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On complementary triples in finite groups

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§1. Introduction.

All groups considered here are finite. Let H be a subgroup of a group G. We say that H *controls fusion* in H with respect to G if H has the property; 'Two elements of H are conjugate in G if and only if they are conjugate in H.' If H has a normal complement (that is, a normal subgroup N of G with $G=HN$ and $H\cap N=1$) in G, then H controls fusion in H with respect to G. But the converse is false. For example, let S_n be the symmetric group on n letters, where n is greater than 4, and let H be the stabilizer of one point. Then we know that H controls fusion in H with respect to S_n and S_n has no normal subgroups of order n.

What conditions on H guarantee that H has a normal complement? The Brauer-Suzuki theorem answered the question for a Hall subgroup H (see, for example, Theorem 8.22 in [2]). In this paper, we shall give a more general criterion for the existence of a normal complement of a subgroup H in a group G. Before stating our result, we shall introduce the following notation:

Let H be a subgroup of a group G which controls fusion in H with respect to G, and let T, M and L be mappings from $H^{\#}$ to the family of subsets of G, where $H^{\#}=H-\{1\}$. Suppose T, M and L satisfy the following conditions. Then we say (T, M, L) a *complementary triple* of H in G.

(1.1) For every $h\in H^{\#}$,

(i) $T(h)$ is a subgroup of G with $T(h)^x=T(h^x)$ for $x\in H$,

(ii) $M(h)=hT(h)$,

(iii) $L(h)=\bigcup_{g\in G} M(h)^g$,

(iv) $N_G(M(h))=T(h)C_H(h)$.

(1.2) Whenever $h\in H^{\#}$ and $g\in G$, $M(h)\cap M(h)^g=\emptyset$ or $M(h)$.

(1.3) $(G-\bigcup_{x\in H^{\#}} L(x))\cap N_G(M(h))=T(h)$ for every $h\in H^{\#}$.

(1.4) Whenever h_1 and h_2 are elements of $H^{\#}$ which are not conjugate in G, then $L(h_1)\cap L(h_2)=\emptyset$.

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REMARK 1. In the above definition, the mappings M and L are determined by T.

REMARK 2. Let h_1 and h_2 be elements of $H^{\#}$. If they are conjugate in H, then there exists an element y of H with $h_1=h_2{}^y$. Since $M(h_1)=M(h_2)^y$ by (i) and (ii) of (1.1), it follows that $L(h_1)=L(h_2)$. If they are not conjugate in H, then by the assumption that H controls fusion in H with respect to G, they are not conjugate in G. Hence by (1.4), $L(h_1)\cap L(h_2)=\emptyset$.

Suppose H has a normal complement N in G. Let $T_0(h)=N$, $M_0(h)=hN$ and $L_0(h)=\bigcup_{g\in G} M_0(h)^g$ for $h\in H^{\#}$. Then (T_0, M_0, L_0) is a complementary triple of H in G. This is a trivial example. In general, complementary triples can be constructed through several ways depending on the structures of H and N (see Theorem 3.3 and Proposition 3.4).

Our main result is the following (which we shall prove in § 2):

THEOREM. *Let H be a subgroup of a group G which controls fusion in H with respect to G. Suppose there exists a complementary triple of H in G. Then H has a normal complement in G.*

REMARK 3. The assumption that H controls fusion in H with respect to G in the above Theorem can not be removed. For example, let G be S_5 and H be a Sylow 3-subgroup of S_5. Set $T(h)=1$, $M(h)=h$ and $L(h)=\bigcup_{g\in S_5} h^g$ for $h\in H^{\#}$. Then it can easily be checked that T, M and L satisfy the conditions (1.1)$\sim$(1.4). However, H does not have a normal complement in G.

Let H be a Frobenius subgroup of a group G. Let $T_1(h)=1$, $M_1(h)=h$ and $L_1(h)=\{h^g \mid g\in G\}$ for $h\in H^{\#}$. Then an easy argument shows that (T_1, M_1, L_1) is a complementary triple of H in G. Hence by applying our theorem to G, we conclude that H has a normal complement. Thus the above Theorem yields Frobenius' theorem (see for example Theorem 7.2 in [2]). Our notation is standard [1].

