre for bringing some of them to his attention.

**2.** We use the following notations:

$k$ : a global field, i.e., either an algebraic number field of finite degree (NF), or an algebraic function field of one variable over a finite field (FF);

$M/k$ : an infinite unramified Galois extension (the unramifiedness refers also to the archimedean primes);

$P$ : runs over all non-archimedean primes of $k$;

$f(P)$ : the residue extension degree of $P$ in $M/k$ $(1\leqq f(P)\leqq\infty)$;

$N(P)$ : the absolute norm of $P$;

$S$ $=\{P; f(P)<\infty\}$;

$S_\infty$ : the set of all archimedean primes of $k$.

In the (FF) case, our goal is to prove the following

THEOREM 1 (FF). *Let $\boldsymbol{F}_q$ be the exact constant field of $k$, and $g$ be the genus. Then*

$$\sum_{P\in S}\frac{\deg P}{N(P)^{(1/2)f(P)}-1}\leqq \mathrm{Max}(g-1,\ 0)\,, \tag{2-1}$$

*the series on the left being convergent. Here,* $\deg P$ *is the degree of $P$ over $\boldsymbol{F}_q$ (so that $N(P)=q^{\deg P}$).*

As for this formula, the presence of the factor 1/2 in the exponent of $N(P)$ is important, and owes to the Weil's Riemann hypothesis for curves [**16**]. Without this, the formula becomes much weaker and easier to prove. When $M/k$ corresponds to a torsion-free co-compact irreducible discrete subgroup $\Gamma$ of $PSL_2(\boldsymbol{R})\times PSL_2(F_\mathfrak{p})$, (2-1) becomes an *equality*. Indeed, in this case, $q=N(\mathfrak{p})^2$, $g\geqq 2$, $S$ is a finite set with cardinality

$$H=(N(\mathfrak{p})-1)(g-1)=(\sqrt{q}-1)(g-1)\,,$$

and moreover, $f(P)=1$ and $\deg P=1$ for all $P\in S$ (cf. [**3**]~[**5**], [**7**]). Thus, the left side of (2-1) is $H\cdot(\sqrt{q}-1)^{-1}$ which is equal to the right side $g-1$. A survey of our study in this direction is given in [**7**].

The equality will no longer (*and never*) hold if we replace this $M/k$ by $Mk'/k'$, where $k'$ is any finite separable extension of $k$ *not contained in* $M$; see § 15.

In the (NF) case, we prove a strong conditional, and a weak unconditional result, as follows.

ASSUMPTION (GRH). *The Riemann hypothesis is valid for the Dedekind zeta* *function* $\zeta_K(s)$ *for all* $K$ *with* $k\subset K\subset M$, $[K:k]<\infty$.

THEOREM 1 (NF; under GRH). *Let* $d$ *be the discriminant of* $k$. *Then*

$$\sum_{P\in S}\frac{\log N(P)}{N(P)^{(1/2)f(P)}-1}+\sum_{P_\infty\in S_\infty}\alpha_{P_\infty}\leqq\frac{1}{2}\log|d|, \tag{2-2}$$

*the first series on the left being convergent. Here,*

$$\alpha_{P_\infty}=\frac{1}{2}\left(\log 8\pi+\gamma+\frac{\pi}{2}\right)\qquad\cdots P_\infty:\ \textit{real}, \tag{2-3}$$

$$=\log 8\pi+\gamma\qquad\cdots P_\infty:\ \textit{imaginary},$$

$\gamma$ *being the Euler's constant*; $\gamma=\lim(1+1/2+\cdots+1/n-\log n)=0.577\cdots$.

PROPOSITION 1 (NF; unconditional). *We have*

$$\sum_{P\in S}\frac{\log N(P)}{N(P)^{f(P)}-1}+\sum_{P\in S_\infty}\beta_{P_\infty}\leqq\frac{1}{2}\log|d|, \tag{2-4}$$

*the first series on the left being convergent. Here,*

$$\beta_{P_\infty}=\frac{1}{2}(\log 4\pi+\gamma)\qquad\cdots P_\infty:\ \textit{real}, \tag{2-5}$$

$$=\log 2\pi+\gamma\qquad\cdots P_\infty:\ \textit{imaginary}.$$

This is much weaker than (2-2). Its proof will be given in § 13. In the (NF) case, one may expect that there exists $M/k$ for which $S$ is infinite and for which the *equality* in (2-2) holds.

The difference (the right side)$-$(the left side) in (2-1) (resp. (2-2)) will be denoted by $\delta(M/k)$ ($\geqq 0$). We can interpret this number in terms of some quantity related to the distribution of zeros of zeta functions of intermediate fields of $M/k$, as follows. (In the (FF) case, we assume for simplicity that $M/k$ contains no constant field extensions.) Let $K$ run over finite Galois extensions of $k$ in $M$, and $\zeta_K(s)$ be its Dedekind zeta function. In the (FF) case, let

$$\{\pi_\nu,\ \bar{\pi}_\nu\ (1\leqq\nu\leqq g(K))\}\qquad \bar{}:\ \text{the complex conjugate,}$$
$$g(K):\ \text{the genus of } K,$$

be its zeros for the variable $q^s$, and in the (NF) case, let

$$\{\rho_\nu,\ \bar{\rho}_\nu\ (\nu=1,2,\cdots)\}$$

be its non-trivial zeros for the variable $s$, each counted with multiplicity. (The (GRH) being assumed, the real non-trivial zeros are only at $s=1/2$, in which case the multiplicity is *even* due to the functional equation, so that there is no ambiguity in this notation.) For each $\sigma>1/2$, put

$$\text{(2-6)}\qquad a_K(\sigma)=\sum_{\nu=1}^{g(K)}\frac{q}{(q^\sigma-\pi_\nu)(q^\sigma-\bar{\pi}_\nu)}\quad(>0)\qquad\text{(FF)},$$

$$=\sum_{\nu=1}^{\infty}\frac{1}{(\sigma-\rho_\nu)(\sigma-\bar{\rho}_\nu)}\quad(>0)\qquad\text{(NF)}.$$

