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Some results on weakly normal ring extensions

By Hiroshi YANAGIHARA

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The purpose of this paper is to give some results on weakly normal ring extensions which correspond to those on seminormal ring extensions obtained by several authors in [4], [7], [10], [12] and [13]. In the paper [8] M. Manresi gave a new characterization of weak normalization and discussed some questions related closely to our results. But our method depends on a criterion for weak normality which was given by S. Itoh in [6] and corresponds to Hamann's criterion for seminormality given in [5] and [7].

In § 1 we shall give a simple proof of this criterion different from the proof given in [6] for convenience' sake and also give a new characterization of weak normalization in the case where the characteristic of rings is a positive prime number. In § 2 we show first faithfully flat descent of weak normality which is a special case of pull-back descent of the property. Then we discuss local properties of weak normality. Furthermore we give some conditions for faithfully flat ring extensions to be weakly normal. These results are all given without any noetherian hypothesis. In the last section we generalize the notions of glueings of prime ideals or primary ideals which were defined in [12] and [11], and we show some basic results on these generalized glueings. In particular we give the notion of a weak glueing of a ring which plays a role for weakly normal ring extensions similar to the one played by ordinary glueings of prime ideals for seminormal ring extensions and show a structure theorem for weak normal ring extensions of noetherian rings corresponding to Theorem 2.1 in [12] for seminormal cases. Lastly we show results related to the going-down of Serre's property (S_2) under generalized glueings of rings corresponding to Theorem 2 and its Corollary in [13].

All the rings in this paper are commutative with unit and we use freely the terminology and results in [9].

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§ 1. Weakly normal rings.

Let B be a ring and A a subring of B such that B is integral over A. We recall the *seminormalization* ${}^{+}_{B}A$ and the *weak normalization* ${}^{*}_{B}A$ *of* A *in* B are defined as follows:

$$ {}^{+}_{B}A=\{b\in B \mid b/1\in A_{\mathfrak{p}}+R(B_{\mathfrak{p}}) \quad \text{for any } \mathfrak{p}\in\mathrm{Spec}(A)\}$$

$$ {}^{*}_{B}A=\{b\in B \mid \text{ for any } \mathfrak{p}\in\mathrm{Spec}(A), \text{ there is an integer } n \text{ such that } (b/1)^{e^n}\in A_{\mathfrak{p}}+R(B_{\mathfrak{p}})\}$$

where $R(B_{\mathfrak{p}})$ is the Jacobson radical of $B_{\mathfrak{p}}$ and e is the characteristic exponent of the quotient field $k(\mathfrak{p})$ of $A/\mathfrak{p}$.

If $A={}^{+}_{B}A$ (resp. ${}^{*}_{B}A$), then we say that A is *seminormal* (resp. *weakly normal*) *in* B.

Let A and B be as above, and let e and f be positive integers satisfying $e>f>1$. Then we say that A is (e, f)*-closed in* B, if the following is satisfied: *Any element* b *in* B *such that* b^e *and* b^f *are contained in* A *belongs to* A. Similarly A is called n*-closed in* B for a positive integer n if *any element* b *in* B *such that* b^n *is in* A *belongs to* A. Then the following result is originally due to E. Hamann [**5**].

THEOREM. *Let* A *and* B *be as above. Then the following are equivalent:*

(i) A *is seminormal in* B.

(ii) *For each* b *in* B, *the conductor of* A *in* $A[b]$ *is a radical ideal of* $A[b]$.

(iii) A *is* $(n, n+1)$*-closed in* B *for some* n.

(iv) A *is* (e, f)*-closed in* B *for a fixed pair of relatively prime integers* e *and* f.

For the proof, see Leahy and Vitulli [**7**].

Now we give a result for weakly normal rings corresponding to the above. First the author obtained the result in the case where A contains a field of positive characteristic, and then S. Itoh proved it in a general case.

THEOREM 1. *Let* A *and* B *be as above. Then the following are equivalent:*

(i) A *is weakly normal in* B.

(ii) A *is seminormal in* B, *and every element* b *in* B *which satisfies* $b^p\in A$ *and* $pb\in A$ *for some prime integer* p *belongs to* A.

