J. Math. Soc. Japan
Vol. 35, No. 1, 1983

On G-functors (II): Hecke operators and G-functors

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(Received May 6, 1981)
(Revised June 17, 1982)

1. Introduction.

There are some facts which suggest the relation between cohomological G-functors and Hecke rings. In cohomology theory of finite groups, Cline, Parshall and Scott showed that Hecke algebras act on cohomology groups of finite groups and described the stability theorem of Cartan-Eilenberg (a generalization of the focal subgroup theorem for finite groups) by the use of the language of fixed points (=common eigenvectors) for an action of the Hecke algebra ([**1**, Section 6]). They pointed out also an example that there is no corresponding Hecke algebra action for algebraic K-theory. It is stated also in [**3**], [**4**, Section 8.3], etc. that Hecke algebras act on cohomology groups of groups.

The purpose of this paper is to study the relation between cohomological G-functors and representation of Hecke rings. In Section 2, we define G-functors. In Section 3, we introduce the concept of Hecke category which is the category of permutation modules. In application, Hecke rings appear frequently as components of additive functors from the Hecke category. In Section 4, it is proved that the concepts of cohomological G-functors and representations of the Hecke category (that is, additive functors from there) are equivalent. The main theorem of this paper is the following:

THEOREM 4.3. *Let $\mathcal{M}_k(G)^c$ be the category of cohomological G-functors over k and let $\mathcal{H}_{kG}$ be the category of permutation modules $k[G/H]$, $H \leq G$. Then*

$$\mathcal{M}_k(G)^c \cong \mathrm{Add}_k(\mathcal{H}_{kG}, \mathcal{M}_k).$$

In general, if $\boldsymbol{a}$ be a cohomological G-functor, then the action of the Hecke ring $k[H\backslash G/H]$ on $\boldsymbol{a}(H)$ is given by

$$\alpha \cdot (HxH) := \alpha^x{}_{Hx \cap H}{}^H, \qquad \alpha \in \boldsymbol{a}(H), \quad x \in G.$$

NOTATION. G denotes a finite group except for Section 5. "$H \leq G$" means that H is a subgroup of G. For a subset or an element X of G and an element g of G, we set $X^g := g^{-1}Xg := \{g^{-1}xg \mid x \in X\}$. The index of a subgroup H of G is denoted by $|G:H|$. The cardinality of a set X is denoted by $|X|$. The notation $H\backslash G/K$ means the set of double cosets HgK and sometimes a complete set of representatives of the set of double cosets. The notation k denotes a commutative ring with unit.

Group and ring operations, homomorphisms and functors are usually on the right. Thus the composition

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is denoted as fg. The category of R-modules is denoted by $\mathcal{M}_R$.

2. G-functors.

In this section, we introduce some concept about G-functors. More details are found in Green [2] and Yoshida [5]. Let G be a finite group and k a commutative ring with unit.

DEFINITION 2.1. A G-functor $\boldsymbol{a}=(\boldsymbol{a}, \tau, \rho, \sigma)$ over k consists of k-modules $\boldsymbol{a}(H)$ for subgroups H of G and k-maps of three kinds:

$$\tau^K=\tau_H^K : \boldsymbol{a}(H) \longrightarrow \boldsymbol{a}(K) : \alpha \longmapsto \alpha^K ,$$

$$\rho_H=\rho_H^K : \boldsymbol{a}(K) \longrightarrow \boldsymbol{a}(H) : \beta \longmapsto \beta_H ,$$

$$\sigma^g=\sigma_H^g : \boldsymbol{a}(H) \longrightarrow \boldsymbol{a}(H^g) : \alpha \longmapsto \alpha^g ,$$

where g is an element of G and H and K are subgroups of G with $H \subseteqq K$. These families must satisfy the following axioms:

AXIOMS FOR G-FUNCTORS. (In these axioms, $D, H, K, L \leq G$; $g, g' \in G$; $\alpha \in \boldsymbol{a}(H)$, $\beta \in \boldsymbol{a}(K)$).

