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On simple groups which are homomorphic images of multiplicative subgroups of simple algebras of degree 2

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Let $M_2(D)$ be the full matrix algebra of degree 2 over a division algebra D of characteristic 0. In [**11**] we proved that if G is a finite multiplicative subgroup of $M_2(D)$ with abelian Sylow 2-subgroups, then G is a solvable group. More generally, in this paper we will determine the non-abelian simple groups S which are homomorphic images of multiplicative subgroups G of $M_2(D)$. In [**10**] we remarked that abelian subgroups of the Sylow 2-subgroups of G are generated by at most 2 elements. In particular, the Sylow 2-subgroups possess no abelian normal subgroups of rank 3, which implies that these 2-groups are generated by at most 4 elements (see MacWilliams [**14**]). All simple groups whose Sylow 2-subgroups are generated by at most 4 elements have been determined in Gorenstein-Harada [**7**]. Using their theorem, we will determine the simple groups S.

Our main result is as follows.

THEOREM. *Let S be a simple group. If there exists a division algebra D of characteristic* 0, *a finite multiplicative subgroup G of $M_2(D)$ and a normal subgroup N of G satisfying $G/N\cong S$, then S is isomorphic to $PSL(2, 5)$ or $PSL(2, 9)$ and $N\neq 1$.*

In the theorem $N\neq 1$ means the following:

COROLLARY. *Let G be a finite group and let K be a field of characteristic* 0. *If one of the simple components of the group ring KG is the full matrix algebra of degree* 2 *over a division algebra, then G is not simple.*

The corollary can not be generalized to the full matrix algebra of degree $\geqq 3$. In fact,

$$\boldsymbol{Q}[PSL(2, 5)]\cong \boldsymbol{Q}\oplus M_3(\boldsymbol{Q}(\sqrt{5}))\oplus M_4(\boldsymbol{Q})\oplus M_5(\boldsymbol{Q})$$

and

$$\boldsymbol{Q}[A_n]\cong \boldsymbol{Q}\oplus M_{n-1}(\boldsymbol{Q})\oplus \cdots, \quad n\geqq 5.$$

1. Preliminaries.

All division algebras considered in this paper are of characteristic 0. As usual $\boldsymbol{Q}$ and $\boldsymbol{C}$ denote respectively the rational number field and the complex number field. By a subgroup of $M_2(D)$ we mean a finite multiplicative subgroup of $M_2(D)$. Let D be a division algebra and let K be a field contained in the center of D. Let G be a subgroup of $M_2(D)$. We define $V_K(G)=\{\sum\alpha_i g_i \mid \alpha_i\in K, g_i\in G\}$ as a K-subalgebra of $M_2(D)$. Then there is a natural epimorphism $KG\rightarrow V_K(G)$. Hence $V_K(G)$ is a semi-simple K-subalgebra of $M_2(D)$. Let $V_K(G)\cong M_{n_1}(D_1)\oplus\cdots\oplus M_{n_t}(D_t)$ be the decomposition of $V_K(G)$ into simple algebras $M_{n_i}(D_i)$. Since $V_K(G)\subseteqq M_2(D)$, there exist at most 2 orthogonal idempotents in $V_K(G)$. Thus we have $\sum_{i=1}^{t} n_i\leqq 2$. This means that $V_K(G)\cong D_1$, $D_1\oplus D_2$ or $M_2(D_1)$.

(1.1) ([**11**]). *Let D be a division algebra and let K be a subfield of the center of D. Let G be a subgroup of $M_2(D)$. Then we have $V_K(G)\cong D_1$, $D_1\oplus D_2$ or $M_2(D_1)$ where D_1, D_2 are some division algebras.*

Now we recall the following results on p-groups.

(1.2) ([**10**], [**11**]). *Let p be a prime number. Let P be a p-group which is a subgroup of $M_2(D)$.*

(1) *If P is abelian, then P is generated by at most 2 elements.*

(2) *If $p\neq 2$, then P is abelian.*

(3) *If $p=2$, then $P/[P, P]$ is generated by at most 4 elements.*

Amitsur proved the following result.