PROOF. Assume that A is weakly normal in B. Then it is clear that A is seminormal in B by definition. Let b be an element of B such that $b^p\in A$ and $pb\in A$ for some prime p. Let $\mathfrak{p}$ be any prime ideal of A. If $\mathrm{char}(k(\mathfrak{p}))\neq p$, then $b/1$ is contained in $A_{\mathfrak{p}}$ because $pb\in A$. If $\mathrm{char}(k(\mathfrak{p}))=p$, then $b/1$ is contained in $A_{\mathfrak{p}}$ by the fact $A={}^{*}_{B}A$. This means that b belongs to A. Therefore the assertion (ii) is satisfied. Conversely assume that the assertion (ii) is satisfied. If A is not weakly normal in B, then there is an element b of ${}^{*}_{B}A$ not belonging to A. Since A is seminormal in B, the conductor ideal $\mathfrak{c}=A:_A A[b]=\{x\in A \mid xA[b]\subset A\}$ is a radical ideal of $C=A[b]$ by the above theorem. If $\mathfrak{p}$ is a minimal prime divisor of $\mathfrak{c}$ in A, then we have $\mathfrak{p}A_S=\mathfrak{c}_S$, where $S=A-\mathfrak{p}$. Since C is a finite A-module, we have $A_S:_{A_S}C_S=\mathfrak{c}_S$. On the other hand $\mathfrak{p}A_S=\mathfrak{c}_S$ coincides with the Jacobson radical $R(C_S)$ of C_S, because $\mathfrak{c}_S$ is a radical ideal of C_S. Moreover $({}^*_BA_S)$ is a quasi-local ring whose residue field with respect to the maximal ideal is a purely inseparable extension of $k(\mathfrak{p})$ (cf. Bombieri [**1**]). This means that C_S is a quasi-local ring with maximal ideal $\mathfrak{p}A_S=\mathfrak{c}_S$ and that the residue field $\kappa=C_S/\mathfrak{c}_S$ is purely inseparable over $k(\mathfrak{p})$. Now if $\kappa\supsetneqq k(\mathfrak{p})$, then $\mathrm{char}(k(\mathfrak{p}))=p$ is positive and there are an element y in C and an element s in S such that $y/s\notin A_S$ and $(y/s)^p\in A_S+\mathfrak{c}_S=A_S$. Then $p\cdot 1$ is an element of $\mathfrak{p}$ and hence $p(y/s)$ is contained in $\mathfrak{c}_S$. Therefore we see easily that $(ty)^p$ and $p(ty)$ are contained in A for some element t in S. Then ty is an element of A by our assumption, and hence y/s is contained in A_S. This is a contradiction. Therefore we see $\kappa=k(\mathfrak{p})$ and so $A_S+\mathfrak{p}A_S=A_S+\mathfrak{c}_S=C_S$. Then we have $A_S=C_S$ by Nakayama's Lemma. But this contradicts the fact that $A_S:_{A_S}C_S=\mathfrak{p}A_S\neq A_S$. Therefore ${}^*_BA$ must be A. q. e. d.

COROLLARY. *Let A and B be as above and assume that A contains a field of positive characteristic p. Then A is weakly normal in B if and only if A is p-closed in B.*

PROOF. The "only if" part is a direct consequence of Theorem 1. Conversely if A is p-closed in B, it is easy to see that A is (2, 3)-closed in B, and hence A is seminormal in B. Since we have $px=0$ for any element x in B, A is weakly normal in B by Theorem 1. q. e. d.

REMARK 1. Let A and B be as above. Then it is well known and easily seen that the weak normalization ${}^*_BA$ of A in B is the largest subring C of B containing A which satisfies the following condition:

(*) *For every prime ideal $\mathfrak{p}$ of A there exists only one prime ideal $\mathfrak{P}$ of C lying over $\mathfrak{p}$, and the natural field extension between the quotient fields of $A/\mathfrak{p}$ and $C/\mathfrak{P}$ is purely inseparable.*

Now, moreover, assume that A contains a field of positive characteristic p. Then ${}^*_BA$ *is equal to the set D of the elements x of B such that there exists a positive integer e satisfying $x^{p^e}\in A$.* In fact D is a p-closed subring of B and hence D is weakly normal in B by the above corollary to Theorem 1. On the other hand it is easy to see that D is contained in ${}^*_BA$ by definition. This means from the above characterization of ${}^*_BA$ that D must be ${}^*_BA$.

§ 2. Faithfully flat descent and local properties.

The purpose of this section is to give some basic properties for weak normality of ring extensions by using Theorem 1.

PROPOSITION 1. *Let the following diagram be a pull-back diagram of commutative rings:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ f\downarrow & & \downarrow g \\ A' & \longrightarrow & B'\,. \end{array}$$

If A' is weakly normal in B', then so is A in B.

PROOF. Let x be an element of B such that x^2 and x^3 are contained in A and put $y=g(x)$. Then we see that y^2 and y^3 belong to A', and hence y is an element of A' by Hamann's criterion (cf. Prop. 1.4 in [7]). This means that x belongs to A by the pull-back property of the given diagram. Therefore A is seminormal in B. Similarly if x is an element of B such that x^p and px are contained in A for some prime integer p, we see that $g(x)^p$ and $pg(x)$ belong to A' and hence that $g(x)$ is contained in A' by Theorem 1. This means that x is contained in A. Therefore A is weakly normal in B by Theorem 1. q.e.d.

COROLLARY. *Let B be a ring and A a subring of B over which B is integral. Let $f: A\rightarrow A'$ be a faithfully flat ring homomorphism. If A' is weakly normal in $B'=A'\otimes_A B$, then so is A in B.*

PROOF. It is enough to show the following diagram

$$\begin{array}{ccl} A & \longrightarrow & B \\ f\downarrow & & \downarrow g=1_B\otimes f \\ A' & \longrightarrow & B'=B\otimes_A A' \end{array}$$

is a pull-back diagram. By Proposition 8 in Chapter 1, § 3 of [2], the following composite h of homomorphisms is injective:

$$h: B/A \longrightarrow (B/A)\otimes_A A' \xrightarrow{\sim} B\otimes_A A'/A\otimes_A A'=B'/A'\,.$$

From this fact we see easily that A is equal to $g^{-1}(A')$ and hence that the above diagram is a pull-back diagram. q.e.d.

PROPOSITION 2. *Let B be a ring and A a subring of B over which B is integral. Let S be a multiplicatively closed subset of A. If A is weakly normal in B, then so is A_S in B_S.*

PROOF. If x/s is an element of B with $x\in B$ and $s\in S$ such that $(x/s)^p$ and $p(x/s)$ are contained in A_S for some prime integer p, then there are elements t and t' in S satisfying $tx^p\in A$ and $pt'x\in A$. Therefore $(tt'x)^p$ and $p(tt'x)$ are contained in A, and hence $tt'x$ belongs to A by Theorem 1. This means that x/s is an element of A_S. Similarly we can see easily that the (2, 3)-closedness of A_S in B_S follows from the (2, 3)-closedness of A in B. Therefore A_S is weakly normal in B_S by Theorem 1 and Hamann's criterion for seminormality quoted as Theorem in the beginning of § 1. q. e. d.

