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On the unit groups of Burnside rings of finite groups

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Introduction.

Let G be a finite group and $\Theta(G)$ the set of G-isomorphism classes of all finite (left) G-sets. Then $\Theta(G)$ is a semi-ring with addition and multiplication induced by disjoint union and cartesian product, respectively. The Burnside ring $A(G)$ of G is defined to be the Grothendieck ring of $\Theta(G)$. Let $A(G)^*$ be the unit group of the Burnside ring $A(G)$.

In this note we shall study $A(G)^*$ and the homomorphism $u: RO(G)\rightarrow A(G)^*$, where $RO(G)$ is the real representation ring of G and u is the homomorphism defined by T. tom. Dieck (see 1.2). By the famous theorem of Feit-Thompson (G is solvable if $|G|$ is odd) and by a result of A. Dress (idempotents of $A(G)$ are determined by perfect subgroups of G, cf. [1] Proposition 1.4.1), we know that

$$|A(G)^*|=2 \qquad \text{if } |G| \text{ is odd}$$

(cf. [1] Proposition 1.5.1). Therefore, it remains to study $A(G)^*$ and the homomorphism $u: RO(G)\rightarrow A(G)^*$ for groups G of even order.

In Section 1, we describe the well known results for $A(G)^*$ and the homomorphism $u: RO(G)\rightarrow A(G)^*$.

Section 2 is the main part of this note, and we obtain the following Theorem A and Theorem B.

THEOREM A (cf. Theorem 2.2, Corollary 2.4 and Lemma 2.5). *$u: RO(G)\rightarrow A(G)^*$ is surjective if and only if $u: RO(G')\rightarrow A(G')^*$ is surjective for every homomorphic image G' of G such that $|C(G')|\leqq 2$, where $C(G')$ is the center of G'.*

THEOREM B (cf. Theorem 2.9 and Theorem 2.11). *Let $1\rightarrow H\rightarrow G\rightarrow K\rightarrow 1$ be a group extension. Then we have*

(i) *K acts on $A(H)^*$* (cf. 2.6) *and $\mathrm{Res}_H^{G*}(A(G)^*)\subset(A(H)^*)^K$, where Res_H^{G*} is the natural restriction homomorphism from $A(G)^*$ to $A(H)^*$,*

(ii) *if $|K|$ is odd and $u: RO(H)\rightarrow A(H)^*$ is surjective, then $u: RO(G)\rightarrow A(G)^*$ is surjective and $\mathrm{Res}_H^{G*}: A(G)^*\rightarrow(A(H)^*)^K$ is an isomorphism,*

(iii) *if the group extension is split and $|K|$ is odd, then $\mathrm{Res}_H^{G*}: A(G)^*\rightarrow (A(H)^*)^K$ is an isomorphism.*

In Section 3, we give a few examples. By Theorem A we obtain the following example.

EXAMPLE (cf. Example 3.1). Let D_m be a dihedral group of order $2m$. We put $G=D_{m_1}\times\cdots\times D_{m_r}$. If $m_1, \cdots, m_r$ are relatively prime integers and $m_i>1$ $(i=1, \cdots, r)$, then $u: RO(G)\to A(G)^*$ is surjective.

The surjectivity of $u: RO(G)\to A(G)^*$ does not necessarily imply the same for subgroups of G. Here is an example.

EXAMPLE (cf. Example 3.4). We put

$$C_{15}=C_3\times C_5=\langle\sigma_1\rangle\times\langle\sigma_2\rangle \quad\text{and}\quad \mathrm{Aut}(C_{15})=C_2\times C_4=\langle\tau_1\rangle\times\langle\tau_2\rangle\,,$$

where C_m is a cyclic group of order m. Moreover, we put

$$H=\langle\sigma_1,\ \sigma_2,\ \tau_1\cdot\tau_2\rangle \quad\text{and}\quad G=\langle\sigma_1,\ \sigma_2,\ \tau_1,\ \tau_2\rangle\,.$$

Then $u: RO(G)\to A(G)^*$ is surjective and $u: RO(H)\to A(H)^*$ is not surjective.