Then

THEOREM 2 (FF, or NF with GRH). *The limit*

$$\text{(2-7)}\qquad a_M(\sigma)=\lim_{K\to M}[K:k]^{-1}a_K(\sigma)$$

*exists for each* $\sigma>1/2$, *and the limit*

$$\text{(2-8)}\qquad a_M=\lim_{\sigma\to1/2}(q^{2\sigma-1}-1)a_M(\sigma)\qquad\text{(FF)},$$

$$=\lim_{\sigma\to1/2}(2\sigma-1)a_M(\sigma)\qquad\text{(NF)}$$

*also exists. We have*

$$\text{(2-9)}\qquad \delta(M/k)=a_M\geqq0\,.$$

*In particular, the equality in* (2-1) (FF) (resp. (2-2) (NF)) *holds if and only if* $a_M=0$.

There is another *essentially different* way of expressing $\delta(M/k)$ in terms of zeros of $\zeta_K(s)$'s in connection with one of the explicit formulae; see § 15 (B).

The influence of the change of base $M/k\to Mk'/k'$ ($k'/k$: a finite separable extension) on both sides of our inequality will also be discussed in § 15 (C). For this purpose, it is better to look at the *ratio* of the two sides instead of their difference.

*Relations with other works.* The present work continues [**3**]~[**6**], but is stimulated also by the Drinfeld-Vladut refinement [**1**] of Theorem (i) of [**6**]. After this work was accomplished and circulated as preprint, the author received a new note [**14**] of Serre referring to this article and leading our Theorem 1 (FF) as a corollary of his new theorem ([**14**]; th. 3 and cor.). He makes a systematic use of the Weil's generalized explicit formula, and gives a similar result on systems of curves $\{C_\alpha\}$ with genus $g_\alpha\to\infty$, which is identical with our Theorem 1 (FF) when $\{C_\alpha\}$ is a family of finite étale coverings of a *fixed* base curve $C$. Such systematic use of the Weil explicit formula has already been developed in the number field case, in order to obtain good lower bounds for the discriminant (cf. Poitou [**10**]). In fact, the infinite local factor $\alpha_{P_\infty}$ in Theorem 1 (NF; under GRH) is exactly the same as the one which appears in [**10**] as coefficients of $r_1$, $r_2$ in the asymptotic lower bound for $\log|d|$ under GRH. About the unconditional result for the (NF) case (Proposition 1), I only mention here that this can be refined to some extent, using [**10**] (b) pp. 6-16, as suggested by

Serre. I would like to add here that Proposition 1 leads immediately to the (perhaps more or less known) fact that the topological Galois group of the maximum unramified Galois extension of a number field is generated by *the conjugacy classes of* finite number of elements, because the sum of $(\log N(P))(N(P)-1)^{-1}$ over all non-archimedean primes $P$ of $k$ is divergent.

**3.** Put $l=[\overline{\boldsymbol{F}}_q\cap M:\boldsymbol{F}_q]$ ((FF)-case), where $\overline{\boldsymbol{F}}_q$ is the algebraic closure of $\boldsymbol{F}_q$. To start the proofs, we first note that in the (FF) case, it suffices to prove Theorem 1 when $l=1$. Indeed, if $l=\infty$, then $f(P)=\infty$ for all $P$; hence $S=\emptyset$ and the left side of (2-1) vanishes. If $l<\infty$, the base change $k\to k\boldsymbol{F}_{q^l}$ reduces the assertion to the case $l=1$. So, we assume from now on that the exact constant field of $M$ is also $\boldsymbol{F}_q$

**4.** Now, returning to the general situation, let $K$ run over the finite Galois extensions of $k$ in $M$, and for each $K$, let $\mathfrak{P}(K)$ denote the set of all non-archimedean primes of $K$. Let $\zeta_K(s)$ be the Dedekind zeta function with the Euler product expansion

$$\zeta_K(s)=\prod_{P_K\in\mathfrak{P}(K)}(1-N(P_K)^{-s})^{-1},\qquad \mathrm{Re}(s)>1. \tag{4-1}$$

Put

$$\begin{aligned} Z_K(s)&=-\frac{\zeta_K'(s)}{\zeta_K(s)}=\sum_{P_K\in\mathfrak{P}(K)}\sum_{m\geq 1}\frac{\log N(P_K)}{N(P_K)^{ms}} \\ &=[K:k]\cdot\sum_{P\in\mathfrak{P}(k)}\sum_{m\geq 1}\frac{\log N(P)}{N(P)^{f(P,K)ms}} \\ &=[K:k]\cdot\sum_{P\in\mathfrak{P}(k)}\frac{\log N(P)}{N(P)^{f(P,K)s}-1},\qquad \mathrm{Re}(s)>1, \end{aligned} \tag{4-2}$$

where $f(P,K)$ is the residue extension degree of $P$ in $K/k$. Dividing $Z_K(s)$ by $[K:k]$ (for averaging purpose) and subtracting the main term at its pole(s) on $\mathrm{Re}(s)=1$, we obtain a function

$$\begin{aligned} Z_K^0(s)&=[K:k]^{-1}\left(Z_k(s)-\frac{\log q}{q^{s-1}-1}\right) &&\text{FF-case} \\ &=[K:k]^{-1}\left(Z_K(s)-\frac{1}{s-1}\right) &&\text{NF-case,} \end{aligned} \tag{4-3}$$

which is holomorphic on $\mathrm{Re}(s)>1/2$ under the assumption (GRH) in the (NF) case. Now put

$$Z_M(s)=\sum_{P\in S}\sum_{m\geq 1}\frac{\log N(P)}{N(P)^{f(P)ms}}=\sum_{P\in S}\frac{\log N(P)}{N(P)^{f(P)s}-1}, \tag{4-4}$$

which is holomorphic on $\mathrm{Re}(s)>1$, being a convergent Dirichlet series. *Assume* (GRH) *for the* (NF)*-case throughout* §§ 4-11. Then our key lemma is:

LEMMA. *$Z_M(s)$ extends to a holomorphic function on* $\mathrm{Re}(s)>1/2$, *and as* $K\to M$, *$Z_K^0(s)$ tends to $Z_M(s)$ absolutely and uniformly in wider sense on* $\mathrm{Re}(s)>1/2$.

