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On spherical space forms which are isospectral but not isometric

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Introduction.

A compact connected Riemannian manifold of constant curvature 1 is said to be a *spherical space form*. If the fundamental group of a spherical space form is cyclic then the spherical space form is called a *lens space*.

Let M and N be spherical space forms. In papers [2], [3], [4] and [5], we studied the spectrum of Laplacian acting on smooth functions of a spherical space form and considered the following problem.

Whether or not M is isometric to N when M is isospectral to N?

In [5], we saw that there are many pairs of lens spaces which are isospectral but not isometric.

In this paper, we consider the above problem in cases which the fundamental groups of M and N are *noncyclic*. First we prove

THEOREM 1. *Let S^{2d-1}/G and S^{2d-1}/G' be spherical space forms with noncyclic fundamental groups of type* 1. *Suppose G and G' are irreducible and that G is isomorphic to G'. Then S^{2d-1}/G is isospectral to S^{2d-1}/G'.* (For the definitions of "type 1" and "irreducible" in Theorem 1, see Sections 2 and 3 respectively).

From this Theorem we can show that there are many pairs of spherical space forms with *noncyclic fundamental groups* which are isospectral but not isometric (for more precise statement, see Theorem 3). And moreover we see that there are spherical space forms which are isospectral but not isometric in every odd dimension not less than 5 (see Theorem 4).

Two lens spaces which are isospectral but not isometric are also not homeomorphic to each other (see [5]). Moreover we obtained in [5] examples of pairs of lens spaces which are isospectral but not even homotopically equivalent. But unfortunately the author don't know whether there are any topological differences between these isospectral non-isometric spherical space forms with noncyclic fundamental groups.

REMARKS. 1. As we have shown in [3], every 3-dimensional spherical space form is completely characterized by its spectrum as Riemannian manifold. Moreover every 5-dimensional spherical space form with *noncyclic fundamental group* is also completely characterized by its spectrum as Riemannian manifold [**4**].

2. By Kitaoka's example [**7**], we see there are flat tori which are isospectral but not isometric in every dimension not less than 8. On the other hand, any 2-dimensional flat torus is completely characterized by its spectrum as Riemannian manifold [**1**].

3. In [**9**], Vigneras constructed examples of pairs of compact hyperbolic spaces which are isospectral but not isometric in every dimension not less than 2.

Theorem 1 and Theorem 3 in this paper were announced in [**6**].

1. Spherical space forms and their generating functions.

Let S^d $(d\geqq 2)$ be the unit sphere centered at the origin in R^{d+1}, the $(d+1)$-dimensional Euclidean space. We denote by $O(d+1)$ the orthogonal group acting on R^{d+1}. A finite subgroup G of $O(d+1)$ is said to be *fixed point free* if for any $g\in G$ $(g\neq 1_{d+1})$ g has not 1 for an eigenvalue. A finite fixed point free subgroup of $O(d+1)$ acts on S^d as fixed point freely. So that the quotient Riemannian manifold S^d/G becomes a spherical space form in a natural way. Conversely any spherical space form is obtained by this way.

It is easy to see that even dimensional spherical space forms are only the canonical spheres and the canonical real projective spaces. Therefore in what follows, we consider only for odd dimensional spherical space forms.

Let $M=S^{2d-1}/G$ $(d\geqq 2)$ be a $(2d-1)$-dimensional spherical space form and Δ the Laplacian acting on the space of smooth functions on M. Then each eigenvalue of Δ is of the form $k(k+2d-2)$ $(k=0, 1, 2, \cdots)$ (see [**1**]). Let E_k be the eigenspace of Δ with eigenvalue $k(k+2d-2)$. We define the generating function associated to the spectrum of Δ by

$$F_G(z)=\sum_{k=0}^{\infty}(\dim E_k)z^k\,. \tag{1.1}$$

Let S^{2d-1}/G and S^{2d-1}/G' be spherical space forms. By the definition of the generating function we have

(1.2) S^{2d-1}/G is isospectral to S^{2d-1}/G' if and only if $F_G(z)=F_{G'}(z)$

PROPOSITION 1.1 (see [**3**]). *We have*

$$F_G(z)=\frac{1}{|G|}\sum_{g\in G}\frac{1-z^2}{\det(z1_{2d}-g)}\,, \tag{1.3}$$

where $|G|$ is the order of G.