COROLLARY. *Let A, B and S be as above. If C is the weak normalization of A in B, then C_S is that of A_S in B_S.*

PROOF. Let D be the weak normalization of A_S in B_S. Then D and C_S are subrings of B_S containing A_S which satisfy the condition (*) in the remark of § 1. Moreover D is the largest subring among such subrings by the remark. Therefore C_S is a subring of D. On the other hand C_S is weakly normal in B_S by Proposition 2. This means that C_S coincides with D by the same remark.

q. e. d.

THEOREM 2. *Let B be a ring and A a subring of B over which B is integral. Then the following are equivalent:*

(i) *A is weakly normal in B.*

(ii) *$A_{\mathfrak{p}}$ is weakly normal in $B_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ of A.*

(iii) *$A_{\mathfrak{m}}$ is weakly normal in $B_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of A.*

(iv) *$A_{\mathfrak{p}}$ is weakly normal in $B_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ in $\mathrm{Ass}_A(B/A)$.*

PROOF. The assertion (ii) follows from (i) by Proposition 2, and the assertions (iii) and (iv) are trivial consequences of (ii). Now let C be the weak normalization ${}^*_BA$ of A in B. Then $C_{\mathfrak{p}}$ is the weak normalization of $A_{\mathfrak{p}}$ in $B_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ of A by Corollary to Proposition 2. Therefore if the assertion (iii) is true, then $C_{\mathfrak{m}}$ is equal to $A_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of A. This means that A is equal to C, and hence the assertion (i) is true. Similarly assume that the assertion (iv) is true. If C is not equal to A, let $\mathfrak{p}$ be a prime ideal in $\mathrm{Ass}_A(C/A)$. Then $\mathfrak{p}A_{\mathfrak{p}}$ is an element of $\mathrm{Ass}_{A_{\mathfrak{p}}}(C_{\mathfrak{p}}/A_{\mathfrak{p}})$. On the other hand $\mathfrak{p}$ is contained in $\mathrm{Ass}_A(B/A)$ and hence we have $C_{\mathfrak{p}}=A_{\mathfrak{p}}$. This is a contradiction. Therefore A coincides with C. q. e. d.

COROLLARY. *Let A and B be as above. Then A is weakly normal in B, if the following are satisfied:*

(i) *$A_{\mathfrak{p}}$ is seminormal in $B_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ of A such that the characteristic of $A/\mathfrak{p}$ is zero.*

(ii) *$A_{\mathfrak{p}}$ is p-closed in $B_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ of A such that the characteristic of $A/\mathfrak{p}$ is $p>0$.*

PROOF. Let $\mathfrak{p}$ be a prime ideal of A such that the characteristic of $A/\mathfrak{p}$ is zero. Then if $\mathfrak{p}'$ is a prime ideal of A contained in $\mathfrak{p}$, the characteristic of $A/\mathfrak{p}'$ is also zero. Therefore the weak normalization of $A_{\mathfrak{p}}$ in $B_{\mathfrak{p}}$ coincides with the seminormalization of $A_{\mathfrak{p}}$ in $B_{\mathfrak{p}}$ by the definition. This means by our assumption (i) that $A_{\mathfrak{p}}$ is weakly normal in $B_{\mathfrak{p}}$ for such a prime ideal of A. On the other hand let $\mathfrak{p}$ be a prime ideal of A such that the characteristic of $A/\mathfrak{p}$ is $p>0$.

Since $A_{\mathfrak{p}}$ is p-closed in $B_{\mathfrak{p}}$, $A_{\mathfrak{p}}$ is (2, 3)-closed in $B_{\mathfrak{p}}$ and hence seminormal in $B_{\mathfrak{p}}$ by Hamann's criterion (cf. Theorem in § 1). Moreover the condition (ii) in Theorem 1 is clearly satisfied. Therefore $A_{\mathfrak{p}}$ is weakly normal in $B_{\mathfrak{p}}$ by Theorem 1. In conclusion A and B satisfy the assertion (ii) in Theorem 2 and hence A is weakly normal in B. q. e. d.

REMARK 2. The converse of the above corollary to Theorem 2 is not true. For example let B be the polynomial ring $\boldsymbol{Z}[X]$ of a variable X over the ring $\boldsymbol{Z}$ of rational integers and A the subring $\boldsymbol{Z}[X^p]$ of B where p is a prime integer. Denoting the zero ideal of A by $\mathfrak{p}_0$, we see that $A_{\mathfrak{p}_0}$ is equal to the rational function field $\boldsymbol{Q}(X^p)$ of X^p over the field $\boldsymbol{Q}$ of rational numbers. Therefore we have $^*_BA\subset\boldsymbol{Q}(X^p)\cap B=\boldsymbol{Z}[X^p]=A$ by the definition of $^*_BA$. This means that A is weakly normal in B. On the other hand if $\mathfrak{p}$ is the prime ideal of A generated by the prime integer p, then the characteristic of $A/\mathfrak{p}$ is p and we see easily that $A_{\mathfrak{p}}$ is not p-closed in $B_{\mathfrak{p}}$. Therefore the condition (ii) of Corollary to Theorem 2 is not satisfied.