§ 2. Proof of the theorem.

To prove the theorem, we need the following lemma which is due to Brauer (see Theorem 8.4 in [2]).

LEMMA 2.0. *Let Θ be a complex-valued class function of a group G. Then Θ is a generalized character of G if and only if $\Theta|_E$ is a generalized character of E for every elementary subgroup E of G.*

Let (T, M, L) be a complementary triple of H in G. Set $N=G-\bigcup_{x\in H^{\#}} L(x)$. We shall show that N is a normal complement of H in G through several steps.

(2.1) *$T(h)$ is a normal complement of $C_H(h)$ in $N_G(M(h))$ for every element h of $H^{\#}$.*

PROOF. Since $T(h)=\langle y^{-1}z \mid y, z\in M(h)\rangle$, $T(h)$ is normal in $N_G(M(h))$. By (1.3), $T(h)\cap H=1$. (2.1) follows from (iv) of (1.1).

(2.2) (i) $|L(h)|=|G:C_H(h)|$ *for every* $h\in H^{\#}$.

(ii) $|N|=|G:H|$.

PROOF. By (1.2), $|L(h)|=|M(h)||G:N_G(M(h))|$. On the other hand, $|N_G(M(h))|=|T(h)||C_H(h)|$ by (iv) of (1.1) and (2.1). Hence we have $|L(h)|=|M(h)||G|/|T(h)||C_H(h)|=|G:C_H(h)|$ from (ii) of (1.1). Thus (i) is proved. For the proof of (ii), let $h_1, h_2, \cdots, h_n$ be the representatives of the conjugacy classes of H, where $h_1=1$. Then $|\bigcup_{x\in H^{\#}} L(x)|=\sum_{i=2}^{n}|G:C_H(h_i)|=|G:H|(|H|-1)$, which implies $|N|=|G:H|$. Thus we have proved (2.2).

(2.3) *Whenever* $h_1\in H^{\#}$ *and* $h_2\in C_H(h_1)^{\#}$, *then* $h_2T(h_1)\subseteqq L(h_2)$.

PROOF. Let $S=T(h_1)\cap T(h_2)$. By (1.2), $N_G(h_2S)\subseteqq N_G(M(h_2))$. Then we have $N_G(h_2S)\cap T(h_1)\subseteqq N_G(M(h_2))\cap T(h_1)\subseteqq S$ by (1.3). Since $S\subseteqq N_G(h_2S)$, we conclude $N_G(h_2S)\cap T(h_1)=S$. Hence $|\bigcup_{x\in T(h_1)}(h_2S)^x|=|h_2S||T(h_1):S|=|T(h_1)|$, which proves (2.3).

(2.4) *Let E be a nilpotent subgroup of G which is not contained in N. Then there exist $h\in H^{\#}$ and $g\in G$ such that $E\subseteqq N_G(M(h)^g)$.*

PROOF. Let $1=E_1\subseteqq E_2\subseteqq \cdots \subseteqq E_m=E$ be the upper central series of E. Then there exists some i, $2\leqq i\leqq m$, such that $E_{i-1}\subseteqq N$ and $E_i\nsubseteqq N$. Let $E_i\cap M(h)^g\neq\emptyset$ for $h\in H^{\#}$ and $g\in G$. Let $z\in E_i\cap M(h)^g$. First we shall show that $E_{i-1}\subseteqq T(h)^g$. Suppose $E_{j-1}\subseteqq T(h)^g$ and $E_j\nsubseteqq T(h)^g$ for some j, $2\leqq j\leqq i-1$. Then $[z, E_j]\subseteqq E_{j-1}\subseteqq T(h)^g$, which implies $E_j\subseteqq N_G(M(h)^g)$. Since $j\leqq i-1$, we have $E_j\subseteqq N$ by the choice of i. Therefore by (1.3), $E_j\subseteqq N_G(M(h)^g)\cap N=T(h)^g$, which contradicts our choice of E_j. Thus we conclude $E_{i-1}\subseteqq T(h)^g$. Then $[z, E]\subseteqq E_{i-1}\subseteqq T(h)^g$. Hence we have $E\subseteqq N_G(M(h)^g)$. Thus (2.4) is proved.