(G. 1) $\alpha^H=\alpha$, $(\alpha^K)^L=\alpha^L$ if $H \subseteqq K \subseteqq L$;

(G. 2) $\beta_K=\beta$, $(\beta_H)_D=\beta_D$ if $D \subseteqq H \subseteqq K$;

(G. 3) $(\alpha^g)^{g'}=\alpha^{gg'}$, $\alpha^h=\alpha$ for $h \in H$;

(G. 4) $(\alpha^K)^g=(\alpha^g)^{K^g}$, $(\beta_H)^g=\beta^g{}_{H^g}$ if $H \subseteqq K$;

(G. 5) (Mackey axiom). If H and K are subgroups of L, then

$$\alpha^L{}_K=\sum_{g \in H \backslash L / K} \alpha^g{}_{H^g \cap K}{}^K ,$$

where g runs over a complete set of representatives of $H \backslash L / K$.

DEFINITION 2.2. A G-functor $(\boldsymbol{a}, \tau, \rho, \sigma)$ is called *cohomological* if it satisfies the following axiom:

(C) If $H \leq K \leq G$ and $\beta \in \boldsymbol{a}(K)$, then $\beta_H{}^K=|K: H| \cdot \beta$.

DEFINITION 2.3. A *pairing* $\boldsymbol{a} \times \boldsymbol{b} \rightarrow \boldsymbol{c}$ of G-functors is a family of k-bilinear maps

$$\boldsymbol{a}(H) \times \boldsymbol{b}(H) \longrightarrow \boldsymbol{c}(H) : \quad (\alpha, \beta) \longmapsto \alpha \cdot \beta$$

for all subgroups H of G which satisfies that if $H \leq K \leq G$, $\alpha \in \boldsymbol{a}(H)$, $\alpha' \in \boldsymbol{a}(K)$,

$\beta\in\boldsymbol{b}(H)$, $\beta'\in\boldsymbol{b}(K)$ and $g\in G$, then

$$\text{(P)}\qquad \begin{aligned}&(\alpha'\cdot\beta')_H=\alpha'_H\cdot\beta'_H\,, \quad (\alpha\cdot\beta)^g=\alpha^g\cdot\beta^g\,,\\ &\alpha^K\cdot\beta'=(\alpha\cdot\beta'_H)^K\,, \quad \alpha'\cdot\beta^K=(\alpha'_H\cdot\beta)^K\,.\end{aligned}$$

If each $\boldsymbol{a}(H)$ becomes a k-algebra through a pairing $\boldsymbol{a}\times\boldsymbol{a}\to\boldsymbol{a}$, then $\boldsymbol{a}$ is called a "*ring*". "*Modules*" over a "ring" are defined by the similar method.

DEFINITION 2.4. A *morphism* $\theta:\boldsymbol{a}\to\boldsymbol{b}$ between G-functors over k is a family of k-linear maps $\theta(H):\boldsymbol{a}(H)\to\boldsymbol{b}(H)$ for $H\leq G$ which commute with τ,ρ,σ. We denote the category of G-functors over k by $\mathcal{M}_k(G)$. The full subcategory of cohomological G-functors in $\mathcal{M}_k(G)$ is denoted by $\mathcal{M}_k(G)^c$.

EXAMPLE 2.1. Let V be a right kG-module and n an integer. Then a G-functor $\boldsymbol{h}_V^n=(\boldsymbol{h}_V^n,\tau,\rho,\sigma)$ is defined by $\boldsymbol{h}_V(H):=H^n(G,V)$, the n-th cohomology group of the subgroup H of G. The maps τ, ρ, σ are corestrictions, restrictions, conjugations, respectively. Then $\boldsymbol{h}_V^n$ is a cohomological G-functor over k. We set $\boldsymbol{c}_V:=\boldsymbol{h}_V^0$. Then we have that for $H\leq K\leq G$ and $g\in G$,

$$\begin{aligned}&\boldsymbol{c}_V(H)=\{v\in V\,|\,vh=v \text{ for all } h \text{ in } H\}\,,\\ &\tau_H^K:\boldsymbol{c}_V(H)\longrightarrow\boldsymbol{c}_V(K):v\longmapsto\sum_{x\in H\backslash K}vx\,,\\ &\rho_H^K:\boldsymbol{c}_V(K)\longrightarrow\boldsymbol{c}_V(H):v\longmapsto v\,,\\ &\sigma_H^g:\boldsymbol{c}_V(H)\longrightarrow\boldsymbol{c}_V(H^g):v\longmapsto vg\,.\end{aligned}$$

If U, V, W are kG-modules and $U\times V\to W$ is a k-bilinear map compatible with the action of G, then we have a pairing $\boldsymbol{h}_U^m\times\boldsymbol{h}_V^n\to\boldsymbol{h}_W^{m+n}$.