REMARK. It is easy to see, by comparing the Dirichlet series expressions for $[K:k]^{-1}Z_K(s)$ and $Z_M(s)$ (valid for $\mathrm{Re}(s)>1$), and by letting $K\to M$, that $Z_K^0(s)$ tends to $Z_M(s)$ for $\mathrm{Re}(s)>1$ uniformly in wider sense. On the other hand, $Z_K^0(s)$ is holomorphic on $\mathrm{Re}(s)>1/2$ (modulo (GRH) in the (NF) case). But it remains to show that the limit $\lim_{K\to M} Z_K^0(s)$ is convergent *also in* $\mathrm{Re}(s)>1/2$ (uniformly in wider sense), and for this purpose, the Dirichlet series expression for $[K:k]^{-1}Z_K(s)$ will not help, as it converges only for $\mathrm{Re}(s)>1$. Here lies the main difficulty to be worked out.

**5.** To prove the lemma, we need some analysis of the inverse Mellin transforms of $Z_K^0(s)$ and $Z_M(s)$. First, let $\psi_K(x)$ ($x$: real, $>1$) be the Chebyshev's step function obtained as the partial sum of coefficients of the Dirichlet series defining $Z_K(s)$; i. e.,

(5-1) $$\psi_K(x)=\sum_{\substack{P_K\in\mathfrak{P}(K)\\ m\geqq 1\\ N(P_K)^m<x}} \log N(P_K)=[K:k]\sum_{\substack{P\in\mathfrak{P}(k)\\ m\geqq 1\\ N(P)^{f(P,K)m}<x}} \log N(P)\,.$$

More precisely, as is usual in Fourier analysis, $\psi_K(x)$ should be replaced by $(1/2)(\psi_K(x+0)+\psi_K(x-0))$ at the points of discontinuity (and similarly for $\psi_K^0(x)$ and $\psi_M(x)$ below). Put

(5-2) $$\psi_K^0(x)=[K:k]^{-1}\Big(\psi_K(x)-\frac{q\log q}{q-1}(q^y-1)\Big) \qquad \text{(FF)}$$

$$=[K:k]^{-1}(\psi_K(x)-x) \qquad \text{(NF)}$$

for $x>1$, where $y=[\log x/\log q]$, the largest integer not exceeding $\log x/\log q$. Finally, put

(5-3) $$\psi_M(x)=\sum_{\substack{P\in S\\ m\geqq 1\\ N(P)^{f(P)m}<x}} \log N(P) \qquad (x>1)\,.$$

Then, as is clear from the definitions, we have for each fixed $x_0>1$,

(5-4) $$[K:k]^{-1}\psi_K(x)=\psi_M(x) \qquad (1<x\leqq x_0)$$

for all sufficiently large $K$ (large w.r.t. $\supset$) depending on $x_0$. Now, it is well-known and easy to prove that

(5-5) $$s^{-1}Z_K(s)=\int_1^\infty \psi_K(x)x^{-s-1}dx\,, \qquad \mathrm{Re}(s)>1\,.$$

Since

$$\int_1^\infty q^y x^{-s-1}dx=(\log q)\int_0^\infty q^{[t]-st}dt \qquad (\text{by } x=q^t)$$

$$=(\log q)\sum_{\nu=0}^{\infty}q^\nu\int_\nu^{\nu+1}q^{-st}dt=\frac{1-q^{-s}}{s(1-q^{1-s})},$$

we obtain

(5-6) $$s^{-1}Z_K^0(s)=\int_1^\infty \psi_K^0(x)x^{-s-1}dx\,, \qquad \mathrm{Re}(s)>1\,.$$

Similarly,

(5-7) $$s^{-1}Z_M(s)=\int_1^\infty \psi_M(x)x^{-s-1}dx\,, \qquad \mathrm{Re}(s)>1\,.$$

**6.** We shall need the following evaluation of $\psi_K^0(x)$ based on the Riemann hypothesis for $\zeta_K(s)$ and the explicit formula for $\psi_K(x)$;

(6-1) $$|\psi_K^0(x)|\leqq C_k x^{1/2}(\log x)^2 \qquad (x\geqq 2)\,,$$

where $C_k$ is a positive constant depending *only on* $k$ (not $K$).

