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# On the Gauss map of a complete minimal surface in $R^m$

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## § 1. Introduction.

Let $x: M \rightarrow R^m$ be a (connected, oriented) minimal surface immersed in $R^m$ ($m \geqq 3$). We may consider $M$ as a Riemann surface by associating a holomorphic local coordinate $z=u+iv$ with each positive isothermal local coordinates $u, v$. We denote by $G$ the (generalized) Gauss map of $M$, which is a map of $M$ into $P^{m-1}(C)$ defined by $G=\pi \cdot(\partial x / \partial \bar{z})$, where $\pi$ is the canonical projection of $C^m-\{0\}$ onto $P^{m-1}(C)$. It is well-known that the map $f=\bar{G}$, the conjugate of $G$, is holomorphic and the image $f(M)$ is contained in the complex quadric $Q_{m-2}(C)$ in $P^{m-1}(C)$ (cf., [7], p. 110). Note that, when $m=3$, we may identify $Q_1(C)$ with the Riemann sphere and the map $f$ may be regarded as a meromorphic function on $M$.

In [9], R. Osserman showed that the Gauss map of a complete non-flat minimal surface in $R^3$ cannot omit a set of positive logarithmic capacity in $Q_1(C)$. Subsequently, in [3], S. S. Chern and R. Osserman proved that the Gauss map of a complete minimal surface $M$ of finite total curvature can fail to intersect at most $(m-1)(m+2)/2$ hyperplanes in general position if it is non-degenerate. Moreover, they showed that the Gauss map of a non-flat complete minimal surface in $R^m$ intersects a dense set of hyperplanes. Recently, in [14], F. Xavier obtained a remarkable result that the Gauss map of a complete non-flat minimal surface in $R^3$ cannot omit 7 points in $Q_1(C)$.

Relating to these results, we shall prove the following theorem in this paper.

MAIN THEOREM. *Let $M$ be a complete minimal surface in $R^m$. If the Gauss map of $M$ is non-degenerate, it can fail to intersect at most $m^2$ hyperplanes in general position.*

It is a very interesting problem to obtain the best estimate $q(m)$ ($\leqq m^2$) of the number of hyperplanes having the property in Main Theorem. In the case $m=3$, R. Osserman showed that there exists a non-flat complete minimal surface in $R^3$ whose Gauss map omits distinct 4 points ([9], p. 72). As its consequence, there exists a complete minimal surface in $R^3$ whose Gauss map, as a map into

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$P^2(\boldsymbol{C})$, is non-degenerate and omits 6 hyperplanes in general position. For, with distinct 4 points $a_1, \cdots, a_4$ in $Q_1(\boldsymbol{C})$ we can associate 6 lines $H_1, \cdots, H_6$ in $P^2(\boldsymbol{C})$ located in general position such that $(\bigcup_{i=1}^{6} H_i) \cap Q_1(\boldsymbol{C}) = \{a_1, \cdots, a_4\}$. Actually, the lines $H_1 = \overline{a_1 a_1}, \cdots, H_4 = \overline{a_4 a_4}$, $H_5 = \overline{a_1 a_2}$ and $H_6 = \overline{a_3 a_4}$ satisfy this condition after suitable changes of indices, where $\overline{a_i a_j}$ denotes the tangent to $Q_1(\boldsymbol{C})$ at $a_i$ if $i=j$ and the line containing $a_i$ and $a_j$ if $i \neq j$. This shows that $6 \leqq q(3) \leqq 9$.

The proof of Main Theorem is based on the result of S. T. Yau ([**15**]) as in [**14**] and those on the value distributions of holomorphic maps of the unit disc into $P^{m-1}(\boldsymbol{C})$. After preparing some results on value distributions of holomorphic maps in § 2 and a basic inequality in § 3, we shall give the proof of Main Theorem in § 4.