Throughout this note, we use the following notation:

1	the unit element of G,
(H)	the conjugate class of a subgroup H of G,
$\Phi(G)$	the set of conjugate classes of all subgroups of G,
$N_G(H)$	the normalizer of a subgroup H of G in G,
X^G	the set of fixed points of a G-set X,
$\lvert X\rvert$	the cardinal number of a set X,
$[X]$	the element of $A(G)$ represented by a finite G-set X,
$\langle Y\rangle$	the subgroup of G generated by a subset Y of G,
$1_{A(G)}$	the unit element [point] of $A(G)$,
$\boldsymbol{R}$	the field of real numbers,
$\boldsymbol{Z}$	the ring of rational integers,
R^*	the unit group of a ring R.

1. Well known results for $A(G)^*$ and $u: RO(G)\to A(G)^*$.

1.1. Any finite G-set X is isomorphic to the disjoint union of some coset G-spaces G/H. Since G/H and G/F are G-isomorphic if and only if $(H)=(F)$ in $\Phi(G)$, $A(G)$ is a free module with basis $\{[G/H]\,|\,(H)\in\Phi(G)\}$. For a finite G-set X, let $\Psi(X): \Phi(G)\to\boldsymbol{Z}$ be the mapping defined by

$$\Psi(X)((H))=|X^H| \qquad \text{for } (H)\in\Phi(G)\,.$$

Let $Hom(\Phi(G), \boldsymbol{Z})$ be the ring of all mappings from $\Phi(G)$ to $\boldsymbol{Z}$ with the ring structure induced by the ring structure of $\boldsymbol{Z}$. It is well known that the assignment $\Psi: X\to\Psi(X)$ induces an injective ring homomorphism

$$\Psi: A(G)\longrightarrow Hom(\Phi(G),\ \boldsymbol{Z})\,.$$

Therefore, we can view $A(G)$ and $A(G)^*$ as a subring of $Hom(\Phi(G), \boldsymbol{Z})$ and a subgroup of $Hom(\Phi(G), \boldsymbol{Z})^* = Hom(\Phi(G), \{\pm 1\})$, respectively.

1.2. Let V be a real representation of G. Let $u(V)$ be an element of $A(G)^*$ defined by

$$u(V)((H)) = (-1)^{\dim_{\boldsymbol{R}} V^H} \quad \text{for } (H) \in \Phi(G)$$

(cf. [1] Proposition 5.5.9). The assignment $u: V \to u(V)$ induces a homomorphism $u: RO(G) \to A(G)^*$ such that $u(V \pm W) = u(V)u(W)$. For a regular representation $V = RG$ we have $\dim_{\boldsymbol{R}} V^G = 1$ and $\dim_{\boldsymbol{R}} V = |G|$. Therefore if $|G|$ is even, then there exists a non-trivial unit of $A(G)$.

1.3. Let $\boldsymbol{Q}$ be the field of rational numbers and $\bar{\boldsymbol{Q}}$ its algebraic closure. Let Γ be the Galois group of $\bar{\boldsymbol{Q}}$ over $\boldsymbol{Q}$. Let $RO(G)^{\mathrm{ab}}$ be the submodule of $RO(G)$ generated by the set, denoted by $\mathrm{ab}(G)$, of all absolutely irreducible real representations of G. The group Γ acts naturally on $RO(G)$, $RO(G)^{\mathrm{ab}}$ and $\mathrm{ab}(G)$. Then we have

$$\mathrm{Image}\, u = u(RO(G)^{\mathrm{ab}}) \quad \text{and} \quad |\mathrm{Image}\, u| \leqq 2^{|\mathrm{ab}(G)/\Gamma|}$$

(cf. [2] Lemma 5.2). Let $\zeta: A(G) \to R(G, \boldsymbol{Q})$ be the natural ring homomorphism defined by $\zeta([G/H]) = 1_H^G$, where $R(G, \boldsymbol{Q})$ is the rational representation ring of G. If ζ is surjective and the Schur index of every element of $\mathrm{ab}(G)$ over $\boldsymbol{Q}$ is odd, then we have

$$|\mathrm{Image}\, u| = 2^{|\mathrm{ab}(G)/\Gamma|} \tag{1.3.1}$$

(cf. [2] Lemma 5.5).

