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Remarks on Sobolev inequalities and stability of minimal submanifolds

Dedicated to Professor Shigeo Sasaki on his 70th birthday

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Sobolev inequalities for Riemannian submanifolds have applications to isoperimetric inequalities, estimates of the first eigenvalue, and the stability of minimal submanifolds. Here we give an improvement of constant of the Sobolev inequality by Hoffman-Spruck-Otsuki and apply it to the stability of minimal submanifolds.

Since the estimates of stability using the Sobolev inequality require the condition that the volume of a domain D is very small, our improvement is now effective. Especially, for the case where $\dim D=2$, to get stability estimate one needs additional estimates, i.e., the estimation of the first eigenvalue. So Hoffman's stability theorem (Theorem 5, (ii), [1]) contains much loss. In this article we carry the volume estimation and we get a nice improvement as Theorem D. As a corollary we obtain

COROLLARY. *Let M be a minimal surface of a unit sphere S^n and D be a compact domain of M. If*

$$\int_D (4-2K)^2 dM < 1/2c_3(2, \alpha_2)^2,$$

then D is stable in S^n, where K denotes the Gauss curvature of M and

$$\alpha_2=(9-\sqrt{57})/2,$$

$$\gamma^{-1}=\{\mathrm{Vol}(D)/(1-\alpha_2)\pi\}^{1/2},$$

$$c_3(2, \alpha_2)=\gamma\cdot\sin^{-1}(1/\gamma)\cdot 2(3-\alpha_2)/\alpha_2(1-\alpha_2)^{1/2}\pi^{1/2}.$$

§ 1. Sobolev inequality for submanifolds.

First we state a Sobolev inequality for submanifolds obtained by D. Hoffman-J. Spruck [2] and T. Otsuki [4]. Let $M\to\bar{M}$ be an isometric immersion of a

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Riemannian manifold M of dimension m into a Riemannian manifold $\bar{M}$ of dimension n. We use the following quantities:

$\bar{K}$=sectional curvature in $\bar{M}$,
H=mean curvature vector field of the immersion,
$R(\bar{M}, M)$=injectivity radius of $\bar{M}$ restricted to M,
w_m=volume of the unit ball in Euclidean m-space,
b=a positive real number or pure imaginary one.

Sobolev inequality. Assume that $\bar{K}\leqq b^2$ and let h be a non-negative C^1-function on M with compact support D and $h|\partial M=0$. Then

$$\left[\int_D h^{m/(m-1)}dM\right]^{(m-1)/m}\leqq c(m)\int_D(|\nabla h|+mh|H|)dM \tag{1.1}$$

holds, provided

$$b\theta(\alpha)\leqq 1, \tag{1.2}$$

$$2\rho_0\leqq R(\bar{M}, M), \tag{1.3}$$

where

$$\theta(\alpha)=\{\mathrm{Vol}(D)/(1-\alpha)w_m\}^{1/m}, \tag{1.4}$$

$$\rho_0=\rho_0(\alpha)=b^{-1}\sin^{-1}[b\theta(\alpha)] \quad \text{for} \quad b: \text{real} \tag{1.5}$$

$$=\theta(\alpha) \qquad \text{for} \quad b: \text{imaginary} \tag{1.6}$$

and α is a parameter, $0<\alpha<1$, and

$$c(m)=c_1(m, \alpha)=P2^m[1/\alpha(1-\alpha)^{1/m}](m/(m-1))w_m^{-1/m}, \tag{1.7}$$

$$c(m)=c_2(m, \alpha)=P\frac{(m-\alpha)2^{m-1}-(1-\alpha)}{(m-1)\alpha(1-\alpha)^{1/m}}(m/(m-1))w_m^{-1/m}, \tag{1.8}$$

where $P=\pi/2$ if b is real, and $P=1$ if b is imaginary. (1.7) is obtained in [**2**], and (1.8) is in [**4**]. (1.8) for the case (b: imaginary) was not explained in [**4**]. But it is easy to check it.

REMARK. (i) For the case where b is a real number, (1.2) and (1.4) imply that $\mathrm{Vol}(D)<b^{-m}w_m$.