## § 2. Some properties of holomorphic maps into $P^n(\boldsymbol{C})$.

Let $f$ be a holomorphic map of the unit disc $\Delta := \{z \in \boldsymbol{C} : |z| < 1\}$ into $P^n(\boldsymbol{C})$. For arbitrary homogeneous coordinates $(w_1 : \cdots : w_{n+1})$ on $P^n(\boldsymbol{C})$, $f$ has a representation $f = (f_1 : \cdots : f_{n+1})$ with holomorphic functions $f_1, \cdots, f_{n+1}$ such that

$$\|f\|^2 := |f_1|^2 + \cdots + |f_{n+1}|^2$$

vanishes nowhere. In the following sections, such a representation of $f$ is referred to as a reduced representation of $f$. Set

$$u(z) := \max_{1 \leq j \leq n+1} \log |f_j(z)| .$$

The characteristic function (in the sense of H. Cartan [**2**]) of $f$ is defined by

$$T(r, f) := \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta - u(0) \qquad (0 \leqq r < 1) .$$

For a non-zero meromorphic function $\varphi$ on $\Delta$, the proximity function and the counting function of $\varphi$ are defined by

$$m(r, \varphi) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\varphi(re^{i\theta})| \, d\theta$$

$$N(r, \varphi) := \int_0^r \frac{n(t) - n(0)}{t} dt + n(0) \log r \qquad (0 < r < 1)$$

respectively, where $\log^+ x = \max(\log x, 0)$ for $x \geqq 0$ and $n(t)$ denotes the number of poles of $\varphi$ in $\{z \in \boldsymbol{C} : |z| \leqq t\}$ counted with multiplicity. We may regard $\varphi$ as a holomorphic map into $P^1(\boldsymbol{C})$. We then have

(2.1) (i) $T(r, \varphi) = m(r, \varphi) + N(r, \varphi) + O(1)$,

(ii) $T(r, 1/\varphi)=T(r, \varphi)+O(1)$.

For the proof, see [2] and [6], p. 5.

We have also

(2.2) *Let* $f: \Delta\rightarrow P^{n}(\boldsymbol{C})$ *be a holomorphic map with a reduced representation* $f=(f_{1}:\cdots:f_{n+1})$ *and*

$$H_{i}:\ a_{i}^{1}w_{1}+\cdots+a_{i}^{n+1}w_{n+1}=0 \qquad (i=1,2)$$

*be hyperplanes in* $P^{n}(\boldsymbol{C})$ *such that* $f(\Delta)\not\subset H_{i}$. *Then, for the meromorphic function* $\varphi:=\sum_{j=1}^{n+1}a_{1}^{j}f_{j}\Big/\sum_{j=1}^{n+1}a_{2}^{j}f_{j}$,

$$T(r, \varphi)\leqq T(r, f)+O(1).$$

For the proof, see [2], p. 10.

DEFINITION 2.3. A holomorphic map $f: \Delta\rightarrow P^{n}(\boldsymbol{C})$ is called *transcendental* if

$$\limsup_{r\rightarrow 1}\frac{T(r, f)}{\log\dfrac{1}{1-r}}=\infty.$$

As a result of the second main theorem of value distribution theory, we have

THEOREM 2.4. *Let* $f: \Delta\rightarrow P^{n}(\boldsymbol{C})$ *be a holomorphic map. Suppose that* $f$ *is non-degenerate, namely, the image of* $f$ *is not contained in any hyperplane in* $P^{n}(\boldsymbol{C})$, *and that* $f$ *omits* $n+2$ *hyperplanes in general position. Then* $f$ *is not transcendental.*

For the proof, see [4], p. 43 and [11], p. 88.