1.4. If e is a non-trivial idempotent of $A(G)$, then $(1-2e) \notin \mathrm{Image}\, u$ (cf. [2] Theorem 5.4). It follows that u is not surjective if G is not solvable. The converse is not always true (cf. [2] Example 5.9). In general $A(G)^*$ is not generated by $\mathrm{Image}\, u$ and $\{(1-2e) \mid e \in A(G) \text{ and } e^2 = e\}$. In fact, for the symmetric group $\mathfrak{S}_5$, there exists only one non-trivial idempotent e of $A(\mathfrak{S}_5)$ and we have

$$A(\mathfrak{S}_5)^* \supsetneqq \langle \mathrm{Image}\, u, (1-2e) \rangle$$

(cf. [2] 5.11.1).

1.5. If G is an abelian group, then $u: RO(G) \to A(G)^*$ is surjective,

$$A(G)^* = \langle -1_{A(G)}, (1_{A(G)} - [G/H]) \mid (H) \in \Phi(G) \text{ and } |G/H| = 2 \rangle$$

and $|A(G)^*| = 2^{m(G)+1}$, where $m(G) = |\{(H) \in \Phi(G) \mid |G/H| = 2\}|$ (cf. [2] Example 4.5 and Example 5.6).

2. The homomorphism $u: RO(G) \to A(G)^*$.

2.1. We put

$S(G)=\{(H)\in\Phi(G) \mid \text{if } H\supset H' \text{ and } H' \text{ is normal in } G, \text{ then } H'=\langle 1\rangle\}\cup\{G\}$,

$A(G)^{+}=\{\alpha\in A(G)^{*} \mid \alpha(G)=1\}$,

$A(G)_{0}=\{\sum_{(H)\in S(G)} n_{(H)}[G/H] \mid n_{(H)}\in \boldsymbol{Z}\}$,

$A(G)_{0}^{+}=A(G)^{+}\cap A(G)_{0}$,

$\mathrm{Min}(G)=\{H \mid H \text{ is a non-trivial minimal normal subgroup of } G\}$.

$A(G)_0$ is a subring of $A(G)$ with the unit element $1_{A(G)}$ (cf. [2] Lemma 3.3), $A(G)_0^+$ a subgroup of $A(G)^*$ and $A(G)^*=\pm A(G)^+$. Let $f: G\to G'$ be a group homomorphism and X a G'-set. Then we may regard X as a G-set via f, which we denote by $f^*(X)$. So f induces a ring homomorphism $f^*: A(G')\to A(G)$ defined by $f^*([X])=[f^*(X)]$. For a subgroup H of G, X^H is a WH-set, where $WH=N_G(H)/H$. The assignment $\omega_H: X\to X^H$ induces a ring homomorphism

$$\omega_H : A(G) \longrightarrow A(WH).$$

If H is normal in G, then the natural projection $p: G\to G/H$ induces an injective ring homomorphism $p^*: A(G/H)\to A(G)$ (cf. [2] Theorem 4.4). So we can view the group $A(G/H)^*$ as a subgroup of $A(G)^*$.

THEOREM 2.2. *We have the following* (i) *and* (ii).

(i)
$$A(G)^{+}=\Big(\prod_{(H)\in \mathrm{Min}(G)} A(G/H)^{+}\Big)\cdot A(G)_0^{+}$$

and

$$\Big(\prod_{(H)\in \mathrm{Min}(G)} A(G/H)^{+}\Big)\cap A(G)_0^{+}=\{1_{A(G)}\}.$$

(ii) *If V is an irreducible faithful real representation of G, then $u(V)\in A(G)_0^+$. Moreover, if $\alpha\in(A(G)_0^+\cap \mathrm{Image}\, u)$, then $\alpha=u(V_1+\cdots+V_r)$ for some irreducible faithful real representations $V_1, \cdots, V_r$ of G.*

PROOF OF (i). Suppose that $\alpha\in A(G)^+$. We put

$$\alpha=\sum_{(H)\in\Phi(G)} n_{(H)}[G/H] \qquad (n_{(H)}\in \boldsymbol{Z}),$$

and $\mathrm{Min}(G)=\{H_1, \cdots, H_s\}$. Since $(G/H)^F$ is non-empty if and only if F is conjugate to a subgroup of H in G,

$$\omega_{H_1}(\alpha)=\sum_{H\supset H_1} n_{(H)}[G/H] \quad \text{and} \quad \omega_{H_1}(\alpha)\in A(G/H_1)^{+}.$$

We put inductively,

$$\alpha_i=\alpha\prod_{j=1}^{i}\omega_{H_j}(\alpha_{j-1}) \qquad (i=1, \cdots, s),$$

where $\alpha_0=\alpha$. Then $\alpha_s\in A(G)_0^+$, $\omega_{H_j}(\alpha_{j-1})\in A(G/H_j)^+$ and

$$\alpha=\left(\prod_{j=1}^{s}\omega_{H_j}(\alpha_{j-1})\right)\cdot\alpha_s .$$

RROOF OF (ii). For a non-trivial normal subgroup H of G, V^H is a real representation of G/H, so $V^H=\{0\}$. It follows that $u(V)\in A(G)_0^+$. The last part is obtained by (i). Q. E. D.