To verify (6-1), first in the (FF) case, write

(6-2) $$\zeta_K(s)=\frac{\prod_{\nu=1}^{g(K)}(1-\pi_\nu q^{-s})(1-\bar{\pi}_\nu q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

with

(6-3) $$|\pi_\nu|=|\bar{\pi}_\nu|=q^{1/2}$$

(the Weil's Riemann hypothesis for curves). Put

(6-4) $$N_i=\sum_{\substack{P_K\in\mathfrak{P}(K)\\ \deg P_K | i}}\deg P_K=q^i+1-\sum_{\nu=1}^{g(K)}(\pi_\nu^i+\bar{\pi}_\nu^i) \qquad (i\geqq 1)\,.$$

Then, as $\log N(P_K)=(\log q)\deg P_K$, we have

(6-5) $$\psi_K(x)=(\log q)\sum_{i=1}^{y}N_i \qquad \left(y=\left[\frac{\log x}{\log q}\right]\right)$$

$$=(\log q)\left\{\frac{q(q^y-1)}{q-1}+y-\sum_{\nu=1}^{g(K)}\sum_{i=1}^{y}(\pi_\nu^i+\bar{\pi}_\nu^i)\right\}.$$

Therefore,

(6-6) $$\psi_K^0(x)=(\log q)[K:k]^{-1}\{y-\sum_{\nu=1}^{g(K)}\sum_{i=1}^{y}(\pi_\nu^i+\bar{\pi}_\nu^i)\}\,.$$

From (6-3) (6-6) and the Hurwitz formula

(6-7) $$g(K)-1=[K:k](g-1),$$

it follows directly that

(6-8) $$|\psi_K^0(x)|\leqq[K:k]^{-1}(\log x+2x^{1/2}\log x)+2(g-1)x^{1/2}\log x;$$

which implies (6-1). (In the (FF) case, log need not be squared.)

To check (6-1) in the (NF) case, we use the Lagarias-Odlyzko estimate for $\psi_K(x)$ based on the explicit formula

(6-9) $$\psi_K(x)=x-\sum_{\substack{\rho\in\mathfrak{Z}(K)\\|\mathrm{Im}(\rho)|<T}}\frac{x^\rho}{\rho}-\frac{r_1(K)}{2}\log(x^2-1)-r_2(K)\log(x-1)$$

$$+\log x+\log|d(K)|-[K:\boldsymbol{Q}](\gamma+\log 2\pi)+\gamma_K+R(x,T),$$

with $R(x,T)\to 0$ for $T\to\infty$, $|R(x,T)|<\cdots$. Here, $\mathfrak{Z}(K)$ is the set of all non-trivial zeros of $\zeta_K(s)$, the sum $\sum_\rho$ being taken with multiplicity, $r_1(K)$ (resp. $r_2(K)$) is the number of real (resp. imaginary) primes of $K$, $d(K)$ is the discriminant of $K$, and

$$\gamma_K=-\lim_{s\to 1}\left(Z_K(s)-\frac{1}{s-1}\right).$$

By giving an explicit estimate for the remainder term $R(x,T)$ and putting $T=x^{1/2}+1$, Lagarias-Odlyzko obtained the following universal estimate for $\psi_K(x)$ (assuming the (GRH)) (cf. [8] Theorem 9.1):

(6-10) $$|\psi_K(x)-x|\leqq C\{[K:\boldsymbol{Q}]x^{1/2}(\log x)^2+\log|d(K)|x^{1/2}\log x\}$$

for all $x\geqq 2$, where $C$ is a positive *absolute* constant. Since

(6-11) $$\log|d(K)|=[K:k]\log|d|$$

($K/k$ being unramified), we obtain (6-1) by dividing (6-10) by $[K:k]$.

**7.** By (6-1), it follows immediately that the integral expression

(7-1) $$s^{-1}Z_K^0(s)=\int_1^\infty\psi_K^0(x)x^{-s-1}dx$$

for $Z_K^0(s)$ is valid for $\mathrm{Re}(s)>1/2$.

Now we shall obtain an estimate of $\psi_M(x)$. Since

(7-2) $$\psi_M(x)=\sum_{\substack{P\in S,\,m\geqq 1\\N(P)^{f(P)m}<x}}\log N(P)\leqq\sum_{\substack{P\in\mathfrak{P}(k),\,m\geqq 1\\N(P)^{f(P,K)m}<x}}\log N(P)$$

$$=[K:k]^{-1}\psi_K(x),$$

we have

$$\psi_M(x) \leqq \psi_K^0(x) + [K:k]^{-1} \frac{q \log q}{q-1} (q^y - 1) \qquad \text{(FF)}, \tag{7-3}$$

$$\leqq \psi_K^0(x) + [K:k]^{-1} x \qquad \text{(NF)}$$

for any $K$. Therefore, by (6-1),

$$\psi_M(x) \leqq C_k x^{1/2} (\log x)^2 + [K:k]^{-1} \frac{q \log q}{q-1} (q^y - 1) \qquad \text{(FF)}, \tag{7-4}$$

$$\leqq C_k x^{1/2} (\log x)^2 + [K:k]^{-1} x \qquad \text{(NF)}.$$

Therefore, by (fixing $x$ and) letting $K \to M$, we obtain

$$\psi_M(x) \leqq C_k x^{1/2} (\log x)^2 \qquad (x \geqq 2)\,. \tag{7-5}$$

(In the (FF) case, combination of (6-8) and (7-3) actually gives

$$\psi_M(x) \leqq 2(g-1) x^{1/2} \log x \qquad (x > 1)\,.) \tag{7-6}$$

By (7-5), the integral on the right side of (5-7):

$$s^{-1} Z_M(s) = \int_1^\infty \psi_M(x) x^{-s-1} dx$$

also converges absolutely and uniformly in wider sense on $\mathrm{Re}(s) > 1/2$, and thus *gives a holomorphic continuation of* $Z_M(s)$ *to* $\mathrm{Re}(s) > 1/2$.