From this proposition, we see easily

COROLLARY 1.2. *Let S^{2d-1}/G and S^{2d-1}/G' be spherical space forms. Suppose there exists a one to one onto map Ψ of G onto G' satisfying* $\det(z1_{2d}-g)=\det(z1_{2d}-\psi(g))$ *for each $g\in G$. Then S^{2d-1}/G is isospectral to S^{2d-1}/G'.*

2. Vincent's results for spherical space forms.

The complete classification of 3-dimensional spherical space forms was obtained by Seifert and Threllfall (see [**11**]). In this section we state Vincent's results for a classification of spherical space forms, according to Wolf's book [**11**].

DEFINITIONS. A finite dimensional orthogonal representation of a finite group is *fixed point free* if it is faithful and its image is a fixed point free subgroup of the orthogonal group. A finite group is called a *fixed point free group* if it has a fixed point free representation.

The following proposition is a fundamental property for Vincent's classification program.

PROPOSITION 2.1 (see [**11**]). *Let K be a finite fixed point free group. Let π_1 and π_2 be fixed point free representations of degree $2d$ of K. Then the spherical space forms $S^{2d-1}/\pi_1(K)$ is isometric to $S^{2d-1}/\pi_2(K)$ if and only if π_1 is equivalent to π_2 modulo automorphisms.*

Owing to Vincent, finite fixed point free groups are divided into two types as abstract groups (see [**10**], [**11**]).

Type 1: Every Sylow subgroup is cyclic.

Type 2: Every Sylow p-subgroup ($p\neq 2$) is cyclic and every Sylow 2-subgroup is a generalized quaternionic group.

For the definition of a generalized quaternionic group, see [**11**]. In this paper we treat only spherical space forms with type 1 fundamental groups. Type 1 groups are not so special because of the following.

PROPOSITION 2.2 (Vincent [**10**], see also [**11**]). *The fundamental group of every $(4k+1)$-dimensional spherical space form is of type* 1.

For any non-zero integer m, K_m denotes the multiplicative group of residue classes modulo m of integers prime to m. The order of K_m is denoted by $\phi(m)$, so called Euler function. For two integers a and b, we denote by (a, b) the greatest common divisor of a and b.

We describe finite fixed point free groups of type 1. Let m, n, d and n' be positive integers and r integer satisfying

$$(2.1)\qquad \begin{cases} ((r-1)n,\ m)=1, \\ r^{n}\equiv 1 \pmod{m}, \\ d \text{ is the order of the residue class of } r \text{ in } K_m, \\ n=n'd, \\ n' \text{ is divisible by any prime divisor of } d. \end{cases}$$

For such integers m, n, d, n' and r, we have the finite group $\Gamma_d(m, n, r)$ of order $N=mn$ generated by two elements A and B with defining relations

$$(2.2)\qquad A^m=B^n=1 \quad \text{and} \quad BAB^{-1}=A^r.$$

Note that the following four conditions are equivalent for the $\Gamma_d(m, n, r)$; (i) $\Gamma_d(m, n, r)$ is cyclic, (ii) $A=1$, (iii) $r\equiv 1 \pmod{m}$, and (iv) $d=1$.

We define automorphisms of $\Gamma_d(m, n, r)$. Whenever s, t and u are integers with $(s, m)=1=(t, n)$ and $t\equiv 1 \pmod{d}$, we put

$$(2.3)\qquad \psi_{s,t,u}(A)=A^s \quad \text{and} \quad \psi_{s,t,u}(B)=B^tA^u.$$

Then we can see easily $\psi_{s,t,u}$ defines an automorphism of $\Gamma_d(m, n, r)$.