COROLLARY 2.3. *We have*

$$A(G)^+=\prod_{H\text{ is normal}} A(G/H)_0^+ .$$

COROLLARY 2.4. $u: RO(G)\to A(G)^*$ *is surjective if and only if* $A(G/H)_0^+\subset$ Image$(u: RO(G/H)\to A(G/H)^*)$ *for any normal subgroup* H *of* G.

LEMMA 2.5. *Let* $C(G)$ *be the center of* G. *If* $|C(G)|\geqq 3$, *then* $A(G)_0^+=\{1_{A(G)}\}$.

PROOF. For a maximal element (H) of $S(G)-\{G\}$, if $\alpha\in A(G)_0^+$, then

$$\alpha((H))=1+n_H|WH|=\pm 1 ,$$

for some $n_H\in \boldsymbol{Z}$. Since $|C(G)|\geqq 3$, we have $|WH|\geqq 3$. It follows that $\alpha=1_{A(G)}$. Q. E. D.

2.6. Let $1\to H\to G\to K\to 1$ be a group extension. We define K-action on $A(H)$ as follows. For each $g\in G$, let $\bar{g}: H\to H$ be the automorphism defined by $\bar{g}(h)=ghg^{-1}$. We define G-action on $A(H)$ by

$$g\cdot\alpha=\bar{g}^*(\alpha) \qquad (g\in G \text{ and } \alpha\in A(H)) .$$

Then H acts trivially on $A(H)$. Therefore $K=G/H$ acts on $A(H)$. Similarly K acts on $RO(H)$.

2.7. For a subgroup H of G, let

$$\mathrm{Res}_H^G: A(G)\longrightarrow A(H) \quad\text{and}\quad \mathrm{Res}_H^G: RO(G)\longrightarrow RO(H)$$

be the natural restriction ring homomorphisms. We put $\mathrm{Res}_H^{G*}=\mathrm{Res}_H^G|_{A(G)^*}$. Let $1\to H\to G\to K\to 1$ be a group extension and X a finite G-set, then $\tilde{g}: \bar{g}^*(X)\to X$ (defined by $\tilde{g}(x)=g\cdot x$) is an H-isomorphism for any $g\in G$. It follows that

$$\text{Image } \mathrm{Res}_H^{G*}\subset(A(H)^*)^K .$$

Similarly,

$$\text{Image }(\mathrm{Res}_H^G: RO(G)\longrightarrow RO(H))\subset RO(H)^K .$$

For each real representation V of G, the diagram

$$\begin{array}{ccc}\Phi(H) & \xrightarrow{\ i_*\ } & \Phi(G)\\ & {\scriptstyle u(\mathrm{Res}_H^G(V))}\searrow \quad \swarrow{\scriptstyle u(V)} & \\ & Z & \end{array}$$

is commutative, where i_* is a mapping induced by the inclusion map $i: H\to G$.

Therefore, the diagram

(2.7.1)
$$\begin{array}{ccc} RO(G) & \xrightarrow{\mathrm{Res}^G_H} & RO(H)^K \\ u\downarrow & & \downarrow u \\ A(G)^* & \xrightarrow{\mathrm{Res}^{G*}_H} & (A(H)^*)^K \end{array}$$

is commutative.