For later use, we give the following:

PROPOSITION 2.5. *Let* $\varphi$ *be a nowhere zero holomorphic function on* $\Delta$ *which is not transcendental. Then, for each positive integer* $l$, *there exists a positive constant* $K_{0}$ *such that*

$$\int_{0}^{2\pi}\left|\frac{d^{l-1}}{dz^{l-1}}\left(\frac{\varphi'}{\varphi}\right)(re^{i\theta})\right|d\theta\leqq\frac{K_{0}}{(1-r)^{l}}\log\frac{1}{1-r} \qquad (0<r<1).$$

PROOF. By assumption, $\log|\varphi(z)|$ is a harmonic function on $\Delta$. Therefore, for arbitrary $z=re^{i\theta}\in\Delta$ and $R$ with $r<R<1$,

$$\log|\varphi(z)|=\frac{1}{2\pi}\int_{0}^{2\pi}\log|\varphi(Re^{i\phi})|\frac{R^{2}-r^{2}}{R^{2}-2Rr\cos(\theta-\phi)+r^{2}}d\phi.$$

Choosing a branch of $\log\varphi(z)$ and a real constant $C$ properly, we have

$$\log\varphi(z)=\frac{1}{2\pi}\int_{0}^{2\pi}\log|\varphi(Re^{i\phi})|\frac{Re^{i\phi}+z}{Re^{i\phi}-z}d\phi+iC,$$

because

$$\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}=\mathrm{Re}\left(\frac{Re^{i\phi}+re^{i\theta}}{Re^{i\phi}-re^{i\theta}}\right).$$

Differentiating this equation $l$ times, we get

$$\frac{d^{l-1}}{dz^{l-1}}\left(\frac{\varphi'}{\varphi}\right)(z)=\frac{l!}{\pi}\int_0^{2\pi}\log|\varphi(Re^{i\phi})|\frac{Re^{i\phi}}{(Re^{i\phi}-z)^{l+1}}d\phi,$$

from which we obtain

$$\begin{aligned}&\int_0^{2\pi}\left|\frac{d^{l-1}}{dz^{l-1}}\left(\frac{\varphi'}{\varphi}\right)(re^{i\theta})\right|d\theta\\&\leqq\frac{l!R}{\pi}\int_0^{2\pi}d\theta\int_0^{2\pi}|\log|\varphi(Re^{i\phi})||\frac{1}{|Re^{i\phi}-re^{i\theta}|^{l+1}}d\phi\\&=\frac{l!R}{\pi}\int_0^{2\pi}\left(|\log|\varphi(Re^{i\phi})||\int_0^{2\pi}\frac{1}{|Re^{i\phi}-re^{i\theta}|^{l+1}}d\theta\right)d\phi.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{|Re^{i\phi}-re^{i\theta}|^{l+1}}&=\int_0^{2\pi}\frac{d\theta}{|R-re^{i\theta}|^{l+1}}\\&\leqq\frac{1}{(R-r)^{l-1}}\int_0^{2\pi}\frac{d\theta}{|R-re^{i\theta}|^2}\\&=\frac{2\pi}{(R-r)^{l-1}(R^2-r^2)}.\end{aligned}$$

Since $|\log|x||=\log^+x+\log^+(1/x)$ for $x>0$, (2.1) gives

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|\log|\varphi(Re^{i\phi})||d\phi&=m(R,\varphi)+m(R,1/\varphi)\\&\leqq 2T(R,\varphi)+O(1).\end{aligned}$$

By the assumption that $\varphi$ is not transcendental, we can easily conclude

$$\int_0^{2\pi}\left|\frac{d^{l-1}}{dz^{l-1}}\left(\frac{\varphi'}{\varphi}\right)(re^{i\theta})\right|d\theta=\frac{1}{(R-r)^l}O\left(\log\frac{1}{1-R}\right).$$

By taking $R=(1+r)/2$, we obtain the desired inequality.