(ii) $c_2(m, \alpha)<c_1(m, \alpha)$ holds, and the optimal choice of α to minimize $c_1(m, \alpha)$ is $\alpha_1=m/(m+1)$ and α to minimize $c_2(m, \alpha)$ is

$$\alpha_2=\{(m+1)(2^{m-1}m-1)-[(m+1)^2(2^{m-1}m-1)^2 -4m(2^{m-1}m-1)(2^{m-1}-1)]^{1/2}\}/2(2^{m-1}-1). \tag{1.9}$$

Therefore, the effective choice of α is as follows: if $b\theta(\alpha_2)\leqq 1$, then $\alpha=\alpha_2$, and if $b\theta(\alpha_2)>1$ then α is determined by $b\theta(\alpha)=1$.

As an improvement we obtain the following.

THEOREM A. *Assume that* $\bar{K}\leqq b^2$ *(b: real) and let h be a non-negative* C^1-*function on M such that*

(i) $\operatorname{supp} h=D$ *is compact and* $\operatorname{Vol}(D)<b^{-m}w_m$,

(ii) $h|\partial M=0$.

Let α_2 $(0<\alpha_2<1)$ *be the real number defined by* (1.9) *which minimizes* $[(m-\alpha)2^{m-1}-(1-\alpha)]/\alpha(1-\alpha)^{1/m}$. *Then*

$$\left[\int_D h^{m/(m-1)}dM\right]^{(m-1)/m}\leqq c_3(m,\ \alpha^*)\int_D(|\nabla h|+mh|H|)dM \tag{1.10}$$

holds, provided

$$b\theta(\alpha^*)=1/\gamma\leqq 1\,,$$

$$2\rho_0(\alpha^*)\leqq R(\bar{M},\ M)\,,$$

where $0<\alpha^*\leqq\alpha_2<1$, *and*

$$\theta(\alpha^*)=\{\operatorname{Vol}(D)/(1-\alpha^*)w_m\}^{1/m}\,,$$

$$\rho_0(\alpha^*)=b^{-1}\sin^{-1}[b\theta(\alpha^*)]\,,$$

$$c_3(m,\ \alpha^*)=\gamma\cdot\sin^{-1}(1/\gamma)\cdot\frac{(m-\alpha^*)2^{m-1}-(1-\alpha^*)}{(m-1)\alpha^*(1-\alpha^*)^{1/m}}(m/(m-1))w_m^{-1/m}\,. \tag{1.11}$$

PROOF. Almost all parts of proof for Theorem A are the same as ones for $\alpha=\alpha^*$ in [2] and [4]. Only one difference is the use of

$$\sin^{-1}\theta\leqq[\gamma\cdot\sin^{-1}(1/\gamma)]\theta \qquad (0\leqq\theta\leqq 1/\gamma\leqq 1)$$

instead of $\sin^{-1}\theta\leqq\pi\theta/2$ in the part (p. 726, ↑2∼p. 727, ↓4) of [2]. Q. E. D.

REMARK. Effective choice of α^* minimizing $c_3(m,\ \alpha^*)$ depends on $\operatorname{Vol}(D)$. So, $\min c_3(m,\ \alpha^*)=c_3(m,\operatorname{Vol}(D))$. If $\operatorname{Vol}(D)$ is sufficiently small, then $\min c_3(m,\ \alpha^*)=c_3(m,\ \alpha_2)$.

EXAMPLE. For $b=1$ and $m=7$ the most effective case for $c_2(7,\ \alpha)$ is as follows:

$$w_7=16\pi^3/105\doteqdot 4.72477$$

$$\alpha_2\doteqdot 0.88892$$

$$\operatorname{Vol}(D)=w_7(1-\alpha_2)\doteqdot 0.5248$$

$$c_2(7,\ \alpha_2)\doteqdot 147.3$$

In this case, if we put $\alpha^*=0.7982$, then $\gamma\doteqdot 1.089$, and

$$c_3(7,\ \alpha^*)\doteqdot 123.3$$

If $\operatorname{Vol}(D)$ becomes smaller, then $c_3(7,\ \alpha^*)$ gets smaller.