3. Hecke operators.

The free k-module with basis X is denoted by kX or $k[X]$. Let G be a finite group. If X is a G-set, then kX becomes a kG-module.

LEMMA 3.1. *Let H and K be a subgroup of G. Then*

$$\begin{aligned}&\Phi: k[H\backslash G/K]\cong\mathrm{Hom}_{kG}(k[G/H],\,k[G/K])\,,\\ &\Phi': k[H\backslash G/K]\cong\mathrm{Hom}_{kG}(k[K\backslash G],\,k[H\backslash G])\,.\end{aligned}$$

By these isomorphisms, $(HxK)\in k[H\backslash G/K]$ is mapped to

$$\Phi(HxK): gH\longrightarrow\textstyle\sum_u guxK=\sum_{u'}gu'K\,,$$

where u (resp. u') runs over a complete set of representatives of $H/H\cap xKx^{-1}$ (resp. HxK/K), and

$$\Phi'(HxK): Kg\longrightarrow\textstyle\sum_v Hxvg=\sum_{v'}Hv'g\,,$$

where v *(resp.* v'*) runs over a complete set of representatives of* $H^x \cap K\backslash K$ *(resp.* $H\backslash HxK$*).*

PROOF. Direct verification ([**4**, Prop. 3.4]).

Let H, K, L be subgroups of G. Then by the above lemma, we see that the composition

$$\alpha\cdot\beta : k[G/H] \xrightarrow{\alpha} k[G/K] \xrightarrow{\beta} k[G/L],$$

or

$$\alpha\cdot\beta : k[L\backslash G] \xrightarrow{\beta} k[K\backslash G] \xrightarrow{\alpha} k[L\backslash G]$$

induces a k-bilinear map

$$k[H\backslash G/K]\times k[K\backslash G/L] \longrightarrow k[H\backslash G/L] : (\alpha, \beta) \longmapsto \alpha\cdot\beta .$$

Then the following holds:

LEMMA 3.2. $(HxK)\cdot(KyL) = \sum_{z\in H\backslash G/L} m(x, y; z)(HzL)$, *where* $m(x, y; z) = |(HxK \cap zLy^{-1}K)/K|$.

PROOF. By Lemma 3.1, we have that

$$\Phi((HxK)\cdot(KyL)) : H \longrightarrow \sum_{u,v} uvL = \sum_{z} m(x, y; z)zL ,$$

where u (resp. v) runs over $H\backslash HxK$ (resp. $K\backslash KyL$) and z runs over $H\backslash G/L$. Thus $m(x, y; z)$ equals to the number of pairs (uK, vL) such that $uK\in HxK/K$, $vL\in KyL/L$, $uvL=zL$, and so we have that

$$\begin{aligned} m(x, y; z) &= \sum_{u\in HxK/K} \#\{vL \mid uvL=zL,\ vL\in KyL/L\} \\ &= \sum_{u} |(u^{-1}zL \cap KyL)/L| . \end{aligned}$$

Since the term $|(u^{-1}zL \cap KyL)/L|=1$ if $u\in zLy^{-1}K$ and $=0$ otherwise, we have that

$$\begin{aligned} m(x, y; z) &= \#\{uK\in HxK/K \mid u\in zLy^{-1}K\} \\ &= |(HxK \cap zLy^{-1}K)/K| , \end{aligned}$$

as required. The lemma is proved.

By the above bilinear map, $k[H\backslash G/H]$ becomes an associated k-algebra with unit $(H1H)$ which is called a *Hecke ring* and isomorphic to $\mathrm{End}_{kG}(k[G/H])$.

DEFINITION 3.1. Let $\mathcal{H}_{kG}$ be the category of which objects are $k[G/H]$, $H\leq G$, and of which hom-sets are defined by

$$\mathrm{hom}(k[G/H], k[G/K]) := k[H\backslash G/K] .$$

The compositions are defined by the above bilinear maps. We call $\mathcal{H}_{kG}$ the *Hecke category* of G.