## § 3. A basic inequality.

Let $f:\Delta\to P^n(\boldsymbol{C})$ be a non-degenerate holomorphic map and

$$H_j:\ a_j^1w_1+\cdots+a_j^{n+1}w_{n+1}=0\qquad(1\leqq j\leqq q)$$

be hyperplanes in general position. Taking a reduced representation $f=(f_1:\cdots:f_{n+1})$, we set

$$F_j=a_j^1 f_1+\cdots+a_j^{n+1}f_{n+1} \qquad (1\leqq j\leqq q)$$

and by $W(f_1, \cdots, f_{n+1})$ we denote the Wronskian of the functions $f_1, \cdots, f_{n+1}$.

The purpose of this section is to prove the following:

PROPOSITION 3.1. *In the above situation, assume that $q>(n+1)^2$ and $f$ omits $q$ hyperplanes $H_1, \cdots, H_q$ in general position. Then, there exists a positive constant $K_1$ such that*

$$\int_0^{2\pi}\left|\frac{W(f_1, \cdots, f_{n+1})}{F_1F_2\cdots F_q}(re^{i\theta})\right|^{2/(q-n-1)}\|f(re^{i\theta})\|^2 d\theta$$

$$\leqq \frac{K_1}{(1-r)^p}\left(\log\frac{1}{1-r}\right)^p \qquad (0<r<1),$$

*where $p=n(n+1)/(q-n-1)$.*

For the proof, we need some lemmas. The following lemma is essentially due to H. Cartan [**2**].

LEMMA 3.2. *Under the same assumption as in Proposition* 3.1, *there is a positive constant $K_2$ such that*

$$\left|\frac{W(f_1, \cdots, f_{n+1})}{F_1F_2\cdots F_q}\right|\|f\|^{q-n-1}\leqq K_2\Big(\sum_{1\leqq i_1<\cdots<i_{n+1}\leqq q}\left|\frac{W(F_{i_1}, \cdots, F_{i_{n+1}})}{F_{i_1}\cdots F_{i_{n+1}}}\right|\Big).$$

PROOF. Take an arbitrary point $z\in\Delta$. Let $i_1, \cdots, i_q$ be a permutation of the indices $1, 2, \cdots, q$ such that

$$|F_{i_1}(z)|\leqq\cdots\leqq|F_{i_{n+1}}(z)|\leqq|F_{i_{n+2}}(z)|\leqq\cdots\leqq|F_{i_q}(z)|.$$

Since we assume that $H_1, \cdots, H_q$ are in general position, $f_1, \cdots, f_{n+1}$ are represented as linear combinations of $F_{i_1}, \cdots, F_{i_{n+1}}$. Hence, we can find a positive constant $C_{i_1\cdots i_{n+1}}$ independent of each $z$ such that

$$|f_i(z)|\leqq C_{i_1\cdots i_{n+1}}\max_{1\leqq k\leqq n+1}|F_{i_k}(z)|\leqq C_{i_1\cdots i_{n+1}}|F_{i_l}(z)|$$

for $i=1, \cdots, n+1$ and $l=n+2, \cdots, q$. We then have

$$\|f(z)\|=\Big(\sum_{i=1}^{n+1}|f_i(z)|^2\Big)^{1/2}\leqq(n+1)^{1/2}C_{i_1\cdots i_{n+1}}|F_{i_l}(z)|$$

for $l=n+2, \cdots, q$ and hence

$$\|f(z)\|^{q-n-1}\leqq K_2'|F_{i_{n+2}}(z)\cdots F_{i_q}(z)|,$$

where $K_2'=((n+1)^{1/2}C_{i_1\cdots i_{n+1}})^{q-n-1}$. On the other hand, we know

$$W(f_1, \cdots, f_{n+1}):=a_{i_1\cdots i_{n+1}}W(F_{i_1}, \cdots, F_{i_{n+1}})$$

for the constant $a_{i_1\cdots i_{n+1}}:=\det(a^j_{i_k} : 1\leqq j,\ k\leqq n+1)^{-1}$. Setting

$$K_2:=\max_{1\leq i_1<\cdots<i_{n+1}\leq q} C_{i_1\cdots i_{n+1}}|a_{i_1\cdots i_{n+1}}|,$$