(1.3) ([**2**]). *Let G be a finite multiplicative subgroup of a division algebra and let N be a normal subgroup of G. If G/N is simple, then $G/N\cong PSL(2, 5)$.*

We recall the following result.

(1.4) ([**11**]). *Let D be a division algebra and let K be a subfield of the center of D. Let G be a subgroup of $M_2(D)$ satisfying $V_K(G)=M_2(D)$ and let N be a normal subgroup of G. If $|N|$ is odd, then one of the following conditions is satisfied:*

(1) *G has a subgroup of index 2.*

(2) *$V_K(G)$ is a division algebra.*

Let S be a non-abelian simple group. We define

$m(S)=\{(D, G, N) \mid D$ is a division algebra of characteristic 0,
G is a finite multiplicative subgroup of $M_2(D)$
and N is a normal subgroup of G such that $G/N\cong S\}$.

We assume $m(S)\neq\emptyset$. Let (D, G, N) be an element of $m(S)$. By (1.2) the 2-rank of G (the maximal rank of an abelian 2-subgroup) is $\leqq 2$. By MacWilliams [**14**] the Sylow 2-subgroups of G are generated by at most 4 elements. Hence S is one of the simple groups which were listed in Gorenstein-Harada [**7**].

2. Basic lemma.

Assume $S\not\cong PSL(2, 5)$ and $m(S)\neq\emptyset$. Let (D_0, G, N) be an element of $m(S)$ satisfying $|G|\leqq|G'|$ for any element $(D', G', N')\in m(S)$. Since $\boldsymbol{Q}\subseteqq$the center of D_0, $V_Q(G)\cong D_1$, $D_1\oplus D_2$ or $M_2(D_1)$ for some division algebras D_1, D_2. By (1.3) if $V_Q(G)\cong D_1$ or $D_1\oplus D_2$, then $S\cong PSL(2, 5)$. Therefore $V_Q(G)\cong M_2(D_1)$. We put $D=D_1$. Then (D, G, N) is an element of $m(S)$ such that $M_2(D)=V_Q(G)$ and $|G|\leqq|G'|$ for any element $(D', G', N')\in m(S)$. In this section we will prove the following basic lemma.

LEMMA 2.1. *Assume* $S\not\cong PSL(2, 5)$ *and* $m(S)\neq\emptyset$.

(1) *There exists an element* (D, G, N) *in* $m(S)$ *such that* $V_Q(G)=M_2(D)$ *and* $|G|\leqq|G'|$ *for any element* $(D', G', N')\in m(S)$.

For (D, G, N) *in* (1) *we have*

(2) $[G, G]=G$.

(3) N *is a 2-group.*

(4) *If* $S\not\cong PSL(2, 7)$, $PSL(2, 9)$, A_7 *nor* A_8, *then* N *is cyclic and* $N=Z(G)$.

To show the lemma we will use the following lemma.

LEMMA 2.2. *Let S be a simple group. If S is a homomorphic image of a subgroup of* $GL(4, 2)$, *then S is isomorphic to one of the following groups:*

$PSL(2, 5)$, $PSL(2, 7)$, $PSL(2, 9)$, A_7 *or* A_8.