LEMMA 2.8. *Let* $1\to H\to G\to K\to 1$ *be a group extension. If* $|K|$ *is odd, then* Res^{G*}_H *is injective.*

PROOF. It is sufficient to prove that

$$(1_{A(G)}+(\mathrm{Res}^G_H)^{-1}(0))\cap A(G)^*=\{1_{A(G)}\}.$$

Suppose that $(1_{A(G)}+\alpha)\in A(G)^*$ and $\mathrm{Res}^G_H(\alpha)=0$. For any subgroup L of G, the diagram

$$\begin{array}{ccc} A(G) & \xrightarrow{\mathrm{Res}^G_H} & A(L) \\ \downarrow \mathrm{Res}^G_H & & \downarrow \mathrm{Res}^L_{H\cap L} \\ A(H) & \xrightarrow{\mathrm{Res}^H_{H\cap L}} & A(H\cap L) \end{array}$$

is commutative, and $|L/H\cap L|$ is odd. Therefore, by induction on $|G|$, we can assume that $\alpha((L))=0$ for every proper subgroup L of G. Suppose that $\alpha(G)\neq 0$. Since $(1_{A(G)}+\alpha)\in A(G)^*$, $\alpha(G)=-2$. Let K_0 be a maximal subgroup of K and L its pre-image. Then

$$\alpha((L))=-2+m|(G/L)^L|=-2+m|G/L|=0$$

for some integer m. Since $|G/L|$ is odd, $\alpha((L))\neq 0$. This contradiction implies that $\alpha(G)=0$. That is, $\alpha=0$. Q. E. D.

THEOREM 2.9. *Let* $1\to H\to G\to K\to 1$ *be a group extension. If* $|K|$ *is odd and* $u: RO(H)\to A(H)^*$ *is surjective, then* $u: RO(G)\to A(G)^*$ *is surjective and* $\mathrm{Res}^{G*}_H: A(G)^*\to (A(H)^*)^K$ *is an isomorphism.*

PROOF. Since $u: RO(H)\to A(H)^*$ is surjective, $u: RO(H)^K\to (A(H)^*)^K$ is surjective. Let V be a K-invariant real representation of H. Then we observe that

$$\mathrm{Res}^G_H(\boldsymbol{R}G \underset{\boldsymbol{R}H}{\otimes} V)=|G/H|\cdot V \quad \text{and} \quad |G/H| \text{ is odd.}$$

It follows that $RO(G)\to RO(H)^K/2\cdot RO(H)^K$ is surjective. Therefore $u\cdot\mathrm{Res}^G_H$: $RO(G)\to (A(H)^*)^K$ is surjective. By the commutative diagram (2.7.1) and Lemma 2.8, the desired result follows. Q. E. D.

COROLLARY 2.10. *Let* $1\to H\to G\to K\to 1$ *be a group extension. If* $|K|$ *is odd and* H *is an abelian group, then* $u: RO(G)\to A(G)^*$ *is surjective and* $|A(G)^*|=2^{m+1}$, *where* $m=|\{(H_0)\in\Phi(G)|H\supset H_0$ *and* $|H/H_0|=2\}|$.

PROOF. It is trivial by 1.5 and Theorem 2.9.

THEOREM 2.11. *Let* $1\to H\to G\to K\to 1$ *be a split group extension. If* $|K|$ *is odd, then* $\mathrm{Res}_H^G: A(G)\to A(H)^K$ *is a split epimorphism and* $\mathrm{Res}_H^{G*}: A(G)^*\to(A(H)^*)^K$ *is an isomorphism.*

PROOF. As an abelian group, $A(H)^K$ is generated by K-orbits of $[H/H_0]$'s. We put $K_0=\{k\in K|kH_0k^{-1}\subset H_0\}$. By the Mackey double coset formula, $\mathrm{Res}_H^G([G/K_0\cdot H_0])$ is the sum of K-orbit of $[H/H_0]$. Therefore Res_H^G is a split epimorphism. By Lemma 2.8, Res_H^{G*} is an isomorphism. Q. E. D.

3. Examples.

EXAMPLE 3.1. Let D_m be a dihedral group of order $2m$. We put $G=D_{m_1}\times\cdots\times D_{m_r}$. If $m_1,\cdots,m_r$ are relatively prime integers and $m_i>1$ $(i=1,\cdots,r)$, then $u: RO(G)\to A(G)^*$ is surjective. Moreover, by (1.3.1), we have

$$|A(G)^*|=2^\rho,$$

where

$$\rho=\begin{cases}(d(m_1)+2)\prod_{j=2}^{r}(d(m_j)+1) & \text{if } m_1 \text{ is even}\\ \prod_{j=1}^{r}(d(m_j)+1) & \text{if } m_j \text{ is odd for each } j,\end{cases}$$

and $d(m)=|\{i\mid i \text{ is a positive divisor of } m\}|$ (cf. [2] Example 5.7).