Next we give results on the weak normality of faithfully flat ring extensions. For this purpose we need the following

PROPOSITION 3. *Let B be a ring and A a subring of B over which B is integral. Then the following are equivalent:*

(i) *A is weakly normal in B.*

(ii) *A contains the nilradical* nil(B) *of B and A_{red} is weakly normal in B_{red}.*

(iii) *$A/\mathfrak{b}$ is weakly normal in $B/\mathfrak{b}$ for some ideal $\mathfrak{b}$ of B contained in A.*

(iv) *$A/\mathfrak{b}$ is weakly normal in $B/\mathfrak{b}$ for any ideal $\mathfrak{b}$ of B contained in A.*

PROOF. First assume that A is weakly normal in B. Let $\mathfrak{b}$ be any ideal of B contained in A. If x is an element of B such that the image $\bar{x}$ of x in $B/\mathfrak{b}$ satisfies $\bar{x}^2\in A/\mathfrak{b}$ and $\bar{x}^3\in A/\mathfrak{b}$. Then x^2 and x^3 are contained in $A+\mathfrak{b}=A$ and hence x is an element of A by Theorem 1. Therefore $\bar{x}$ belongs to $A/\mathfrak{b}$. Similarly if $\bar{x}^p$ and $p\bar{x}$ are contained in $A/\mathfrak{b}$ for some prime integer p, we can see easily that $\bar{x}$ belongs to $A/\mathfrak{b}$. Therefore $A/\mathfrak{b}$ is weakly normal in $B/\mathfrak{b}$ by Theorem 1. Conversely assume that $A/\mathfrak{b}$ is weakly normal in $B/\mathfrak{b}$ for some ideal $\mathfrak{b}$ of B contained in A. Then we can see easily that the (2, 3)-closedness of A in B follows from that of $A/\mathfrak{b}$ in $B/\mathfrak{b}$ and that the condition (ii) in Theorem 1 for A and B follows from that for $A/\mathfrak{b}$ and $B/\mathfrak{b}$. This means by Theorem 1 that A is weakly normal in B. Therefore the assertions (i), (iii) and (iv) are equivalent to each other, and (i) follows from (ii). Now assume that A is weakly normal in B, and let x be a nilpotent element of B. Then $x/1$ is contained in $A_{\mathfrak{p}}+R(B_{\mathfrak{p}})$ for any prime ideal $\mathfrak{p}$ of A. Therefore x belongs to $^*_BA=A$. Therefore nil(B) is contained in A and hence (ii) is true by the equivalence between (i) and (iv). q. e. d.

REMARK 3. It is clear that Proposition 3 holds if we replace "weakly normal" with "seminormal".

PROPOSITION 4. *Let B be a ring and A a subring of B over which B is integral and flat. Assume that A has a finite number of minimal prime ideals and contains* nil(B). *Then A is seminormal in B. Moreover A is weakly normal in B if and only if $A_{\mathfrak{p}}$ is weakly normal in $B_{\mathfrak{p}}$ for any minimal prime ideal $\mathfrak{p}$ of A.*

PROOF. By our assumption, the ideal nil(B) is an ideal of A and hence $B_{\text{red}}=B/\text{nil}(B)=B\otimes_A(A/\text{nil}(B))$ is flat and integral over $A_{\text{red}}=A\otimes_A(A/\text{nil}(B))$. Therefore by Proposition 3 and Remark 3 we may assume that B is reduced. Let K and L be the total quotient rings of A and B, respectively. Since B is flat over A, any element of A which is not a zero divisor in A is also not a zero divisor in B. Therefore we may consider K as a subring of L. Since B is flat and integral over A, B is faithfully flat over A by Chapter 1, § 3, Proposition 9 in [2], and hence we have $aB\cap A=aA$ for any element a in A by Proposition 19, ibid. From this fact we see easily that the following commutative diagram of subrings of L is a pull-back diagram:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ K=A_S & \longrightarrow & B_S \end{array}$$

where S is the set of non-zero divisors in A. Since K has only a finite number of minimal prime ideals corresponding to those of A, K is a direct product of a finite number of fields. In particular K has Krull dimension 0. Moreover the localization of K with respect to any prime ideal of K is equal to $A_{\mathfrak{p}}$ for some minimal prime ideal $\mathfrak{p}$ of A. Therefore if $A_{\mathfrak{p}}$ is weakly normal in $B_{\mathfrak{p}}$ for any minimal prime ideal $\mathfrak{p}$ of A, then $K=A_S$ is weakly normal in B_S by Theorem 2. This means that A is weakly normal in B by Proposition 1. The converse is a direct consequence of Theorem 2. On the other hand $A_{\mathfrak{p}}$ is a field for any minimal prime ideal $\mathfrak{p}$ of A, because A is reduced. Therefore $A_{\mathfrak{p}}$ is seminormal in $B_{\mathfrak{p}}$ for such a prime ideal $\mathfrak{p}$ and hence $K=A_S$ is seminormal in B_S by an argument similar to the above and by using Proposition 1.7 in [7] instead of Theorem 2. Therefore A is seminormal in B as is seen in the proof of Proposition 1. q. e. d.