PROOF. This may be well known. Here we give a proof. Since $|S|\,|\,|GL(4, 2)|=2^6\cdot3^2\cdot5\cdot7$, by [**3**] we have that $S\cong PSL(2, 5)$, $PSL(2, 7)$, $PSL(2, 8)$, $PSL(2, 9)$, A_7, A_8 or $PSL(3, 4)$. But $S\not\cong PSL(3, 4)$, because $|PSL(3, 4)|=|GL(4, 2)|$ and $PSL(3, 4)\not\cong GL(4, 2)$. Let $\mathcal{L}=\{(G, N)\,|\,GL(4, 2)\supseteqq G\triangleright N$ and $G/N\cong PSL(2, 8)\}$. We show that $\mathcal{L}=\emptyset$. Suppose that $\mathcal{L}\neq\emptyset$. Let (G, N) be an element of $\mathcal{L}$ satisfying $|G|\leqq|G'|$ for any $(G', N')\in\mathcal{L}$. Since $G\subseteqq GL(4, 2)\cong A_8$, we may regard G as a permutation group on $\mathcal{X}=\{1, 2, \cdots, 8\}$. We decompose $\mathcal{X}$ into the orbits of G: $\mathcal{X}=\mathcal{X}_1\cup\mathcal{X}_2\cup\cdots\cup\mathcal{X}_n$. And we may assume $|\mathcal{X}_1|\neq1$. Let a be an element of $\mathcal{X}_1$. We put $G_a=\{g\in G\,|\,g(a)=a\}$. Then G_a is a proper subgroup of G and $1<|G:G_a|=|\mathcal{X}_1|\leqq8$. If $G/N=G_a/G_a\cap N$, then $(G_a, G_a\cap N)\in\mathcal{L}$. But it is impossible. Therefore $8\geqq|G:G_a|\geqq|G/N:G_a/G_a\cap N|>1$. This shows that $PSL(2, 8)$ has a proper subgroup of index $\leqq8$. But the minimal index of a proper subgroup of $PSL(2, 8)$ is 9 (see [**12**] (8.28)). Thus we conclude that $S\not\cong PSL(2, 8)$.

PROOF OF LEMMA 2.1. Step 1. We show first that $G=[G, G]$. Since $G/[G, G]$ is an abelian group, $(D, [G, G], [G, G]\cap N)\in m(S)$. The assumption on (D, G, N) implies $G=[G, G]$.

Step 2. Let P be a Sylow p-subgroup of N for a prime p. We show that P is a normal subgroup of G. Since $N_G(P)N=G$, we have $N_G(P)/N_G(P)\cap N\cong N_G(P)N/N=G/N\cong S$. Then $(D, N_G(P), N_G(P)\cap N)\in m(S)$, which implies $G=N_G(P)$. Thus $P\triangleleft G$.

Step 3. Next we show that N is a 2-group. Let p be an odd prime and let P be a Sylow p-subgroup of N. If G has a subgroup H of index 2, then $H/H\cap N\cong S$. This means $(D, H, H\cap N)\in m(S)$, which contradicts the assumption on (D, G, N).

Then it follows from (1.4) that $V_Q(P)$ is a division algebra. Since a p-subgroup of a division algebra is cyclic, P is cyclic. On the other hand $C_G(P)\lhd G$, because $P\lhd G$. Since $G/C_G(P)$ is isomorphic to a subgroup of the automorphism group of the cyclic group P, we have $(D, C_G(P), C_G(P)\cap N)\in m(S)$. Hence $G=C_G(P)$, which implies $P\subseteqq Z(G)$. Now let Q be a Sylow p-subgroup of G. Put $R=Q\cap Z(N_G(Q))$. By (1.2) Q is abelian, and by ([**5**], (20.12)) there exists a normal subgroup G_0 of G such that $G/G_0\cong R$. Since $G/G_0\not\cong S$, we have $G=G_0$, and $R=1$. Hence $P=1$, because $R\supseteqq P$.