**8.** Now we shall prove the rest of the key lemma. By § 7, we have

$$s^{-1}(Z_K^0(s) - Z_M(s)) = \int_1^\infty (\psi_K^0(x) - \psi_M(x)) x^{-s-1} dx \tag{8-1}$$

for $\mathrm{Re}(s) > 1/2$. Let $\mathfrak{F}$ be a compact set in $\mathrm{Re}(s) > 1/2$, and $\delta$ be the distance from $\mathfrak{F}$ to the line $\mathrm{Re}(s) = 1/2$. For any given $\varepsilon > 0$, take $x_0 \geqq 2$ so large that

$$2C_k \cdot \int_{x_0}^\infty (\log x)^2 x^{-1-\delta} dx < \varepsilon\,. \tag{8-2}$$

Then take $K$ so large that

$$[K:k]^{-1} \psi_K(x) = \psi_M(x)\,, \qquad \text{all } x \leqq x_0 \quad \text{(see (5-4))}.$$

Then

$$\int_1^\infty (\psi_K^0(x) - \psi_M(x)) x^{-s-1} dx = \int_{x_0}^\infty (\psi_K^0(x) - \psi_M(x)) x^{-s-1} dx \tag{8-3}$$

$$- \begin{cases} [K:k]^{-1} \displaystyle\int_1^{x_0} \frac{(\log q) q}{q-1} (q^y - 1) x^{-s-1} dx & \text{(FF)} \\ [K:k]^{-1} \displaystyle\int_1^{x_0} x \cdot x^{-s-1} dx & \text{(NF)}. \end{cases}$$

If $K$ is further sufficiently large, the second term on the right side becomes arbitrarily small in absolute value (uniformly for $s\in\mathfrak{F}$), say $<\varepsilon$. On the other hand, by (6-1) (7-5) and (8-2), the absolute value of the first term on the right side of (8-3) also does not exceed $\varepsilon$. Therefore, the right side of (8-1) tends to 0 as $K\to M$, uniformly for $s\in\mathfrak{F}$. This settles the proof of the lemma.

**9.** By definition, we have

$$Z_M(s)=\sum_{P\in S,\, m\geq 1}\frac{\log N(P)}{N(P)^{f(P)ms}} \tag{9-1}$$

for $\mathrm{Re}(s)>1$, and $Z_M(s)$ is holomorphic on $\mathrm{Re}(s)>1/2$, as we have shown. Since the right side of (9-1) is a Dirichlet series with real *non-negative* coefficients, we conclude via Landau's lemma (cf. e.g. [**13**] Ch. VI Prop. 7) that the *right side of* (9-1) *converges also for* $\mathrm{Re}(s)>1/2$, *and* (9-1) *is valid for* $\mathrm{Re}(s)>1/2$.

**10.** So, for $s=\sigma>1/2$, we have

$$Z_M(\sigma)=\sum_{P\in S}\frac{\log N(P)}{N(P)^{f(P)\sigma}-1}=\lim_{K\to M}Z_K^0(\sigma)\,. \tag{10-1}$$

Now let $a_K(\sigma)$ be the *positive* number defined by (2-6). We shall show that

$$Z_K^0(\sigma)=\frac{\log q}{[K:k]}\Big(\frac{1}{q^\sigma-1}+1\Big)+(\log q)(g-1)-\frac{\log q}{[K:k]}(q^{2\sigma-1}-1)a_K(\sigma) \qquad \text{(FF)} \tag{10-2}$$

$$=\frac{1}{[K:k]}\frac{1}{\sigma}+\log A_k+\frac{r_1}{2}g\Big(\frac{\sigma}{2}\Big)+r_2 g(\sigma)-\frac{2\sigma-1}{[K:k]}a_K(\sigma) \qquad \text{(NF)}$$

for $\sigma>1/2$, where $r_1$ (resp. $r_2$) is the number of real (resp. imaginary) primes of $k$, $A_k=\pi^{-r_1/2}(2\pi)^{-r_2}\cdot|d|^{1/2}$, and $g(s)=\Gamma'(s)/\Gamma(s)$.

In the (FF) case, we have (by (6-2))

$$Z_K(s)=(\log q)\Big\{\frac{1}{q^s-1}+\frac{1}{q^{s-1}-1}-\sum_{\nu=1}^{g(K)}\Big(\frac{\pi_\nu}{q^s-\pi_\nu}+\frac{\bar\pi_\nu}{q^s-\bar\pi_\nu}\Big)\Big\}\;; \tag{10-3}$$

hence

$$Z_K^0(\sigma)=\frac{(\log q)}{[K:k]}\Big\{\frac{1}{q^\sigma-1}-\sum_{\nu=1}^{g(K)}\Big(\frac{\pi_\nu}{q^\sigma-\pi_\nu}+\frac{\bar\pi_\nu}{q^\sigma-\bar\pi_\nu}\Big)\Big\}\,. \tag{10-4}$$

But since

$$\frac{\pi_\nu}{q^\sigma-\pi_\nu}+\frac{\bar\pi_\nu}{q^\sigma-\bar\pi_\nu}=-1+\frac{q^{2\sigma}-q}{(q^\sigma-\pi_\nu)(q^\sigma-\bar\pi_\nu)} \tag{10-5}$$

and $g(K)-1=[K:k](g-1)$, we obtain (10-2). In the (NF) case, we use the partial fractional decomposition of $Z_K(s)$ (used extensively by Stark and Odlyzko;

cf. [15], [9]):

$$(10\text{-}6)\qquad Z_K(s)=\log A_K+\frac{r_1^{(K)}}{2}g\left(\frac{s}{2}\right)+r_2^{(K)}g(s)+\left(\frac{1}{s}+\frac{1}{s-1}\right)-\sum_{\rho\in\mathfrak{Z}(K)}{}'\frac{1}{s-\rho},$$

where the terms for $\rho$ and $\bar{\rho}$ should be summed together. Since $\log A_K$, $r_1^{(K)}$, $r_2^{(K)}$ are proportional with $[K: k]$ ($K/k$ being unramified), we obtain

$$(10\text{-}7)\qquad Z_K^0(\sigma)=\log A_k+\frac{r_1}{2}g\left(\frac{\sigma}{2}\right)+r_2g(\sigma)+\frac{1}{[K: k]}\cdot\frac{1}{\sigma}-\frac{1}{[K: k]}\sum_{\rho\in\mathfrak{Z}(K)}{}'\frac{1}{\sigma-\rho}.$$

But since

$$(10\text{-}8)\qquad \frac{1}{\sigma-\rho}+\frac{1}{\sigma-\bar{\rho}}=\frac{2\sigma-1}{(\sigma-\rho)(\sigma-\bar{\rho})},$$

we obtain (10-2) also for the (NF) case.