§ 2. Lower bounds on λ_1 for submanifolds.

For a domain D of a Riemannian manifold M, let $\lambda_1(D)$ denote the first eigenvalue of the Laplacian acting on functions (with Dirichlet condition). Theorem 3 in [1] is improved as follows:

THEOREM B. *Let M be a submanifold of $\bar{M}$ whose curvature is bounded above by a positive constant b^2. Let D be a compact domain of M. Assume the following:*

$$m|H|\leqq\kappa, \qquad \kappa^m \operatorname{Vol}(D)\leqq c_3(m, \alpha^*)^{-m},$$

$$b\theta(\alpha^*)=1/\gamma\leqq 1,$$

$$\rho_0(\alpha^*)=b^{-1}\sin^{-1}[b\theta(\alpha^*)]\leqq R(\bar{M}, M)/2,$$

$$\theta(\alpha^*)=\{\operatorname{Vol}(D)/(1-\alpha^*)w_m\}^{1/m}.$$

Then

$$\lambda_1(D)\geqq[c_3(m, \alpha^*)^{-1}(\operatorname{Vol}(D))^{-1/m}-\kappa]^2/4. \tag{2.1}$$

Proof is similar to one in [1] and we use the Sobolev inequality stated in Theorem A. If M is minimal in $\bar{M}$, then we put $\kappa=0$ in (2.1).

§ 3. Stability of minimal submanifolds.

The second fundamental form of a submanifold M of $\bar{M}$ is denoted by B and its norm is denoted by $|B|$.

PROPOSITION 3.1 ([1], [3]). *Let $\bar{M}$ be a Riemannian manifold with the following properties:*

(i) $\bar{K}\leqq b^2$ *(b: real),*

(ii) *injectivity radius of* $\bar{M}\geqq b^{-1}\pi$.

Let M ($m\geqq 3$) be a minimal submanifold of $\bar{M}$ and D be a compact domain of M. If

$$\left[\int_D(|B|^2+mb^2)^{m/2}dM\right]^{1/m}<(m-2)/2(m-1)c_2(m, \alpha) \tag{3.1}$$

then D is stable in $\bar{M}$, where $c_2(m, \alpha)$ is given by (1.8).

REMARK. In the paper [3] one finds a mistake in the coefficient $c(m, \alpha)$ of the Sobolev inequality at page 12 and hence $c_2(m)$ at the same page should be multiplied by $1/\sqrt{2}$.

Contrary to the starting assumption $\operatorname{Vol}(D)\leqq(1-\alpha)b^{-m}w_m$ of the Sobolev inequality, (3.1) requires stronger restriction on $\operatorname{Vol}(D)$. In fact, by $mb^2\leqq|B|^2+mb^2$ and

$$c(m)>[(m-\alpha)2^{m-1}-(1-\alpha)]m/(m-1)^2\alpha(1-\alpha)^{1/m}w_m^{1/m}$$

(where $c(m)$ is one of $c_2(m, \alpha)$, $c_3(m, \alpha)$), (3.1) implies

(3.2) $$b[\mathrm{Vol}(D)/(1-\alpha)w_m]^{1/m}<(m-1)(m-2)\alpha/2m^{3/2}[(m-\alpha)2^{m-1}-(1-\alpha)],$$

and hence,

(3.3) $$b[\mathrm{Vol}(D)/w_m]^{1/m}<(m-1)(m-2)/2m^{3/2}[(m-1)2^{m-1}-1].$$

For example, if $b=1$ and $m=7$, (3.3) implies

$$\mathrm{Vol}(D)<8.94\times 10^{-19}.$$

This shows that stability theorem (Proposition 3.1 above) works only for very small domains of a minimal submanifold M of $\overline{M}$.

If we apply Theorem A to stability of a minimal submanifold, we may put $\alpha^*=\alpha_2$, since the right hand side of (3.2) for $\alpha=\alpha_2$ is smaller than 1.