we conclude

$$\begin{aligned}&\left|\frac{W(f_1,\cdots,f_{n+1})}{F_1\cdots F_q}(z)\right|\|f(z)\|^{q-n-1}\\ &\leqq C_{i_1\cdots i_{n+1}}|a_{i_1\cdots i_{n+1}}|\left|\frac{W(F_{i_1},\cdots,F_{i_{n+1}})F_{i_{n+2}}\cdots F_{i_q}}{F_1F_2\cdots F_q}(z)\right|\\ &\leqq K_2\left|\frac{W(F_{i_1},\cdots,F_{i_{n+1}})}{F_{i_1}\cdots F_{i_{n+1}}}(z)\right|\\ &\leqq K_2\Big(\sum_{1\leq i_1<\cdots<i_{n+1}\leq q}\left|\frac{W(F_{i_1},\cdots,F_{i_{n+1}})}{F_{i_1}\cdots F_{i_{n+1}}}(z)\right|\Big).\end{aligned}$$

This completes the proof.

LEMMA 3.3. *Let $F_1,\cdots,F_{n+1}$ be non-zero holomorphic functions on the unit disc $\Delta$ in $\mathbf{C}$, and set $\varphi_i:=F_i/F_{n+1}$ $(1\leqq i\leqq n)$. Then, there is a polynomial $P(\cdots, u_{il},\cdots)$ with positive real coefficients not depending on each $F_1,\cdots,F_{n+1}$ such that*

$$\left|\frac{W(F_1,\cdots,F_{n+1})}{F_1\cdots F_{n+1}}\right|\leqq P\Big(\cdots,\left|\Big(\frac{\varphi_i'}{\varphi_i}\Big)^{(l-1)}\right|,\cdots\Big).$$

*More precisely, if we associate weight $l$ with each indeterminate $u_{il}$, $P$ can be chosen so as to be isobaric of weight $n(n+1)/2$.*

PROOF. It is easy to see that

$$\begin{aligned}\frac{W(F_1,\cdots,F_{n+1})}{F_1\cdots F_{n+1}}&=(-1)^n\det\Big(\frac{\varphi_i^{(l)}}{\varphi_i} : 1\leqq i,\ l\leqq n\Big)\\ &=\sum_{(l_1,\cdots,l_{n+1})}(-1)^n\operatorname{sgn}\begin{pmatrix}1 & 2\cdots & n+1\\ l_1 & l_2\cdots & l_{n+1}\end{pmatrix}\frac{\varphi_1^{(l_1)}}{\varphi_1}\cdots\frac{\varphi_n^{(l_n)}}{\varphi_n}.\end{aligned}$$

On the other hand, each $\varphi_i^{(l)}/\varphi_i$ can be represented as a polynomial of $\varphi_i'/\varphi_i$, $(\varphi_i'/\varphi_i)'$, $\cdots$, $(\varphi_i'/\varphi_i)^{(l-1)}$ which is isobaric of weight $l$ if we associate weight $m$ with each $(\varphi_i'/\varphi_i)^{(m-1)}$ (cf., the proof of Lemma 4.2 in [**5**]). From these facts, Lemma 3.3 follows immediately.

LEMMA 3.4. *Let $\varphi_1,\cdots,\varphi_k$ be nowhere zero holomorphic functions on $\Delta$, $l_1,\cdots,l_k$ be positive integers and $t$ be a positive real number with $kt<1$. Assume that $\varphi_1,\cdots,\varphi_k$ are not transcendental. Then there exists a positive constant $K_3$ such that*

$$\int_0^{2\pi}\left|\Big(\Big(\frac{\varphi_1'}{\varphi_1}\Big)^{(l_1-1)}\cdots\Big(\frac{\varphi_k'}{\varphi_k}\Big)^{(l_k-1)}\Big)(re^{i\theta})\right|^t d\theta$$

$$\leqq \frac{K_3}{(1-r)^s}\left(\log \frac{1}{1-r}\right)^s \quad (0<r<1),$$

*where* $s=t(l_1+l_2+\cdots+l_k)$.