PROPOSITION 2.3 (see [**11**]). 1. *The automorphisms of $\Gamma_d(m, n, r)$ are just the $\psi_{s,t,u}$'s.*

2. *$\Gamma_d(m, n, r_1)$ is isomorphic to $\Gamma_d(m, n, r_2)$ if and only if there exists an integer c such that $r_1\equiv r_2^c \pmod{m}$.*

3. *A finite fixed point free group of type* 1 *is isomorphic to some $\Gamma_d(m, n, r)$.*

As for fixed point free representations of $\Gamma_d(m, n, r)$, we have

PROPOSITION 2.4 (see [**11**]). *Let $K=\Gamma_d(m, n, r)$, and let $R(\theta)$ denote the rotational matrix on the plane;*

$$R(\theta)=\begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

Given integers k and l with $(k, m)=1=(l, n)$, let $\pi_{k,l}$ be the representation of degree $2d$ of K defined by

$$\pi_{k,l}(A)=\begin{pmatrix} R(k/m) & & & \mathbf{0} \\ & R(kr/m) & & \\ & & \ddots & \\ \mathbf{0} & & & R(kr^{d-1}/m) \end{pmatrix}$$

and

$$\pi_{k,l}(B)=\begin{pmatrix} 0 & I & & \mathbf{0} \\ & \ddots & I & \\ & & \ddots & \ddots \\ & \mathbf{0} & & \ddots & I \\ R(l/n') & & & & 0 \end{pmatrix},$$

where each matrix is a block matrix consisting of 2×2*-matrices,* I *is the unit* 2×2*-matrix and all other components are zero. Then* $\pi_{k,l}$ *is irreducible and a real representation of* K *is fixed point free if and only if it is equivalent to a sum of these representations* $\pi_{k,l}$. $\pi_{k,l}$ *is equivalent to* $\pi_{k',l'}$ *if and only if there exist numbers* $e=\pm 1$ *and* $c=0, 1, \cdots, d-1$ *such that* $k'\equiv kr^c \pmod{m}$ *and* $l'\equiv el \pmod{n'}$. $\pi_{k,l}\circ\phi_{s,t,u}$ *is equivalent to* $\pi_{sk',tl'}$ *where* $\phi_{s,t,u}$ *is the automorphism of* K *defined before.*

REMARK. Any irreducible fixed point free representation of $\Gamma_d(m, n, r)$ has the same degree $2d$.

LEMMA 2.5. *Let* $K=\Gamma_d(m, n, r)$ *be a finite fixed point free group of type* 1 *with* $n'=d$. *Then the number of isometry classes in* $(2d-1)$*-dimensional spherical space forms with the same fundamental group* K *is at least* 2 *if and only if* $d=5$ *or* $d>6$.

PROOF. Let $\pi_{k,l}$ and $\pi_{k',l'}$ be fixed point free representations of K as in Proposition 2.4. Then $\pi_{k,l}$ is equivalent to $\pi_{k',l'}$ modulo automorphisms if and only if there exists an integer t with $(t, n)=1$, $t\equiv 1 \pmod{d}$ and $l\equiv \pm tl' \pmod{n'}$. Since $n'=d$, the number of isometry classes in $(2d-1)$-dimensional spherical space forms with the fundamental group K is $\phi(d)/2$ if $d>2$, and 1 if $d\leqq 2$. Now the Lemma follows easily from this fact. q. e. d.

LEMMA 2.6. *For fixed* $d\geqq 2$, *there are infinitely many finite fixed point free groups* $\Gamma_d(m, n, r)$ *of type* 1 *with* $n'=d$.

PROOF. It is well known that there are infinitely many prime numbers of forms $kd+1$. Let $m=kd+1$ be a prime number. Then K_m is a cyclic group of order kd. So there exists an integer r whose order in K_m is d. Put $n=d^2$, then we have a finite fixed point free group of type 1 $\Gamma_d(m, n, r)=\Gamma_d(m, d^2, r)$. q. e. d.