Step 4. Finally we show that if $S\not\cong PSL(2, 5)$, $PSL(2, 7)$, $PSL(2, 9)$, A_7 nor A_8, then N is a cyclic 2-group and $N=Z(G)$. If $N=Z(G)$, then $N\subseteqq$the center of $V_Q(G)=$the center of $M_2(D)=$the center of D. Since any finite multiplicative subgroup of a field is cyclic, N is cyclic. Hence it suffices to show that $N\subseteqq Z(G)$ (the converse $N\supseteqq Z(G)$ can be easily checked). Let us consider a chain of subgroups of N, $N=N_s\supseteqq N_{s-1}\supseteqq \cdots \supseteqq N_1\supseteqq N_0=1$ such that $N_i\lhd G$ and N_i/N_{i-1} is an elementary abelian 2-group for any i, $1\leqq i\leqq s$. By the induction on i we will prove that $N_i\subseteqq Z(G)$. We assume that $N_{i-1}\subseteqq Z(G)$. By (1.2) N_i/N_{i-1} is generated by at most 4 elements. We can regard $\mathrm{Aut}(N_i/N_{i-1})$ as a subgroup of $GL(4, 2)$. By (2.2) and by our assumption on S it is easy to see that $S\cong C_G(N_i/N_{i-1})/C_G(N_i/N_{i-1})\cap N$. Then we get $G=C_G(N_i/N_{i-1})$. We now put $|N_{i-1}|=2^t$. Let $g\in G$ and $x\in N_i$. Since $G=C_G(N_i/N_{i-1})$, $x^{-1}g^{-1}xg\in N_{i-1}$. We set $y=x^{-1}g^{-1}xg$. Then $g^{-2^t}xg^{2^t}=xy^{2^t}=x$ because $y\in N_{i-1}\subseteqq Z(G)$ and $|N_{i-1}|=2^t$. Thus we have $g^{2^t}\in C_G(N_i)$ for any $g\in G$. This shows that $G/C_G(N_i)$ is a 2-group. Hence $S\cong C_G(N_i)/C_G(N_i)\cap N$ and $(D, C_G(N_i), C_G(N_i)\cap N)\in m(S)$. By the assumption on G we conclude that $G=C_G(N_i)$, i.e. $N_i\subseteqq Z(G)$. The proof of the lemma is completed.

3. Quasisimple group of 2-rank $\leqq 2$.

Let S be a simple group. In this section we assume that $m(S)\neq\emptyset$ and $S\not\cong PSL(2, 5)$, $PSL(2, 7)$, $PSL(2, 9)$, A_7 nor A_8. By (2.1) there exists an element (D, G, N) in $m(S)$ such that $G=[G, G]$, $N=Z(G)$ and N is a cyclic 2-group. Therefore $O(G)$ (the largest normal subgroup of G of odd order) $=1$ and G is a quasisimple group (i.e. $G=[G, G]$ and $G/Z(G)$ is simple) of 2-rank $\leqq 2$ (cf. (1.2)). These groups G have been studied by Alperin, Brauer, Gorenstein and Harada.

We recall their theorems.

(3.1) (Alperin-Brauer-Gorenstein [**1**]). *If S is a finite simple group of 2-rank 2, then one of the following holds:*

(1) *S has dihedral Sylow 2-subgroups, and $S\cong PSL(2, q)$, q odd, or A_7;*

(2) *S has quasi-dihedral Sylow 2-subgroups, and $S\cong PSL(3, q)$, $q\equiv -1 \pmod 4$, $PSU(3, q^2)$, $q\equiv 1 \pmod 4$, or M_{11};*

(3) *S has wreathed Sylow 2-subgroups, and $S\cong PSL(3, q)$, $q\equiv 1 \pmod 4$ or $PSU(3, q^2)$, $q\equiv -1 \pmod 4$; or*

(4) $S\cong PSU(3, 4^2)$.

(3.2) (Gorenstein-Harada [7]). *If G is a quasisimple group of 2-rank 2 with $O(G)=1$, then either G is simple or G is isomorphic to $Sp(4, q)$, q odd.*

In the case where 2-rank of G is 1, it is known that a Sylow 2-subgroup P of G is cyclic or generalized quaternion. Since $S\cong G/N$ is simple, P/N is dihedral. Then by (3.1) $S\cong PSL(2, q)$, q odd. In the case where 2-rank of G is 2, by (3.2), and by (3.1), $G\cong PSL(2, q)$, $PSL(3, q)$, $PSU(3, q^2)$, q odd, M_{11}, $PSU(3, 4^2)$ or $Sp(4, q)$, q odd. If q is a power of an odd prime p, the Sylow p-subgroups of $PSL(3, q)$, $PSU(3, q^2)$ and $Sp(4, q)$ are not abelian. Therefore by (1.2) $G\not\cong PSL(3, q)$, $PSU(3, q^2)$ nor $Sp(4, q)$. Hence we have