COROLLARY. *Let B be a ring and A a subring of B over which B is flat and of finite presentation as an A-module. Assume that A has a finite number of minimal prime ideals and contains* nil(B). *If A contains the ring $\boldsymbol{Z}$ of rational integers and any non-zero element of $\boldsymbol{Z}$ is not a zero-divisor in A, then A is weakly normal in B.*

PROOF. If $\mathfrak{m}$ is any maximal ideal of A, then $B_{\mathfrak{m}}$ is flat and of finite presentation over $A_{\mathfrak{m}}$. This means that $B_{\mathfrak{m}}$ is free over $A_{\mathfrak{m}}$ by Chapter II, § 3, Corollary 2 to Proposition 5 in [2], because $A_{\mathfrak{m}}$ is a quasi-local ring. Moreover there is a free basis $\{b_1=1, b_2, \cdots, b_n\}$ of $B_{\mathfrak{m}}$ over $A_{\mathfrak{m}}$ by Proposition 5, ibid. If $x=a_1b_1+\cdots+a_nb_n$ is an element of $B_{\mathfrak{m}}$ with $a_i\in A_{\mathfrak{m}}$ for $i=1, \cdots, n$ such that x^p and px are contained in $A_{\mathfrak{m}}$ for some prime number p, then we see $pa_i=0$ for $i\geqq 2$. This means that we have $a_i=0$ for $i\geqq 2$ and have $x=a_1b_1=a_1$ is an element of $A_{\mathfrak{m}}$. On the other hand A is seminormal in B by Proposition 4, and hence $A_{\mathfrak{m}}$ is seminormal in $B_{\mathfrak{m}}$ as is seen in the proof of Proposition 2. Therefore $A_{\mathfrak{m}}$ is weakly normal in $B_{\mathfrak{m}}$ by Theorem 1, and so A is weakly normal in B by Theorem 2. q. e. d.

§ 3. Generalized glueings of rings.

Let B be a ring and A a subring of B such that B is a finite A-module. Let $\mathfrak{p}$ be a prime ideal of A and let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be the prime ideals of B lying over $\mathfrak{p}$. Moreover let $\mathfrak{q}_i$ be a $\mathfrak{p}_i$-primary ideal of B containing $\mathfrak{p}B$ for $i=1, \cdots, n$. Then we see easily $\mathfrak{q}_i\cap A=\mathfrak{p}$ for each i. We denote by $k(\mathfrak{p})$ the quotient field $Q(A/\mathfrak{p})$ of $A/\mathfrak{p}$ and by Q_i the total quotient ring $Q(B/\mathfrak{q}_i)$ of $B/\mathfrak{q}_i$ for each i.

LEMMA 1. *If $S=A/\mathfrak{p}-\{0\}$, then the canonical image in $B/\mathfrak{q}_i$ of any element of S is not a zero-divisor of $B/\mathfrak{q}_i$.*

PROOF. If s is a representative in A of an element $\bar{s}$ of S, then s is not contained in $\mathfrak{p}$ and hence not in $\mathfrak{p}_i$. If b is an element of B such that the class $\bar{b}$ of b in $B/\mathfrak{q}_i$ satisfies $\overline{bs}=0$, then we see that bs is contained in $\mathfrak{q}_i$. This means that b is an element of $\mathfrak{q}_i$ since s is not contained in $\mathfrak{p}_i$. Therefore $\bar{s}$ is not a zero-divisor of $B/\mathfrak{q}_i$. q. e. d.

By Lemma 1 we have a canonical injection j_i of $k(\mathfrak{p})$ into Q_i for each $i=1, \cdots, n$ and hence a ring homomorphism f of $k(\mathfrak{p})$ into $Q=\prod_{i=1}^{n} Q_i$ such that the composition of f and the canonical projection ρ_i of Q to Q_i is j_i for each i. Since f is injective, we may assume that $k(\mathfrak{p})$ is a subfield of Q. Let K be a subfield of Q containing $k(\mathfrak{p})$ and define a ring D by the following pull-back diagram of commutative rings:

$$\begin{array}{ccc} D & \xrightarrow{\ i\ } & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow[\ h\ ]{} & Q \end{array}$$

where g is the composite of canonical homomorphisms $B\rightarrow\prod_{i=1}^{n} B/q_i\rightarrow Q=\prod_{i=1}^{n} Q_i$ and h is the inclusion. Then i is an injection and $i(D)$ is the set of elements b of B satisfying $g(b)\in K$. In particular $i(D)$ contains A, since $A/\mathfrak{p}$ is a subring of K. We call D the *ring obtained from* $(B; \mathfrak{q}_1, \cdots, \mathfrak{q}_n)$ *by glueing over* K or simply the *glueing of* $\mathfrak{q}_1, \cdots, \mathfrak{q}_n$ *over* K. In the following we identify D with $i(D)$.

PROPOSITION 5. *If the notations are as above, then we have the following:*

(i) $\mathfrak{p}'=\bigcap_{s=1}^{n}\mathfrak{q}_s$ *is a prime ideal of* D.

(ii) $\mathfrak{q}_t\cap D=\mathfrak{p}'$ *for any* $t=1, \cdots, n$.

(iii) $\mathfrak{p}'$ *is the unique prime ideal of* D *lying over* $\mathfrak{p}$.

(iv) *The quotient field* $k(\mathfrak{p}')$ *of* $D/\mathfrak{p}'$ *is naturally isomorphic to* K, *i.e., the quotient field of the image* $gi(D)$ *of* D *in* Q *is equal to* $K=h(K)$.