3. Spherical space forms which are isospectral but not isometric.

LEMMA 3.1. *Let* $A=(a_{i,j})$ *be a* $d\times d$*-matrix and let* d_1 *be an integer with* $0\leqq d_1<d$. *Suppose that* $a_{i,j}=0$ *if* $j\not\equiv i+d_1 \pmod{d}$. *Then the characteristic polynomial of* A *is*

$$\det(z1_{2d}-A)=\prod_{i=1}^{(d,d_1)}\left\{z^{d/(d,d_1)}-\prod_{j=1}^{d/(d,d_1)}a_{i+(d,d_1)(j-1),i+(d,d_1)(j-1)+d_1}\right\}, \tag{3.1}$$

where $a_{i,j}=a_{i',j'}$ *if* $i\equiv i' \pmod{d}$ *and* $j\equiv j' \pmod{d}$.

PROOF. It is easy to see the Lemma in case $d_1=0$ or 1. Now we regard the matrix A as the linear transformation on a d-dimensional complex vector space V with a basis $\{e_1, e_2, \cdots, e_d\}$ such that

$$Ae_i=\sum_{j=1}^{d}a_{i,j}e_j \tag{3.2}$$

$$=a_{i,i+d_1}e_{i+d_1} \qquad (i=1, 2, \cdots, d),$$

where $e_j=e_{j'}$ if $j\equiv j'$ (mod d). Put

$$f_{i,j}=e_{i+d_1(j-1)} \qquad (1\leqq i\leqq (d, d_1), 1\leqq j\leqq d/(d, d_1)). \tag{3.3}$$

Let V_i ($i=1, 2, \cdots, (d, d_1)$) be the subspace of V generated by $f_{i,1}, f_{i,2}, \cdots, f_{i,d/(d,d_1)}$. Then we have

(i) $V=V_1\oplus V_2\oplus \cdots \oplus V_{(d,d_1)}$ (direct sum),

(ii) $AV_i\subset V_i \qquad (i=1, 2, \cdots, (d, d_1))$

and

(iii) $Af_{i,j}=a_{i+d_1(j-1),i+d_1j}f_{i,j+1} \qquad (1\leqq j<d/(d, d_1))$,

$Af_{i,d/(d,d_1)}=a_{i-d_1,i}f_{i,1}$.

Hence we have the characteristic polynomial of A is

$$\prod_{i=1}^{(d,d_1)}\left\{z^{d/(d,d_1)}-\prod_{j=1}^{d/(d,d_1)}a_{i+d_1(j-1),i+d_1j}\right\}. \tag{3.4}$$

Now the Lemma follows from the fact that

$$\{d_1(j-1); j=1, 2, \cdots, d/(d, d_1)\} \tag{3.5}$$
$$\equiv\{(d, d_1)(j-1); j=1, 2, \cdots, d/(d, d_1)\} \qquad (\mathrm{mod}\, d). \qquad \text{q.e.d.}$$

For a positive integer p, we put

$$\zeta_p=\exp(2\pi\sqrt{-1}/p). \tag{3.6}$$

Let G be a subgroup of $SO(2d)$. We say G is *irreducible* when the representation $G\subset SO(2d)$ is real irreducible.