**11.** Now, from (10-1) and (10-2) we see, by letting $K\to M$, that the limit $a_M(\sigma)=\lim_{K\to M}[K: k]^{-1}a_K(\sigma)\geqq 0$ exists and that

$$(11\text{-}1)\qquad Z_M(\sigma)=(\log q)(g-1)-(\log q)(q^{2\sigma-1}-1)a_M(\sigma)\qquad \text{(FF)}$$

$$=\log A_k+\frac{r_1}{2}g\left(\frac{\sigma}{2}\right)+r_2g(\sigma)-(2\sigma-1)a_M(\sigma)\qquad \text{(NF)}$$

for all $\sigma>1/2$. Since $a_M(\sigma)\geqq 0$, this implies

$$(11\text{-}2)\qquad \sum_{P\in S}\frac{\log N(P)}{N(P)^{f(P)\sigma}-1}\leqq(\log q)(g-1)\qquad \text{(FF)},$$

$$\leqq\log A_k+\frac{r_1}{2}g\left(\frac{\sigma}{2}\right)+r_2g(\sigma)\qquad \text{(NF)},$$

for all $\sigma>1/2$. Therefore, for any *finite* subset $S'$ of $S$, we have (11-2) with $S'$ in place of $S$. Letting $\sigma\to 1/2$ then gives

$$(11\text{-}3)\qquad \sum_{P\in S'}\frac{\log N(P)}{N(P)^{f(P)/2}-1}\leqq(\log q)(g-1)\qquad \text{(FF)}$$

$$\leqq\log A_k+\frac{r_1}{2}g\left(\frac{1}{4}\right)+r_2g\left(\frac{1}{2}\right)\qquad \text{(NF)}.$$

Since $S'$ is an arbitrary finite subset of $S$, (11-3) is valid for $S$ in place of $S'$, the series over $S$ being convergent. Since $\log N(P)=(\log q)\deg P$ (FF), $g(1/2)=-\log 4-\gamma$ and $g(1/4)=-\gamma-\log 8-\pi/2$, we obtain our Theorem 1.

Finally, the convergence of the series

$$(11\text{-}4)\qquad \sum_{P\in S}\frac{\log N(P)}{N(P)^{f(P)/2}-1}$$

just proved implies that

(11-5) $$\lim_{\sigma\to 1/2} Z_M(\sigma)$$

exists and is equal to (11-4), since they are essentially the Dirichlet series

$$\sum_{P\in S,\, m\geqq 1}\sum (\log N(P))\cdot N(P)^{-f(P)\sigma m} \qquad \left(\sigma\geqq\frac{1}{2}\right).$$

Therefore, by letting $\sigma\to 1/2$ in (11-1), we obtain the rest of Theorem 2.

**12.** For the proof of Proposition 1, we look at (10-6) just for $s=\sigma>1$. Since

$$Z_M(\sigma)=\sum_{P\in S}\frac{\log N(P)}{N(P)^{f(P)\sigma}-1}\leqq[K:k]^{-1}Z_K(\sigma),$$

and since

$$\frac{1}{\sigma-\rho}+\frac{1}{\sigma-\bar{\rho}}>0 \qquad \text{for } \sigma>1 \text{ (unconditionally)},$$

we obtain (by letting $K\to M$)

$$Z_M(\sigma)\leqq\log A_k+\frac{r_1}{2}g\left(\frac{\sigma}{2}\right)+r_2 g(\sigma) \qquad (\sigma>1).$$

So, by the same argument as above, we obtain

$$\sum_{P\in S}\frac{\log N(P)}{N(P)^{f(P)}-1}\leqq\log A_k+\frac{r_1}{2}g\left(\frac{1}{2}\right)+r_2 g(1).$$

Since $g(1)=-\gamma$, this gives Proposition 1.

**13.** Here, we shall indicate what modifications are necessary if we allow some ramifications in $M/k$. Assume now that a finite number of primes (including possibly the archimedean ones) of $k$ are ramified in $M$, but *tamely*, with ramification index $e(P)$ ($P$: non-archimedean; $1\leqq e(P)\leqq\infty$) or $e(P_\infty)$ ($P_\infty$: archimedean; $1\leqq e(P_\infty)\leqq 2$). Then Theorem 1 holds under the following modifications: (2-1) should be replaced by

(13-1) $$\sum_{P\in S}\frac{1}{e(P)}\frac{\deg P}{N(P)^{f(P)/2}-1}\leqq\operatorname{Max}\left(g-1+\frac{1}{2}\sum_P\left(1-\frac{1}{e(P)}\right)\deg P,\ 0\right),$$

and (2-2) by

(13-2) $$\sum_{P\in S}\frac{1}{e(P)}\frac{\log N(P)}{N(P)^{f(P)/2}-1}+\sum_{\substack{P_\infty;\\ e(P_\infty)=1}}\alpha_{P_\infty}+\sum_{\substack{P_\infty;\\ e(P_\infty)=2}}\alpha'_{P_\infty}$$

$$\leqq\frac{1}{2}\left(\log|d|+\sum_P\left(1-\frac{1}{e(P)}\right)\log N(P)\right),$$

where

$$\alpha'_{P_\infty}=\frac{1}{2}(\log 8\pi+\gamma)\,. \tag{13-3}$$

When $e(P)<\infty$ for all $P$, these formulae follow directly from (2-1) resp. (2-2) by changing the base $k$. When $e(P)=\infty$ for some $P$, this does not work, but our proof carries through with but slight modifications. The modified zeta function $Z_M(s)$ is given by

$$Z_M(s)=\sum_{\substack{P\in S\\ e(P)\neq\infty}}\frac{\log N(P)}{e(P)(N(P)^{f(P)s}-1)}\,. \tag{13-4}$$