PROOF. By Corollary 2.4 and Lemma 2.5, it is sufficient to prove that

$$A(G')_0^+\subset\mathrm{Image}(u: RO(G')\longrightarrow A(G')^*) \tag{3.1.1}$$

for each homomorphic image G' of G such that $|C(G')|\leqq 2$. G' has one of the following three types of groups (mutually exclusive).

(I) $D_{m_1}\times\cdots\times D_{m_r}$, where $m_1,\cdots,m_r$ are relatively prime odd integers and $m_i>1$ $(i=1,\cdots,r)$.

(II) $C_2\times H$, where H has type (I) and C_2 is a cyclic group of order 2.

(III) $D_{m_1}\times\cdots\times D_{m_r}$, where $m_1,\cdots,m_r$ are relatively prime integers, $m_i>1$ $(i=1,\cdots,r)$ and $4|m_1$.

If G' has type (II), then (3.1.1) is true by the following Lemma 3.2. For the other two types, it will be proved by the same way.

LEMMA 3.2. *If* $m_1,\cdots,m_r$ *are relatively prime odd integers, then* (3.1.1) *is true for* $G=C_2\times D_{m_1}\times\cdots\times D_{m_r}$.

PROOF. We put

$$D_{m_i}=\langle \sigma_i, \tau_i \mid \sigma_i^{m_i}=\tau_i^2=1 \textit{ and } \tau_i^{-1}\cdot\sigma_i\cdot\tau_i=\sigma_i^{-1}\rangle ,$$

$$C_2=\langle\mu\rangle \quad\text{and}\quad L=\langle\mu, \tau_i \mid i=1, \cdots, r\rangle .$$

Since each subgroup of $\langle\mu\cdot\tau_1\cdot\tau_2\cdot\cdots\cdot\tau_r\rangle$ is normal in G, if $(H)\in(S(G)-\{G\})$, then H is an elementary abelian 2-group conjugate to a subgroup of L. Suppose that $\alpha\in A(G)_0^+$. We can put

$$\alpha=(\sum_{H\subset L \text{ and } \mu\notin H} n_H[G/H])+1_{A(G)} ,$$

where $n_H\in \boldsymbol{Z}$. Let $L_1, \cdots, L_s$ be all subgroups of L such that $\mu\notin L_i$ and $|L/L_i|=2$ for each i. Considering $\alpha((L_i))$, we have

$$n_{L_i}=0 \quad\text{or}\quad -1 \qquad \text{for each } i .$$

Moreover, we have

(3.2.1) $\qquad$ if $n_{L_i}=0$ for some i, $\quad$ then $\alpha=1_{A(G)}$.

PROOF OF (3.2.1). We proceed by induction on r. If $r=1$, then

$$A(G)_0^+=\{1_{A(G)}, (1_{A(G)}-[G/\langle\tau_1\rangle]-[G/\langle\mu\cdot\tau_1\rangle]+[G])\} .$$

So, (3.2.1) is true for $r=1$. Suppose that $n_{L_1}=0$ and $r>1$. We observe that

$$\omega_{\langle\tau_1\rangle}(\alpha)=1_{A(G')}+(\sum_{\mu\notin H\subset L \text{ and } \tau_1\in H} n_H[G'/H']) ,$$

where $G'=C_2\times D_{m_2}\times\cdots\times D_{m_r}$ and $H'=H/\langle\tau_1\rangle$. By the assumption of induction, if $\tau_1\in L_1$, $\mu\notin H\subset L$ and $\tau_1\in H$, then $n_H=0$. In particular,

$$n_{\langle\tau_1,\cdots,\tau_r\rangle}=0 \qquad \text{if } \tau_1\in L_1 .$$

Similarly,

$$n_{\langle\tau_1,\cdots,\tau_r\rangle}=0 \qquad \text{if } \tau_i\in L_i \text{ for some } i .$$

Therefore we have

$$n_H=0 \qquad \text{if } \tau_i\in H \text{ for some } i .$$

If L_2 is a subgroup of L such that $|L/L_2|=2$ and $L_2\cap\{\mu, \tau_1, \cdots, \tau_r\}$ is empty, then $L_2=\langle\mu\cdot\tau_1, \cdots, \mu\cdot\tau_r\rangle$. For a maximal proper subgroup H of L_2,

$$\alpha((H))=1+n_{L_2}|(G/L_2)^H|+n_H|(G/H)^H|=\pm 1 .$$

Since $|(G/L_2)^H|$ is even and $|(G/H)^H|$ is divisible by 4, $n_H=0$. It follows that $\alpha=1_{A(G)}$ or $(1_{A(G)}-[G/L_2])$. Since $(1_{A(G)}-[G/L_2])$ is not in $A(G)^*$, $\alpha=1_{A(G)}$. Therefore we obtain (3.2.1).