THEOREM 1. *Let S^{2d-1}/G and S^{2d-1}/G' be spherical space forms with noncyclic fundamental groups of type* 1. *Suppose G and G' are irreducible and that G is isomorphic to G'. Then S^{2d-1}/G is isospectral to S^{2d-1}/G'.*

PROOF. By Proposition 2.3, G and G' are isomorphic to a finite fixed point free group $K=\Gamma_d(m, n, r)$. For the proof, we may assume $G=\pi_{1,l}(K)$ and $G'=\pi_{1,1}(K)$, where $\pi_{1,l}$ and $\pi_{1,1}$ are fixed point free representations of K as in Proposition 2.4. The complexification of $\pi_{1,l}$, $(\pi_{1,l})_C$ is decomposed into two irreducible mutually conjugate complex representations of K;

$$(\pi_{1,l})_C=\rho_l\oplus\bar{\rho}_l \tag{3.7}$$

where

$$\rho_l(A)=\begin{pmatrix} \zeta_m & & & \mathbf{0} \\ & \zeta_m^r & & \\ & \mathbf{0} & \ddots & \\ & & & \zeta_m^{r^{n-1}} \end{pmatrix}$$

and

$$\rho_l(B)=\begin{pmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots\ 1 & 0 \\ 0 & \cdots\cdots & & 0 & 1 \\ \zeta_{n'}^l & 0 & \cdots\cdots & & 0 \end{pmatrix}.$$

Hence for each d_1 with $0\leqq d_1<d$, the components of the matrix $\rho_l(A^sB^{vd+d_1})$ $(1\leqq v\leqq n')$ are as follows;

(3.8)
$$\begin{cases} (\rho_l(A^sB^{vd+d_1}))_{i,i+d_1} =\zeta_m^{sr^{i-1}}\cdot\zeta_{n'}^{lv} & 1\leqq i\leqq d-d_1, \\ (\rho_l(A^sB^{vd+d_1}))_{i,i+d_1-d}=\zeta_m^{sr^{i-1}}\cdot\zeta_{n'}^{l(v+1)} & d-d_1<i, \\ (\rho_l(A^sB^{vd+d_1}))_{i,j}=0 & \text{otherwise}. \end{cases}$$

Applying Lemma 3.1 to the matrix $\rho_l(A^sB^t)$ $(1\leqq s\leqq m,\ 1\leqq t\leqq n)$, we have

(3.9)
$$\det(z-\rho_l(A^sB^t))=\prod_{i=1}^{(d,t)}(z^{d/(d,t)}-\zeta_m^{sr^{i-1}r(t)}\cdot\zeta_{n'}^{tl/(d,t)}),$$

where $r(t)=\sum_{j=1}^{d/(d,t)} r^{(d,t)j}$.

We define the map ψ of G onto G' by

(3.10)
$$\psi(\pi_{1,l}(A^sB^t))=\pi_{1,1}(A^sB^{lt}).$$

Then ψ is clearly one to one onto. Then from (3.7) and (3.9) we have

(3.11)
$$\det(z1_{2d}-g)=\det(z1_{2d}-\psi(g)) \quad \text{for each} \quad g\in G.$$

By Corollary 1.2, this implies S^{2d-1}/G is isospectral to S^{2d-1}/G'. q.e.d.

THEOREM 2. *Let S^{2d-1}/G and S^{2d-1}/G' be $(2d-1)$-dimensional spherical space forms with noncyclic fundamental groups. Suppose d is odd prime. Then S^{2d-1}/G is isospectral to S^{2d-1}/G' if and only if G is isomorphic to G'.*

PROOF. If d is odd, then $2d-1\equiv 1 \pmod 4$. By Proposition 2.2, G and G' are of type 1. Moreover if d is odd prime, and G, G' are not cyclic, then G and G' are irreducible by Proposition 2.4. Therefore, if part of the theorem follows from Theorem 1. On the other hand, only if part of the theorem was already proved in [4]. q.e.d.

Combining Theorem 1 with Lemma 2.5 and Lemma 2.6, we have

THEOREM 3. *Suppose that $d=5$ or $d>6$. Then there exist infinitely many pairs of $(2d-1)$-dimensional spherical space forms which are isospectral but not isometric.*

Since there exist lens spaces which are isospectral but not isometric in dimensions 5, 7 and 11 (see [5]), we have

THEOREM 4. *There exist spherical space forms which are isospectral but not isometric in every odd dimension not less than* 5.

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