PROPOSITION 3.3. *Let S be a simple group. Assume that $m(S)\neq\emptyset$. Then we have*

(1) *$S\cong PSL(2, q)$, q odd, $PSU(3, 4^2)$, A_7, A_8 or M_{11}.*

(2) *If $S\cong PSU(3, 4^2)$ or M_{11}, then there exists a division algebra D such that $(D, S, 1)\in m(S)$ and $V_{\boldsymbol{Q}}(S)=M_2(D)$.*

4. Proof of theorem.

Let χ be an irreducible character of a finite group G. By $m(\chi)$ we denote the Schur index of χ over $\boldsymbol{Q}$.

LEMMA 4.1. *Let G be a finite group. Then the following conditions are equivalent:*

(1) *There exist a division algebra D and a normal subgroup N of G such that $G/N\cong M_2(D)$ and $V_{\boldsymbol{Q}}(G/N)=M_2(D)$.*

(2) *There exists an irreducible character χ of G satisfying $\chi(1)=2\,m(\chi)$.*

PROOF. Let $M_n(D)$ be a simple component of $\boldsymbol{Q}G$ and let χ be an irreducible character of G corresponding to $M_n(D)$. Then $\chi(1)=n\,m(\chi)$. From this relation we can easily see that the conditions (1) and (2) are equivalent.

The character table of $SL(2, q)$, q odd, is well known (see [4], § 38), and the Schur indices of $SL(2, q)$ have been determined in Janusz [13].

We use the same notation as in Dornhoff [4], § 38.

(4.2) ([13]). *The degrees and the Schur indices of the irreducible character of $SL(2, q)$, q odd, are as follows;*

(1)	$1(1)=1$,	$m(1)=1$,
(2)	$\psi(1)=q$,	$m(\psi)=1$,
(3)	$\chi_i(1)=q+1$,	$m(\chi_i)=1$ *if i is even,*
		$m(\chi_i)=2$ *if i is odd,*
(4)	$\theta_j(1)=q-1$,	$m(\theta_j)=1$ *if j is even,*
		$m(\theta_j)=2$ *if j is odd,*
(5)	$\xi_k(1)=(q+1)/2$,	$m(\xi_k)=1$,

(6) $\eta_k(1)=(q-1)/2$, $m(\eta_k)=1$ *if* $q\equiv -1 \pmod 4$,
$m(\eta_k)=2$ *if* $q\equiv 1 \pmod 4$,

where $1\leqq i\leqq (q-3)/2$, $1\leqq j\leqq (q-1)/2$, $1\leqq k\leqq 2$.

By (4.2) we can easily find all irreducible characters of $SL(2, q)$ satisfying $\chi(1)=2m(\chi)$.

COROLLARY 4.3. *Let* χ *be an irreducible character of* $SL(2, q)$, q *odd, satisfying* $\chi(1)=2m(\chi)$. *Then* χ *is one of the following;*

(1) $\chi=\xi_k$ *and* $q=3$, $1\leqq k\leqq 2$,

(2) $\chi=\theta_1$ *and* $q=5$,

(3) $\chi=\eta_k$ *and* $q=9$, $1\leqq k\leqq 2$.

PROPOSITION 4.4. *If* $m(PSL(2, q))\neq\emptyset$, q *odd, then* $q=5$, 7 *or* 9.

PROOF. We assume $m(PSL(2, q))\neq\emptyset$ and $q\neq 5$, 7 nor 9. Let (D, G, N) be an element of $m(PSL(2, q))$. By (2.1) we may assume that $V_{\boldsymbol{Q}}(G)=M_2(D)$ and G is a central extension of $PSL(2, q)$ with $G=[G, G]$. It is well known that there exists an epimorphism from $SL(2, q)$ onto G. (See [**12**] (25.7).) Therefore $V_{\boldsymbol{Q}}(G)=M_2(D)$ is a simple component of $\boldsymbol{Q}[SL(2, q)]$. By (4.1) and (4.3) $q=5$ or 9 (cf. $PSL(2, 3)$ is not simple), which is a contradiction.