THEOREM C. *Let $\overline{M}$ be a Riemannian manifold with the properties* (i) *and* (ii) *of Proposition* 3.1. *Let M* $(m\geqq 3)$ *be a minimal submanifold of $\overline{M}$ and D be a compact domain of M. If*

(3.4) $$\left[\int_D(|B|^2+mb^2)^{m/2}dM\right]^{1/m}<(m-2)/2(m-1)c_3(m,\alpha_2),$$

then D is stable in $\overline{M}$, where

$$c_3(m,\alpha_2)=\gamma\cdot\sin^{-1}(1/\gamma)\cdot\frac{(m-\alpha_2)2^{m-1}-(1-\alpha_2)}{(m-1)\alpha_2(1-\alpha)^{1/m}}(m/(m-1))w_m^{-1/m}.$$

$$\gamma^{-1}=b\{\mathrm{Vol}(D)/(1-\alpha_2)w_m\}^{1/m}.$$

EXAMPLE. For $b=1$, $m=7$ and $\mathrm{Vol}(D)=8\times 10^{-19}$, we get

$$\gamma\doteqdot 350.974$$

$$\gamma\cdot\sin^{-1}(1/\gamma)\doteqdot 1.000001$$

Then $c_3(7,\alpha_2)\doteqdot 93.779$ and $1/c_3(7,\alpha_2)\doteqdot 0.01066$. On the other hand, we get $1/c_2(7,\alpha_2)\doteqdot 0.00679$.

THEOREM D. *For $m=2$, under the same assumption as in Theorem* C, *if*

(3.5) $$\int_D(|B|^2+2b^2)^2dM<\lambda_1(D)/4c_3(2,\alpha_2)^2,$$

or

(3.6) $$\int_D(|B|^2+2b^2)^2dM<b^2/2c_3(2,\alpha_2)^2,$$

then D is stable in $\overline{M}$.

Proof for (3.5) is similar to one in page 69 of [1] where some misprints should be corrected ($\|\beta\|_2\to\|\beta^2\|_2$, etc.).

If one assumes (3.6) one gets

$$b^2 \operatorname{Vol}(D) < 1/8c_3(2, \alpha_2)^2 . \tag{3.7}$$

By (3.6) and (3.7), we obtain

$$\int_D (|B|^2 + 2b^2)^2 dM < 1/16c_3(2, \alpha_2)^4 \operatorname{Vol}(D) . \tag{3.8}$$

By (3.8) and Theorem B we get (3.5). This completes the proof.

COROLLARY. *In Theorem* D, *if $\overline{M}$ is a unit sphere S^n and if*

$$\int_D (4-2K)^2 dM < 1/2c_3(2, \alpha_2)^2 , \tag{3.9}$$

then D is stable in S^n, where K denotes the Gauss curvature of M.

PROOF. By Gauss equation we get $|B|^2+2=4-2K$.

REMARK. D. Hoffman's estimate is as follows: If

$$\int_D (|B|^2 + 2b^2)^2 dM < (1-\alpha)\alpha^4 b^2/(16\pi)^3 , \tag{3.10}$$

then D is stable (p. 69, ↓6, [1]).

REMARK. Let $b=1$. Since $c_3(2, \alpha_2) > 2(3-\alpha_2)/\alpha_2(1-\alpha_2)^{1/2}\pi^{1/2} \doteqdot 6.752$ ($\alpha_2 \doteqdot 0.725$), (3.6) or (3.7) implies $\operatorname{Vol}(D) < 0.002742$. For example, consider the case where $\operatorname{Vol}(D) = 0.0027$. Then

$$\theta(\alpha_2) = \{\operatorname{Vol}(D)/(1-\alpha_2)\pi\}^{1/2} \doteqdot 0.0559$$

Since $\gamma = 1/\theta(\alpha_2) \doteqdot 17.8879$ and $\gamma \cdot \sin^{-1}(1/\gamma) \doteqdot 1.00052$, we get

$$c_3(2, \alpha_2) \doteqdot 6.756$$

Therefore $\int_D (|B|^2+2)^2 dM < 0.01095$ implies that D is stable in S^n.

On the other hand, Hoffman type estimate (3.10) requires

$$\operatorname{Vol}(D) < 0.000000129 \qquad (\alpha = \alpha_1 = 2/3) .$$

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