Let Ψ be an irreducible complex character of H. Then by (1.4), we can define a class function $\hat{\Psi}$ on G as follows:

$$\hat{\Psi}(x)=\begin{cases}\Psi(h) & \text{if} \quad x\in L(h) \quad \text{for} \quad h\in H^{\#},\\ \Psi(1) & \text{if} \quad x\in N.\end{cases}$$

(2.5) *The restriction of $\hat{\Psi}$ to $N_G(M(h)^g)$ is a character of $N_G(M(h)^g)$ for $h\in H^{\#}$ and $g\in G$.*

PROOF. Since $\hat{\Psi}$ is a class function, it suffices to show that the restriction of $\hat{\Psi}$ to $N_G(M(h))$ is a character of $N_G(M(h))$ for every $h\in H^{\#}$. Since $h_1T(h)\subseteqq L(h_1)$ for $h_1\in C_H(h)^{\#}$ by (2.3), $\hat{\Psi}$ takes the constant value $\Psi(h_1)$ on $h_1T(h)$. Also $\hat{\Psi}$ takes the constant value $\Psi(1)$ on $T(h)$. Considering $N_G(M(h))/T(h)\cong C_H(h)$ by (2.1), we can regard $\hat{\Psi}$ as a character of $N_G(M(h))/T(h)$. Thus we conclude (2.5).

(2.6) *$\hat{\Psi}$ is an irreducible character of G.*

PROOF. First we shall show that $\hat{\Psi}$ is a generalized character of G. By

Lemma 2.0, it suffices to show that the restriction of $\hat{\Psi}$ to any nilpotent subgroup E of G is a generalized character of E. If E is contained in N, then the restriction of $\hat{\Psi}$ to E is a multiple of the principal character of E by the definition of $\hat{\Psi}$. So we may assume that E is not contained in N. Then by (2.4), there exist $h\in H^{\#}$ and $g\in G$ such that $E\subseteqq N_G(M(h)^g)$. Since the restriction of $\hat{\Psi}$ to $N_G(M(h)^g)$ is a character of $N_G(M(h)^g)$ by (2.5), we conclude that the restriction of $\hat{\Psi}$ to E is a character of E. Hence $\hat{\Psi}$ is a generalized character of G. Next, we shall show that $\hat{\Psi}$ is an irreducible character of G. Let $h_1=1, h_2, \cdots, h_n$ be representatives of the conjugacy classes of H. Then by (2.2),

$$\begin{aligned}\sum_{g\in G}\hat{\Psi}(g)\hat{\Psi}(g^{-1}) &= \sum_{g\in N}\Psi(1)^2+\sum_{i=2}^{n}\Big(\sum_{g\in L(h_i)}|\Psi(h_i)|^2\Big)\\ &=\Psi(1)^2|N|+\sum_{i=2}^{n}|L(h_i)|\,|\Psi(h_i)|^2\\ &=|G:H|\Big(\Psi(1)^2+\sum_{i=2}^{n}|H:C_H(h_i)|\,|\Psi(h_i)|^2\Big)=|G|.\end{aligned}$$

Considering $\hat{\Psi}(1)=\Psi(1)$, we conclude that $\hat{\Psi}$ is an irreducible character of G. Hence (2.6) is proved.

(2.7) *N is a normal complement of H in G.*

PROOF. Let $\Psi_1, \Psi_2, \cdots, \Psi_n$ be the irreducible characters of H. Set $M=\bigcap_{i=1}^{n}\mathrm{Ker}(\hat{\Psi}_i)$. Then N is contained in M. On the other hand, $M\cap H=1$, which implies $|M|\leqq|G:H|$. Thus we have $N=M$, which proves (2.7). This completes the proof of the theorem.

§3. Constructions of complementary triples.