Let V_i $(i=1, \cdots, r)$ be the real representation of D_{m_i} $(i=1, \cdots, r)$ determined by

$$\sigma_i \longrightarrow \begin{pmatrix} \xi_{m_i} & 0 \\ 0 & \xi_{m_i}^{-1} \end{pmatrix}, \quad \tau_i \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where ξ_m is a primitive m-th root of 1. We put $V=V_1\times\cdots\times V_r$. Then V is an irreducible faithful real representation of G, where μ acts on V by $\mu(v)=-v$. If $\alpha\in A(G)_0^+$ and $\alpha\neq 1_{A(G)}$, then

$$\alpha\cdot u(V)=1_{A(G)} \qquad \text{(by (3.2.1))}.$$

Q. E. D.

EXAMPLE 3.3. Let $1\rightarrow C_p\times C_p\rightarrow G\rightarrow C_2\rightarrow 1$ be a split group extension, where p is an odd prime and C_p is a cyclic group of order p. We put $C_p\times C_p=\langle\sigma_1\rangle\times\langle\sigma_2\rangle$ and $C_2=\langle\tau\rangle$. If $\tau^{-1}\cdot\sigma_i\cdot\tau=\sigma_i^{-1}$ $(i=1, 2)$, then $u: RO(G)\rightarrow A(G)^*$ is not surjective.

PROOF. Any subgroup of $C_p\times C_p$ is normal in G. It follows that there is no irreducible faithful real representation of G. Since $(1_{A(G)}-2[G/\langle\tau\rangle]+[G])$ is an element of $A(G)_0^+$, the desired result follows from 2.2, (ii). Q. E. D.

Similarly, For each of the following groups G, $u: RO(G)\rightarrow A(G)^*$ is not surjective:

$D_p\times D_p$ (p is an odd prime), $D_4\times D_4$ and D_4*D_4
($*$ means the central product).

EXAMPLE 3.4. We put

$$C_{15}=C_3\times C_5=\langle\sigma_1\rangle\times\langle\sigma_2\rangle \quad\text{and}\quad \mathrm{Aut}(C_{15})=C_2\times C_4=\langle\tau_1\rangle\times\langle\tau_2\rangle.$$

Moreover, we put

$$H=\langle\sigma_1, \sigma_2, \tau_1\cdot\tau_2\rangle \quad\text{and}\quad G=\langle\sigma_1, \sigma_2, \tau_1, \tau_2\rangle.$$

Then $u: RO(H)\rightarrow A(H)^*$ is not surjective and $u: RO(G)\rightarrow A(G)^*$ is surjective.

PROOF. We put $\sigma=\sigma_1\cdot\sigma_2$ and $\tau=\tau_1\cdot\tau_2$. Since $(1_{A(H)}-2[H/\langle\tau\rangle]+[H/\langle\tau^2\rangle])$ is an element of $A(H)_0^+$, it is sufficient to prove that there is no absolutely irreducible faithful real representation of H. Since

$$\boldsymbol{Q}C_{15}\cong\boldsymbol{Q}[\xi_{15}]+\boldsymbol{Q}[\xi_5]+\cdots,$$

every irreducible faithful representation appears in $\boldsymbol{Q}[\xi_{15}][\langle\tau\rangle]$, where ξ_m is a primitive m-th root of 1 and $\boldsymbol{Q}[\xi_{15}][\langle\tau\rangle]$ is a twisted group ring. Since $\tau_1\cdot\tau_2^2$ is the complex conjugation and $\langle\tau_1\cdot\tau_2^2\rangle\not\subset\langle\tau\rangle$, no absolutely irreducible faithful representation of H is defined over $\boldsymbol{R}$. It follows that $u: RO(H)\rightarrow A(H)^*$ is not surjective. The surjectivity of $u: RO(G)\rightarrow A(G)^*$ will be proved by the use of Corollary 2.4 and by similar calculations as in Example 3.1. Q. E. D.

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