**14.** Now, we shall give some example of $M/k$ with $S\neq\emptyset$ in the (NF) case, in connection with Golod-Šafarevič theory [**12**], [**2**]. Let $k$ be a number field, $\mathfrak{S}$ be a given set of non-archimedean prime divisors of $k$, and $l$ be any prime number. Let $M$ be the maximum unramified pro $l$-extension of $k$ in which all primes of $k$ belonging to $\mathfrak{S}$ decompose completely. Let $E$ be the group of units of $k$, and $E(\mathfrak{S})$ be the group of all elements $a\in k^\times$ with the property that all prime constituents of $(a)$ belong to $\mathfrak{S}$. Put

$$\begin{aligned}\rho&=\mathrm{rank}_l(E(\mathfrak{S})/E(\mathfrak{S})^l)\\ &=H+\mathrm{rank}_l(E/E^l)=H+(r_1+r_2-1)+\delta\,,\end{aligned} \tag{14-1}$$

where $H=|\mathfrak{S}|$ and $\delta=1$ or 0, according to whether $k$ contains a primitive $l$-th root of unity or not. Let $Cl(\mathfrak{S})$ be the quotient of the ideal class group of $k$ modulo the subgroup generated by the classes of elements of $\mathfrak{S}$, and put

$$t=\mathrm{rank}_l(Cl(\mathfrak{S})/Cl(\mathfrak{S})^l)\,. \tag{14-2}$$

Then the Gaschütz-Wienberg refinement of Golod-Šafarevič theory (cf. [**11**]) says that if

$$\rho\leqq\frac{1}{4}t^2-t\,, \tag{14-3}$$

then $M/k$ is infinite. (Actually, only the case $\mathfrak{S}=\emptyset$ is presented in [**11**], but this generalization follows immediately just by considering the elements of $\mathfrak{S}$ as additional "infinite primes".) In particular, let $k=\boldsymbol{Q}(\sqrt{d})$ be imaginary quadratic and $l=2$. Then $\rho=H+1$ and $t$ is calculated by the genus theory. For instance, if $t=6$ and $H=2$, then $M/k$ is infinite.

For a numerical example, let

$$d=-3.5.7.11.13.17.23.31 \qquad (\equiv 1\ (\mathrm{mod}\ 8))$$

and $\mathfrak{S}$ be the set of two distinct prime factors of (2). Then $H=2$ and $t=8-2$

$=6$. The contribution of $\mathfrak{S}$ on the left side of (2-2) is $2(\log 2)(\sqrt{2}-1)^{-1}=3.34681\cdots$, and that of $S_\infty$ is $\log 8\pi+\gamma=3.80138\cdots$; hence the left side of (2-2) is at least $7.14819\cdots$, while the right side is $(1/2)\log 181996815=9.50975\cdots$. This ratio $7.14\cdots/9.50\cdots=0.7517\cdots$ is the largest among the examples that I know[1]. On the other hand, I do not know whether the set $S$ for this example contains any primes outside $\mathfrak{S}$.

**15.** In this section, we shall give some remarks and discussions.

(A) One may define $Z_M(s)$ also for $\mathrm{Re}(s)<1/2$, in a rather obvious way using $[K:k]^{-1}Z_K(s)$ for $\mathrm{Re}(s)<1/2$, and then one has a "functional equation" which, e.g., in the (FF) case reads as $Z_M(s)+Z_M(1-s)=2(g-1)\log q$ $(\mathrm{Re}(s)\neq 1/2)$. But this does *not* give a part of a correct analytic continuation of $Z_M(s)$ even when $M/k$ corresponds to a $\Gamma$-classfield, because then $Z_M(s)=H\cdot(q^s-1)^{-1}\log q$.

(B) There is another aspect related to the difference $\delta(M/k)$, (the right side) $-$(the left side) of our fundamental inequality (2-1) (resp. (2-2)). It is directly connected with one of the explicit formulae. We state it only in the (NF) case.

PROPOSITION 2 (NF; under GRH). *Let $K$ run over the intermediate fields $k\subset K\subset M$; $[K:k]<\infty$, and $\rho=(1/2)+\gamma i$ run over the set $\mathfrak{Z}(K)$ of non-trivial zeros of $\zeta_K(s)$. Let $x>1$. Then the limit*

$$b(M,x)=\lim_{K\to M}[K:k]^{-1}\cdot\sum_{\rho\in\mathfrak{Z}(K)}\frac{\sin(\gamma\cdot\log x)}{\gamma}$$

*exists, and its value is connected with $\delta(M/k)$ by:*

$$b(M,x)=\delta(M/k)+\sum_{\substack{P\in S,\, m\geq 1\\ N(P)^{f(P)m}>x}}\frac{\log N(P)}{N(P)^{f(P)m/2}}+\frac{[k:\mathbf{Q}]}{2}\log\frac{\sqrt{x}+1}{\sqrt{x}-1}+r_1\operatorname{Arctan}\frac{1}{\sqrt{x}}. \tag{15-1}$$

Since the second$\sim$fourth terms on the right side of (15-1) are positive and tend to 0 as $x\to\infty$, we obtain

COROLLARY. *We have $b(M,x)>0$, and*

$$\delta(M/k)=\lim_{x\to\infty}b(M,x). \tag{15-2}$$

Since $b(M,x)$ is an oscillating sum, the equality (15-2) *alone* does *not* seem to give an alternative proof of $\delta(M/k)\geqq 0$.