LEMMA 4.5. *Let* H *be a non-abelian group of order* 21. *Let* ε_n *be a primitive n-th root of unity. Then*

$$\boldsymbol{Q}H\cong\boldsymbol{Q}\oplus\boldsymbol{Q}(\varepsilon_3)\oplus M_3(\boldsymbol{Q}(\varepsilon_7+\varepsilon_7^2+\varepsilon_7^4)).$$

In particular H *is not a subgroup of* $M_2(D)$ *for any division algebra* D.

PROOF. We put $H=\langle a, b\,|\,a^7=1, b^3=1, bab^{-1}=a^2\rangle$. Let σ be the automorphism of $\boldsymbol{Q}(\varepsilon_7)$ over $\boldsymbol{Q}$ defined by $\sigma(\varepsilon_7)=\varepsilon_7^2$. Since there exists an epimorphism from $\boldsymbol{Q}H$ to the cyclic algebra $(\boldsymbol{Q}(\varepsilon_7), \sigma, 1)$ determined by the mapping $a\to\varepsilon_7$ and $b\to\sigma$, we have

$$\begin{aligned}\boldsymbol{Q}H&\cong\boldsymbol{Q}\oplus\boldsymbol{Q}(\varepsilon_3)\oplus(\boldsymbol{Q}(\varepsilon_7), \sigma, 1)\\ &\cong\boldsymbol{Q}\oplus\boldsymbol{Q}(\varepsilon_3)\oplus M_3(\boldsymbol{Q}(\varepsilon_7+\varepsilon_7^2+\varepsilon_7^4)).\end{aligned}$$

Now we prove the theorem.

THEOREM. *Let* S *be a simple group. Then*

(1) $m(S)\neq\emptyset$ *if and only if* $S\cong PSL(2, 5)$ *or* $PSL(2, 9)$.

(2) *If* $(D, G, N)\in m(S)$, *then* $N\neq 1$.

PROOF. We assume that $m(S)\neq\emptyset$. It follows from (3.3) and (4.4) that $S\cong PSL(2, 5)$, $PSL(2, 7)$, $PSL(2, 9)$, $PSU(3, 4^2)$, A_7, A_8 or M_{11}. First we suppose that $S\cong PSL(2, 7)$, A_7 or A_8. Let $(D, G, N)\in m(S)$. By (2.1) we may assume that N is a 2-group. It is easily checked that S contains a non-abelian group of order 21. Thus G contains a non-abelian group of order 21, which contradicts (4.5). Therefore $m(PSL(2, 7))=m(A_7)=m(A_8)=\emptyset$. Since $PSL(2, 11)$ is isomorphic to a subgroup of M_{11} (see [**6**]) and $m(PSL(2, 11))=\emptyset$ by (4.4), we obtain $m(M_{11})=\emptyset$.

Finally we assume that $m(PSU(3, 4^2))\neq\emptyset$. By (3.3) we can find a division algebra D such that $(D, PSU(3, 4^2), 1)\in m(PSU(3, 4^2))$ and $V_Q(PSU(3, 4^2))=M_2(D)$. Let χ be an irreducible character of $PSU(3, 4^2)$ corresponding to $M_2(D)$. Then, as shown by Gow [8], $m(\chi)=1$ except only one character χ of degree 12 with $m(\chi)=2$. By (4.1) we have $m(\chi)=1$, and D is an algebraic number field. Hence $PSU(3, 4^2)$ is a subgroup of $GL(2, \boldsymbol{C})$, but it is impossible (see [4] (26.1)). Therefore $m(PSU(3, 4^2))=\emptyset$. Thus we find that if $m(S)\neq\emptyset$, then $S\cong PSL(2, 5)$ or $PSL(2, 9)$.

The assertion (2) and the converse of (1) follow directly from (4.3).

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