PROOF. For brevity, we set $\psi_j:=(\varphi_j'/\varphi_j)^{(l_j-1)}$ $(1\leqq j\leqq k)$. Using the Hölder's inequality, we have

$$\int_0^{2\pi} |(\psi_1\cdots\psi_k)(re^{i\theta})|^t d\theta$$

$$\leqq\left(\int_0^{2\pi} |\psi_1(re^{i\theta})|^{kt}d\theta\right)^{1/k}\cdots\left(\int_0^{2\pi} |\psi_k(re^{i\theta})|^{kt}d\theta\right)^{1/k}$$

and

$$\int_0^{2\pi} |\psi_j(re^{i\theta})|^{kt}d\theta\leqq(2\pi)^{1-kt}\left(\int_0^{2\pi} |\psi_j(re^{i\theta})|\, d\theta\right)^{kt}.$$

On the other hand, it follows from Proposition 2.5 that

$$\int_0^{2\pi} |\psi_j(re^{i\theta})|\, d\theta\leqq\frac{K_3'}{(1-r)^{l_j}}\log\frac{1}{1-r}$$

for a suitable constant $K_3'$ $(1\leqq j\leqq k)$. Therefore,

$$\int_0^{2\pi} |(\psi_1\cdots\psi_k)(re^{i\theta})|^t d\theta\leqq\left(\left(\frac{K_3''}{(1-r)^{l_1+\cdots+l_k}}\left(\log\frac{1}{1-r}\right)^k\right)^{kt}\right)^{1/k}$$

$$\leqq\frac{K_3}{(1-r)^s}\left(\log\frac{1}{1-r}\right)^s$$

for suitable constants $K_3''$, $K_3$. This completes the proof of Lemma 3.4.

PROOF OF PROPOSITION 3.1. Since $2/(q-n-1)<1$, Lemma 3.2 gives

$$\left|\frac{W(f_1,\cdots,f_{n+1})}{F_1\cdots F_q}\right|^{2/(q-n-1)}\|f\|^2$$

$$\leqq K_4\left(\sum_{1\leqq i_1<\cdots<i_{n+1}\leqq q}\left|\frac{W(F_{i_1},\cdots,F_{i_{n+1}})}{F_{i_1}\cdots F_{i_{n+1}}}\right|^{2/(q-n-1)}\right)$$

for some constant $K_4$. It suffices to find a constant $K_4'$ such that

$$\int_0^{2\pi}\left|\frac{W(F_{i_1},\cdots,F_{i_{n+1}})}{F_{i_1}\cdots F_{i_{n+1}}}(re^{i\theta})\right|^{2/(q-n-1)}d\theta\leqq\frac{K_4'}{(1-r)^p}\left(\log\frac{1}{1-r}\right)^p \tag{3.5}$$

for all $i_1,\cdots,i_{n+1}$ $(1\leqq i_1<\cdots<i_{n+1}\leqq q)$. There is no harm in asssuming that $i_1=1$, $\cdots,i_{n+1}=n+1$. According to Lemma 3.3, we can estimate $|W(F_1,\cdots,F_{n+1})/F_1\cdots F_{n+1}|$ from above by a positive constant multiple of the sum of some functions of type

$$\left(\frac{\varphi_{i_1}'}{\varphi_{i_1}}\right)^{(l_1-1)}\cdots\left(\frac{\varphi_{i_k}'}{\varphi_{i_k}}\right)^{(l_k-1)} \tag{3.6}$$

where $1 \leqq i_1, \cdots, i_k \leqq n+1$ and $l_1, \cdots, l_k$ are positive integers with $l_1+\cdots+l_k=n(n+1)/2$. By the assumption, $f$ is not transcendental by the help of Theorem 2.4. So, each $\varphi_i$ also is not transcendental because of (2.2). We now apply Lemma 3.4 to the functions $\varphi_{i_1}, \cdots, \varphi_{i_k}$ and $t=2/(q-n-1)$. For the function $\psi$ given by (3.6), we have

$$\int_0^{2\pi} |\psi(re^{i\theta})|^{2/(q-n-1)} d\theta \leqq \frac{K_4''}{(1-r)^p} \left(\log \frac{1}{1-r}\right)^p .$$

Consequently, we obtain (3.5) and hence complete the proof of Proposition 3.1.