PROOF OF PROPOSITION 2. We use the following explicit formula:

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1) *Added in proof.* Recently, K. Yamamura improved this to $0.9059\cdots$. In his example, $[k,\mathbf{Q}]=52$ and $|\mathfrak{S}|=105$.

$$\sum_{\substack{P_K\in\mathfrak{P}(K),\, m\geq 1\\ N(P_K)^m\leq x}} \frac{\log N(P_K)}{N(P_K)^{m/2}}=2\Big(\sqrt{x}-\frac{1}{\sqrt{x}}\Big)-\sum_{\rho\in\mathfrak{Z}(K)}\frac{\sin(\gamma\cdot\log x)}{\gamma} \tag{15-3}$$
$$+\frac{[K:\boldsymbol{Q}]}{2}\log\frac{\sqrt{x}+1}{\sqrt{x}-1}+r_1^{(K)}\operatorname{Arctan}\frac{1}{\sqrt{x}}$$
$$+\frac{1}{2}\log|d_K|-\frac{r_1^{(K)}}{2}\Big(\log 8\pi+\gamma+\frac{\pi}{2}\Big)$$
$$-r_2^{(K)}(\log 8\pi+\gamma)\qquad (x>1).$$

This is a special case of Weil [17]; formula (11) for

$$\begin{cases} F(t)=\begin{cases} 1 & \cdots 0<t<\log x,\\ 1/2 & \cdots t=0, \text{ or } t=\log x,\\ 0 & \cdots \text{otherwise};\end{cases}\\ \Phi(s)=\dfrac{x^{s-(1/2)}-1}{s-1/2}.\end{cases} \tag{15-4}$$

Fix $x$, divide the both sides by $[K:k]$, and let $K\to M$. Then the left side converges to

$$\sum_{\substack{P\in S,\, m\geq 1\\ N(P)^{f(P)m}\leq x}} \frac{\log N(P)}{N(P)^{f(P)m/2}} \tag{15-5}$$

which is a partial sum for our

$$\sum_{P\in S}\frac{\log N(P)}{N(P)^{f(P)/2}-1}=\sum_{P\in S,\, m\geq 1}\frac{\log N(P)}{N(P)^{f(P)m/2}}. \tag{15-6}$$

On the other hand, the right side converges to

$$-b(M,\, x)+\frac{[k:\boldsymbol{Q}]}{2}\log\frac{\sqrt{x}+1}{\sqrt{x}-1}+r_1\operatorname{Arctan}\frac{1}{\sqrt{x}}+\frac{1}{2}\log|d| \tag{15-7}$$
$$-\frac{r_1}{2}\Big(\log 8\pi+\gamma+\frac{\pi}{2}\Big)-r_2(\log 8\pi+\gamma).$$

(The existence of the limit $b(M,\, x)$ follows at this step.) Our Proposition follows immediately from these. q. e. d.

(C) Here, we discuss the effect of changing bases $M/k\to Mk'/k'$, where $k'/k$ is any finite separable extension. For this purpose, it is more convenient to look at the *ratio* of the two sides of (2-1) (resp. (2-2)) instead of the difference. Thus put

$$\rho(M/k)=\sum_{P\in S}\frac{\deg P}{N(P)^{f(P)/2}-1}\Big/(g-1) \qquad \text{(FF)} \tag{15-8}$$
$$=\Big\{\sum_{P\in S}\frac{\log N(P)}{N(P)^{f(P)/2}-1}+\sum_{P_\infty\in S_\infty}\alpha_{P_\infty}\Big\}\Big/\frac{1}{2}\log|d| \qquad \text{(NF)}.$$

The trivial cases $g=0$, 1, or $k=\boldsymbol{Q}$ will be excluded. Our fundamental inequality implies

$$\rho(M/k) \leqq 1 \tag{15-9}$$

(assuming (GRH) for (NF)). First, it is easy to see that $\rho(M/k)$ is determined only by $M$ and is independent of the choice of the base field $k$ satisfying (i) $k$ is a global field, (ii) $M/k$ is an infinite unramified Galois extension. So, *we shall write* $\rho(M)$ *instead of* $\rho(M/k)$. Secondly, note that any finite separable extension $M'/M$ is of the form $M'=Mk'$ with some finite separable extension $k'/k$. Since $M'/k'$ is also an infinite unramified Galois extension, $\rho(M')$ is defined for any such $M'$. Note that the set $S'$ ("$S$ for $M'/k'$") consists of all extensions of elements of $S$ to $k'$. Thirdly, for each $M/k$, let $\tilde{M}$ denote the maximum unramified Galois extension of $k$ containing $M$ such that for each $P \in S$, the residue extension degree of $P$ in $\tilde{M}$ coincides with that in $M$. When $M/k$ corresponds to a discrete subgroup $\Gamma$ as in §2, we have $\tilde{M}=M$ (cf. [**5**] or [**7**]). In general, we have

PROPOSITION 3. *Let $M'$ be any finite separable extension of $M$. Then*

$$\rho(M') \leqq \rho(M)\,, \tag{15-10}$$

*and the equality holds if and only if $M' \subset \tilde{M}$.*

The proof is straightforward and will be left to the readers. (Look at the effect of the base change on the numerator and the denominator of $\rho(M/k)$ separately, and in calculating the effect on the numerator, use the inequality

$$\frac{f}{t^f-1} \leqq \frac{1}{t-1} \qquad (t>1,\ f=1, 2, \cdots);$$

with the equality $=$ if and only if $f=1$.)

This Proposition suggests that the extensions $M/k$ satisfying $\rho(M)=1$ are *very rare.*

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