PROOF. Since h is an injection, we see that $\mathfrak{p}'=\bigcap_{s=1}^{n}\mathfrak{q}_s$ is equal to $\ker g=\ker g\cap D=\ker gi=\ker hj=\ker j$. This shows that $\mathfrak{p}'$ is a prime ideal of D, because K is a field. Therefore the assertion (i) is true. Now if b is an element of $\mathfrak{q}_t\cap D$, then we have $\pi_t(b)=0$, where $\pi_t\colon B\to B/\mathfrak{q}_t$ is the canonical homomorphism. Since $gi(b)$ is contained in the subfield $K=h(K)$ of $Q=\prod_{s=1}^{n}Q_s$, we see easily that $gi(b)=0$ if and only if any component of $gi(b)$ in the direct product $\prod_{s=1}^{n}Q_s$ is zero. This means that $\rho_s gi(b)=0$ follows from $\rho_t gi(b)=\pi_t(b)=0$ for each $s=1, \cdots, n$. Therefore $\mathfrak{p}'=\ker g\cap D$ is equal to $D\cap\mathfrak{q}_t$. Next if $\mathfrak{p}''$ is a prime ideal of D lying over $\mathfrak{p}$, then we see $\mathfrak{p}_s\cap D=\mathfrak{p}''$ for some s and hence $\mathfrak{p}''\supset\mathfrak{q}_s\cap D=\mathfrak{p}'$. Since $\mathfrak{p}'$ is clearly lying over $\mathfrak{p}$ and D is integral over A, we see $\mathfrak{p}'=\mathfrak{p}''$. Therefore $\mathfrak{p}'$ is the unique prime ideal of D lying over $\mathfrak{p}$. To show the assertion (iv) we remark that Q is obtained from $B/\bigcap_{s=1}^{n}\mathfrak{q}_s$ by the localization with respect to $S=A/\mathfrak{p}-\{0\}$, where we identify S with a subset of $B/\bigcap_{s=1}^{n}\mathfrak{q}_s$. Therefore if x is an element of K, then there are elements b in B and s in $A-\mathfrak{p}$ such that $x=\bar{b}/\bar{s}$ where $\bar{b}$ and $\bar{s}$ are the classes of b and s in $B/\bigcap_{s=1}^{n}\mathfrak{q}_s$ and $A/\mathfrak{p}$, respectively. Then it is easy to see that b is an element of D by the definition of D and hence that $K=h(K)$ is contained in the quotient field of $gi(D)$. The inverse inclusion is obvious. q.e.d.

COROLLARY. *Let the notations be as above, and let* D' *be a subring of* B *containing* A. *If the image* $g(D')$ *of* D' *in* Q *is contained in* K, *then* D' *is a subring of* D *and* $D'\cap(\bigcap_{s=1}^{n}\mathfrak{q}_s)$ *is the unique prime ideal of* D' *lying over* $\mathfrak{p}$. *In particular* D *is the largest ring among subrings* D' *of* B *satisfying* $g(D')\subset K$.

PROOF. Since $g(D)$ is a subring of K, the kernel $D'\cap(\bigcap_{s=1}^{n}\mathfrak{q}_s)$ of the composite of g and the inclusion $D'\hookrightarrow B$ is a prime ideal of D' and D' is a subring of D by the definition of D. If $\mathfrak{P}$ is a prime ideal of D' lying over $\mathfrak{p}$, then there is a prime ideal $\mathfrak{p}''$ of D lying over $\mathfrak{P}$. This means that $\mathfrak{p}''$ is lying over

$\mathfrak{p}$ and hence that $\mathfrak{p}''$ coincides with $\mathfrak{p}'=\bigcap_{s=1}^{n}\mathfrak{q}_s$. Therefore we see that $D'\cap(\bigcap_{s=1}^{n}\mathfrak{q}_s)$ is the unique prime ideal of D' lying over $\mathfrak{p}$. q. e. d.

Next we shall show a structure theorem for weakly normal ring extensions, which corresponds to Theorem 2.1 in [**11**] for seminormal ring extensions. For this purpose it is necessary to show some lemmas.

LEMMA 2. *Let k be a field of positive characteristic p, and let R be a commutative k-algebra of finite k-rank. Then we have the following:*

(i) *There exists a subalgebra R_0 of R such that R is the direct sum of R_0 and the nilradical* $\mathrm{nil}(R)$ *of R.*

(ii) *If q is the canonical homomorphism of R to $R_0\cong R/\mathrm{nil}(R)$, then there exists the largest subfield K among subfields of R_0 purely inseparable over $q(k)$, and the weak normalization of k in R coincides with $K+\mathrm{nil}(R)$.*

PROOF. Since R is an artinian semilocal ring, R is a direct product $\prod_{i=1}^{n} A_i$ of artinian local rings A_i. By Cohen's theorem A_i has a coefficient field L_i for $i=1, \cdots, n$, because A_i is an equicharacteristic complete local ring. Moreover we see that A_i is the direct sum of L_i and $\mathrm{nil}(A_i)$ for each i. Therefore if we denote by R_0 the direct product $\prod_{i=1}^{n} L_i$, then we see $R=R_0+\mathrm{nil}(R)$. If p_i is the canonical projection of R_0 to L_i for each $i=1, \cdots, n$, and if F_1 and F_2 are subfields of R_0 purely inseparable over $q(k)$, the F_1 and F_2 are isomorphic to $p_i(F_1)$ and $p_i(F_2)$, respectively, for each i and hence there are isomorphisms $g_{ij}: p_i(F_1)\to p_j(F_1)$ and $h_{ij}: p_i(F_2)\to p_j(F_2)$ for each i and j. Since $p_i(R_0)=L_i$ is a field algebraic over $p_iq(k)$, the composite ring M_i of $p_i(F_1)$ and $p_i(F_2)$ is a subfield of L_i for each i. Moreover g_{ij} and h_{ij} give an isomorphism of M_i and M_j for any i and j, because $p_i(F_1)$ and $p_i(F_2)$ are purely inseparable over $p_iq(k)$ for each i (cf. Chapter 6, § 10, Example (4) in [**3**]). Therefore the composite ring M of F_1 and F_2 in R_0 is also a field isomorphic to M_i for each $i=1, \cdots, n$. This means that there exists the largest subfield K among subfields of R_0 purely inseparable over $q(k)$. To show that $K+\mathrm{nil}(R)$ is the weak normalization of k in R, it is enough by Proposition 3 and Remark 1 to see that K is equal to the weak normalization of $q(k)$ in $R_0=R_{\mathrm{red}}$. Since K is the largest purely inseparable extension of $q(k)$ in R_0, the characterization of the weak normalization given in Remark 1 shows that our assertion is true. q. e. d.