LEMMA 3.1. *Let N be a normal π-subgroup of a group G, where π is a set of primes. Let x be an element of G. Set $S=\{g\in N\,|\,[g, x, \cdots, x]=1\}$. Suppose N is generated by S. Then $[N, y]=1$ for every π'-element y of $\langle x\rangle$.*

PROOF. Suppose false. Then there exists a π'-element y of $\langle x\rangle$ with $C_N(y)\neq N$. Since N is generated by S, there exists an element z of N such that $z\notin C_N(y)$ and $z^{-1}z^x\in C_N(y)$. Considering $C_N(y)=C_N(y)^x$, we have $z^{-1}z^y\in C_N(y)$. Let $w=z^{-1}z^y$. Then $w^n=ww^y\cdots w^{y^{n-1}}=1$, where n is the order of y, which contradicts the choice of y. Thus Lemma 3.1 is proved.

LEMMA 3.2. *Let N be a nilpotent normal subgroup of a group G. Set $T^*(g)=\{x\in N\,|\,[x, g, \cdots, g]=1\}$ for $g\in G$. Then the following hold:*

(i) *$T^*(g)$ is a subgroup of G.*

(ii) *$\langle g\rangle T^*(g)$ is a nilpotent subgroup of G.*

(iii) *Whenever $\langle g\rangle K$ is a nilpotent subgroup of G for a subgroup K of N, $K\subseteqq T^*(g)$.*

PROOF. For the proof of (i) and (ii), it suffices to show that $\langle g\rangle\langle T^*(g)\rangle$ is a nilpotent subgroup. Set $H=\langle g\rangle\langle T^*(g)\rangle$. Let p be a prime divisor of $|H|$ and let $\langle g_p\rangle$ be the Sylow p-subgroup of $\langle g\rangle$. Since $\langle T^*(g)\rangle$ is nilpotent, the Sylow p-subgroup P of $\langle T^*(g)\rangle$ is normal in H. Set $\bar{H}=H/P$. Then $\langle T^*(g)\rangle/P$ is a normal p'-subgroup of $\bar{H}$. Hence by Lemma 3.1, we have $[\langle T^*(g)\rangle, g_p]\subseteqq P$, which implies $\langle g_p\rangle P$ is a normal subgroup of H. Therefore H is nilpotent and we have proved (i) and (ii). On the other hand, (iii) is obvious by the definition of $T^*(g)$. Thus we have Lemma 3.2.

THEOREM 3.3. *Let H be a subgroup of a group G which has a nilpotent normal complement N in G. Set $T^*(h)=\{g\in N|[g, h, \cdots, h]=1\}$, $M^*(h)=hT(h)$ and $L^*(h)=\bigcup_{g\in G}M^*(h)^g$ for $h\in H^\#$. Then (T^*, M^*, L^*) is a complementary triple of H in G.*

PROOF. By Lemma 3.2, (i) of (1.1) is satisfied. Also by the definition of M^* and L^*, (ii) and (iii) of (1.1) are satisfied. Suppose $h_1x\in N_G(M^*(h))$ for $h_1\in H$ and $x\in N$. Then $x^{-1}h_1^{-1}hh_1x\in M^*(h)$. Hence $h_1^{-1}hh_1\in H\cap hN$, which implies $h_1\in C_H(h)$. Then $[x, h]$ is contained in $T^*(h)$, and it follows that x is an element of $T^*(h)$. Therefore $N_G(M^*(h))$ is contained in $T^*(h)C_H(h)$. Since the converse inclusion is obvious, (iv) of (1.1) is satisfied. Suppose $M^*(h_1)\cap M^*(h_2)^g \neq\emptyset$ for h_1, $h_2\in H^\#$ and $g\in G$. Let x be an element of $M^*(h_1)\cap M^*(h_2)^g$. Then both $\langle x, T^*(h_1)\rangle$ and $\langle x, T^*(h_2)^g\rangle$ are nilpotent. Then by Lemma 3.2, $\langle x, T^*(h_1), T^*(h_2)^g\rangle$ is nilpotent. Considering $\langle x, T^*(h_1), T^*(h_2)^g\rangle=\langle h_1, T^*(h_1), T^*(h_2)^g\rangle=\langle h_2{}^g, T^*(h_1), T^*(h_2)^g\rangle$, we have $T^*(h_1)=T^*(h_2)^g$ by (iii) of Lemma 3.2. Hence $M^*(h_1)=M^*(h_2)^g$. Let $g=h_3z$ for $h_3\in H$ and $z\in N$. Since $h_2{}^g\in M^*(h_1)$, we have $h_1^{-1}h_2{}^g=h_1^{-1}h_4h_4^{-1}h_4{}^z\in T^*(h_1)$, where $h_4=h_3^{-1}h_2h_3$. Obviously, $h_4^{-1}h_4{}^z$ is contained in N. Hence we have $h_1^{-1}h_4\in N\cap H$. Thus $h_1=h_4$, which proves (1.2) and (1.4). For (1.3), let $h\in H^\#$. Since $|N: N_G(M^*(h))\cap N|=|N: T^*(h)|$, we have $|\bigcup_{x\in N}M^*(h)^x|=|N|$ by (1.2). Therefore $hN\subseteqq L^*(h)$. Then $G-\bigcup_{x\in H^\#}L^*(x)$ coincides with N, which implies (1.3). Thus we have proved Theorem 3.3.