## §4. Proof of Main Theorem.

For the proof of Main Theorem, we first recall the following result of S. T. Yau ([**15**]), which plays an essential role in the following.

THEOREM 4.1. *Let $M$ be a complete Riemannian manifold and $h$ a non-negative and non-constant $C^\infty$-function on $M$ such that $\Delta \log h=0$ almost everywhere. Then, $\int_M h^p d\sigma=\infty$ for $p>0$, where $d\sigma$ denotes the volume form of $M$.*

As in Main Theorem, let $M$ be a complete minimal surface in $\boldsymbol{R}^m$ $(m \geqq 3)$. For our purpose, it suffices to prove that the conjugate $f$ of the Gauss map is necessarily degenerate if $f$ omits hyperplanes $H_1, \cdots, H_q$ in general position, where $q=m^2+1$. Take the universal covering surface $\varpi: \tilde{M} \rightarrow M$. The Riemann surface $\tilde{M}$ is considered also as a complete minimal surface in $\boldsymbol{R}^m$. There is no loss of generality in assuming that $\tilde{M}=M$. Then, $M$ is biholomorphic either to $\boldsymbol{C}$ or to the unit disc $\Delta$, because there is no compact minimal surface in $\boldsymbol{R}^m$. We may assume $M=\boldsymbol{C}$ or $M=\Delta$. For the case $M=\boldsymbol{C}$, $f: \boldsymbol{C} \rightarrow P^n(\boldsymbol{C})-\bigcup_{i=1}^q H_i$ $(n=m-1)$ is necessarily degenerate by the classical result of E. Borel (cf., [**1**], [**2**], [**12**] or [**13**]).

Now, we consider the case $M=\Delta$. Assume that $f$ is non-degenerate. It is easily seen that the area form of the metric on $M$ induced from the flat metric on $\boldsymbol{R}^m$ is given by

$$d\sigma=2\|f\|^2 du \wedge dv .$$

Taking a reduced representation $f=(f_1: \cdots : f_{n+1})$, we consider the functions $F_1, \cdots, F_q$ defined in §3 and set

$$h=\left|\frac{W(f_1, \cdots, f_{n+1})}{F_1 F_2 \cdots F_q}\right| .$$

Clearly, $h \not\equiv 0$ and $\Delta \log h=0$ except the set $\{z \in \Delta : h(z)=0\}$. On the other hand, $\Delta$ has the infinite area with respect to the metric induced from $\boldsymbol{R}^m$ because it is complete, simply connected and of non-positive curvature. By the help of Theorem 4.1, we have

$$\iint_{\Delta} h^{2/(q-n-1)}\|f\|^2 du dv=\infty . \tag{4.2}$$

We now apply Proposition 3.1. Then

$$\iint_{\Delta} h^{2/(q-n-1)}\|f\|^2 du dv=\int_0^1 r dr\left(\int_0^{2\pi} h(re^{i\theta})^{2/(q-n-1)}\|f(re^{i\theta})\|^2 d\theta\right)$$

$$\leqq K_1\int_0^1 \frac{r}{(1-r)^p}\left(\log \frac{1}{1-r}\right)^p dr ,$$

which is finite because $p=n(n+1)/(q-n-1)<1$ by assumption. This contradicts (4.2). The map $f$ is necessarily degenerate. The proof of Main Theorem is completed.

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