Let A, B, $\mathfrak{p}$ and $\mathfrak{p}_i$ $(i=1, \cdots, n)$ be as in the beginning of this section. Moreover let Q_i be the quotient field of $B/\mathfrak{p}_i$ and let Q be the direct product $\prod_{i=1}^{n} Q_i$, which contains the quotient field $k(\mathfrak{p})$ of $A/\mathfrak{p}$ as before. Then there exists the largest purely inseparable field extension K of $k(\mathfrak{p})$ in Q by Lemma 2. If D is the ring obtained from $(B;\mathfrak{p}_1, \cdots, \mathfrak{p}_n)$ by glueing over K, we call D the *weak* *glueing* of B *over* $\mathfrak{p}$. It is easy to see from the definition of weak normalization and Corollary to Proposition 5 that D is weakly normal in B.

LEMMA 3. *Let A and B be as above, and let $\mathfrak{p}$ be a prime divisor of the conductor $\mathfrak{c}=A:_A B$ of A in B. Assume that A is weakly normal in B. If there is only one prime ideal $\mathfrak{p}'$ of B lying over $\mathfrak{p}$, then the quotient field of $B/\mathfrak{p}'$ is not purely inseparable over $k(\mathfrak{p})$.*

PROOF. If S is the multiplicatively closed subset $A-\mathfrak{p}$ of A, then A_S is weakly normal in B_S by Proposition 2. Since A is seminormal in B by Theorem 1 and hence $\mathfrak{c}$ is equal to its radical in B by Lemma 1.3 in [**12**], $\mathfrak{p}_S$ is equal to $\mathfrak{c}_S$, which is the conductor of A_S in B_S and coincides with its radical in B_S. This means that B_S is not equal to A_S and that B_S is a local ring with maximal ideal $\mathfrak{p}_S=\mathfrak{c}_S$. Therefore we see $B_S=B_{\mathfrak{p}'}$ and $\mathfrak{c}_S=\mathfrak{p}'_{\mathfrak{p}'}$. On the other hand if $\mathfrak{q}$ is the prime ideal of A_S different from $\mathfrak{p}_S$, then we have $(B_S)_{\mathfrak{q}}=(A_S)_{\mathfrak{q}}$ because $\mathfrak{q}\neq\mathfrak{p}_S=\mathfrak{c}_S$. Therefore if the quotient field of $B/\mathfrak{p}'$ is purely inseparable over $k(\mathfrak{p})$, B_S is the weak normalization of A_S in B_S by the characterization of the weak normalization given in Remark 1. But B_S is not equal to A_S and A_S is weakly normal in B_S as seen in the above. This is a contradiction. q.e.d.

THEOREM 3. *Let A be a noetherian ring and let B be a ring containing A which is a finite A-module. If A is weakly normal in B, then there is a sequence*

$$B=B_0\supset B_1\supset\cdots\supset B_{n-1}\supset B_n=A$$

of subrings of B such that B_{i+1} is the weak glueing of B_i over a prime ideal of A.

PROOF. Suppose that B_i has already been obtained. If $B_i=A$, all is done. Otherwise let $\mathfrak{c}_i$ be the conductor of A in B_i. Since A is weakly normal in B_i, $\mathfrak{c}_i$ is equal to its radical in B_i as is seen in the proof of Lemma 3. Let $\mathfrak{p}$ be a prime divisor of $\mathfrak{c}_i$ in A of the smallest height and define B_{i+1} as the weak glueing of B_i over $\mathfrak{p}$. If $\mathfrak{c}_{i+1}$ is the conductor of A in B_{i+1}, we see $\mathfrak{c}_{i+1}\supset\mathfrak{c}_i$. Assume that $\mathfrak{p}$ is a prime divisor of $\mathfrak{c}_{i+1}$. Then since A is weakly normal in B_{i+1}, there are at least two prime ideals of B_{i+1} lying over $\mathfrak{p}$ or there is a unique prime ideal $\mathfrak{p}'$ of B_{i+1} lying over $\mathfrak{p}$ such that the quotient field of $B_{i+1}/\mathfrak{p}'$ is not purely inseparable over $k(\mathfrak{p})$ by Lemma 3. But this contradicts the fact that B_{i+1} is the weak glueing of B_i over $\mathfrak{p}$. Therefore $\mathfrak{p}$ is not a prime divisor of $\mathfrak{c}_{i+1}$ and hence we have $\mathfrak{c}_{i+1}\neq\mathfrak{c}_i$. By noetherian hypothesis we cannot have an infinitely increasing chain of $\mathfrak{c}_i$, and hence there is n such that $B_n=A$. q.e.d.