REMARK 4. The complementary triple constructed in Theorem 3.3 is applicable to investigate a finite group which admits an automorphism of prime order (see [**3**]).

PROPOSITION 3.4. *Let H be a Hall π-subgroup of a group G which controls fusion in H with respect to G, where π is a set of primes. Suppose for every $h\in H^\#$, $C_G(h)$ is π-nilpotent and $C_H(h)$ is a Hall π-subgroup of $C_G(h)$. Let $T(h)$ be the normal π-complement of $C_G(h)$, $M(h)=hT(h)$ and $L(h)=\bigcup_{g\in G}M(h)^g$ for $h\in H^\#$. Then (T, M, L) is a complementary triple of H in G. Consequently, G is π-nilpotent.*

PROOF. (i), (ii) and (iii) of (1.1) are satisfied by the definition of T, M and L. Let $h\in H^\#$. Since $T(h)$ is contained in $C_G(h)$, h is the only π-element of $M(h)$, hence we have $N_G(M(h))\subseteqq C_G(h)$. This implies (iv) of (1.1). Suppose

$M(h_1)\cap M(h_2)^g\neq\emptyset$ for h_1, $h_2\in H^\#$ and $g\in G$. Let $x\in M(h_1)\cap M(h_2)^g$. Then there exist $t_1\in T(h_1)$ and $t_2\in T(h_2)^g$ with $x=h_1t_1=h_2{}^gt_2$. Since such a decomposition of x is unique, we have $h_1=h_2{}^g$. Thus (1.2) and (1.4) are satisfied. By the assumption of Proposition 3.4, every π-element of G is conjugate to an element of H. Therefore $G-\bigcup_{h\in H^\#}L(h)$ coincides with the set of all π'-elements of G, which shows that (1.3) is satisfied. This proves Proposition 3.4.

COROLLARY 3.5. *Let A be a solvable group of automorphisms of a group G with $(|A|,|G|)=1$. Suppose $C_G(A)$ is a Hall π-subgroup of G. Then G is π-nilpotent.*

PROOF. Let G be a minimal counterexample. Then obviously, G contains no proper A-invariant normal subgroups. Let $h\in C_G(A)^\#$. Then $C_G(h)$ is a proper A-invariant subgroup. Since every A-invariant p-subgroup is contained in $C_G(A)$ for a prime $p\in\pi$, we conclude that $C_G(A)\cap C_G(h)$ is a Hall π-subgroup of $C_G(h)$. Hence by the minimality of G, $C_G(h)$ is π-nilpotent. Thus setting $H=C_G(A)$, we have that G satisfies the assumption of Proposition 3.4, and therefore G is π-nilpotent. This contradicts the choice of G. Therefore Corollary 3.5 is proved.

REMARK 5. Note that the assumption that A is solvable in Corollary 3.5 can be dropped by the Feit-Thompson theorem. Also note that both Proposition 3.4 and Corollary 3.5 can be obtained from the Brauer-Suzuki theorem.

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