Lastly we terminate this section by giving a result on the going-down of the (S_2)-*property of Serre* from a ring to a glueing of it. For this purpose we need the following

LEMMA 4. *Let A be a noetherian ring whose prime ideals of height $\geqq 1$ contain regular elements. Let $A^{(1)}$ be the set of elements z in the total quotient* *ring $Q(A)$ of A such that any prime ideal of A containing $A:_A z$ has height $\geqq 2$. Then A has* (S_2) *if and only if $A=A^{(1)}$.*

For the proof and the definition of (S_2), see § 3 in [**13**].

THEOREM 4. *Let A, B, $\mathfrak{p}$, $\mathfrak{p}_i$, $\mathfrak{q}_i$, Q_i and $Q=\prod_{i=1}^{n} Q_i$ be as in the beginning of this section, and let K be a subfield of Q containing $k(\mathfrak{p})$. Assume that A is noetherian and let D be the ring obtained from $(B; \mathfrak{q}_1, \cdots, \mathfrak{q}_n)$ by glueing over K. Then we have the following:*

(i) *If any $\mathfrak{p}_i$ has height* 1 *for $i=1, \cdots, n$ and if B has* (S_2), *then D has also* (S_2).

(ii) *Assume that D is not equal to B. If any $\mathfrak{p}_i$ contains a regular element of B for each i, then $D_{\mathfrak{p}'}$ has depth* 1, *where $\mathfrak{p}'$ is the unique prime ideal $\bigcap_{i=1}^{n} \mathfrak{q}_i$ of D lying over $\mathfrak{p}$. Furthermore if some $\mathfrak{p}_i$ has height >1, then D does not have* (S_2).

PROOF. We give only an outline of our proof, because the idea is very similar to the proof of Theorem 2 in [**13**].

(i) First the ideal $\mathfrak{p}'=\bigcap_{i=1}^{n} \mathfrak{q}_i$ is the unique prime ideal of D lying over $\mathfrak{p}$ and we can see easily that $\mathrm{ht}(\mathfrak{p}')=1$ from the hypothesis $\mathrm{ht}(\mathfrak{p}_i)=1$ for any $i=1, \cdots, n$. Next if $\mathfrak{Q}$ is a prime ideal of B and if $\mathfrak{q}=\mathfrak{Q}\cap D$, then we can see in the same way as in the proof of Theorem 2, ibid. that $\mathrm{ht}(\mathfrak{Q})\geqq 2$ if and only if $\mathrm{ht}(\mathfrak{q})\geqq 2$. Since B has no embedded prime divisor of zero and $\mathfrak{p}'$ is contained in the conductor of D in B, we see easily that the total quotient ring of D may be considered to coincide with that of B. Denote by $D^{(1)}$ and $B^{(1)}$ the rings for D and B, respectively, like $A^{(1)}$ for A given in Lemma 4. Then we can see that any prime ideal $\mathfrak{P}$ of B containing $B:_B z$ has height $\geqq 2$ for any element z in $D^{(1)}$, because we have $B:_B z \supset (D:_D z)B$ and $\mathrm{ht}(\mathfrak{P}\cap D)\geqq 2$. Therefore we see $D^{(1)}\subset B^{(1)}=B$ by Lemma 4 and the assumption that B has (S_2). If x is an element of $D^{(1)}$, then x is an element in B and $\mathfrak{p}'$ does not contain $D:_D x$. So there is an element s of $D:_D x$ not belonging to $\mathfrak{p}'$. Let a be the element xs in D. Then we see that the image of x in the total quotient ring Q of $B/\bigcap_{i=1}^{n} \mathfrak{q}_i$ belongs to the quotient field K of $D/\mathfrak{p}'$. Therefore x must be an element of D by the definition of D. This means that D coincides with $D^{(1)}$ and hence we see easily from Lemma 4 that D has (S_2).

(ii) Let S be the multiplicatively closed subset $D-\mathfrak{p}'$ of D. Then we see that D_S is the ring obtained from $(B_S; (\mathfrak{q}_1)_S, \cdots, (\mathfrak{q}_n)_S)$ by glueing over K. In fact D is obtained as the pull-back with respect to ring homomorphisms $B\to Q$ and $K\to Q$. Therefore if we tensor this pull-back diagram with the flat ring extension D_S over D, we have also a pull-back diagram by Lemma 4.2 in [**4**].

This means that our assertion is true. Therefore we may assume that B is a semilocal ring with maximal ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ and that D is a local ring with maximal ideal $\mathfrak{p}'$. Moreover we see, in the same way as in the proof of Theorem 2 in [**13**], that the total quotient ring $Q(A)$ of A is equal to the total quotient ring $Q(B)$ of B and that if $z=a/b$ in $Q(A)=Q(B)$ with an element a of D and a regular element b of D is an element of B not contained in D, then $\mathfrak{p}'$ is contained in $\mathrm{Ass}_D(D/bD)$. Therefore we have $\mathrm{depth}\, D=1$. Consequently if some of $\mathfrak{p}_i$ has height >1, then $\mathfrak{p}'$ has also height >1 and hence D does not have (S_2). q.e.d.

COROLLARY. *Let B be a noetherian ring, and let A be a subring of B which is weakly normal in B. Assume that B is a finite A-module. If a prime divisor in A of the conductor $A:_A B$ of A in B has height >1 and if $A:_A B$ contains a regular element of B, then A does not have* (S_2).

The proof is exactly the same as that of Corollary to Theorem 2 in [**13**], if we use our Theorem 3 instead of Theorem 2.1 in [**12**]. Therefore we omit the detail.

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Hiroshi YANAGIHARA
Department of Mathematics
Hyogo University of Teacher Education
Shimokume, Yashiro-cho, Kato-gun,
Hyogo 673-14, Japan