In the remainder of this section, we state some formulas about Hecke rings which we associate with the axioms for cohomological G-functors. We define some morphisms of the Hecke category as follows:

$$T^K := T_H^K := (H\cdot 1\cdot K) \quad \text{for} \quad H\leq K\leq G,$$
$$R_H := R_H^K := (K\cdot 1\cdot H) \quad \text{for} \quad H\leq K\leq G,$$
$$S^g := S_H^g := (H\cdot g\cdot H^g) \quad \text{for} \quad H\leq G,\ g\in G,$$
$$I := I_H := (H\cdot 1\cdot H) \quad \text{for} \quad H\leq G.$$

LEMMA 3.3. *Let H and K be subgroups of G and g an element of G. Set $D=H^g\cap K$. Then*

$$(HgK)=(HgH^g)(H^g\cdot 1\cdot D)(D\cdot 1\cdot K)$$
$$=S_H^g\cdot R_D\cdot T^K.$$

PROOF. Direct verification.

LEMMA 3.4. *Let D, H, K, L be subgroups of G and let g, g' be elements of G. Then the following hold:*

(H. 1) $T_H^H=I$, $T_H^K T_K^L=T_H^L$ *if* $H\leq K\leq L$;

(H. 2) $R_K^K=I$, $R_H^K R_D^H=R_D^K$ *if* $D\leq H\leq K$;

(H. 3) $S_H^g S^{g'}=S_H^{gg'}$, $S_H^h=I$ *if* $h\in H$;

(H. 4) $T_H^K S^g=S_H^g T^{K^g}$, $R_H^K S^g=S_K^g R_{H^g}$ *if* $H\leq K$;

(H. 5) (Mackey decomposition). *If H and K are subgroups of L, then*

$$T_H^L R_K^L=\sum_{g\in H\backslash L/K} S_H^g R_{H^g\cap K} T^K.$$

(H. C) (Cohomologicality). *If H is a subgroup of K, then*

$$R_H^K T_H^K=|K:H|I.$$

PROOF. These equalities are easy conclusion from Lemma 3.2. We will show only (H. 5). Let H and K be subgroups of L. Then by Lemma 3.2, we have that

$$T_H^L R_K^L=\sum_{z\in H\backslash G/K} m(z)(HzK),$$

where

$$m(z)=|(HL\cap zKL)/L|$$
$$=|(L\cap zL)/L|,$$

and so $m(z)=1$ if z is in L and $m(z)=0$ if z is not in L. Thus (H. 5) holds.

4. Hecke operators and G-functors.

In this section, we prove the main theorem of this paper which states that cohomological G-functors and additive functors from the Hecke category are the same concepts (Theorem 4.3).

THEOREM 4.1. *Let $\boldsymbol{a}$ be a cohomological G-functor over k. Then a k-additive functor* $A: \mathcal{H}_{kG} \to \mathcal{M}_k$ *is defined by*

$$A: k[G/H] \longmapsto \boldsymbol{a}(H);$$

$$A: k[H\backslash G/K] \longrightarrow \mathrm{Hom}_k(\boldsymbol{a}(H), \boldsymbol{a}(K)) : \lambda \longmapsto \bar{\lambda},$$

where

$$\overline{(HxK)} : \boldsymbol{a}(H) \longrightarrow \boldsymbol{a}(K) : \alpha \longmapsto \alpha^x{}_{H^x \cap K}{}^K .$$

PROOF. We must show that $\overline{\lambda\mu} = \bar{\lambda}\bar{\mu}$ for any morphisms λ, μ in $\mathcal{H}_{kG}$ with composition $\lambda\mu$. Define a set $\mathcal{S}$ of morphisms in $\mathcal{H}_{kG}$ as follows:

$$\mathcal{S} = \{\mu \mid \overline{\lambda\mu} = \bar{\lambda}\bar{\mu} \text{ for any } \lambda \text{ with } \mathrm{dom}\,\mu = \mathrm{cod}\,\lambda\} .$$

Then $\mathcal{S}$ is a subcategory of $\mathcal{H}_{kG}$. We will show that $\mathcal{S} = \mathcal{H}_{kG}$. For any subgroups $H \leq K \leq L \leq G$ and $g \in G$, morphisms $R_H^K = (K1H)$, $T_K^L = (K1L)$, $S_H^g = (HgH^g)$ in $\mathcal{H}_{kG}$ are called morphisms of *type r, t, s*, respectively. By Lemma 2.2, we have that for any subgroups H, K, L of G,

(1) $(HxK)\cdot(KyL) = \sum_{HzL} m(x, y; z)(HzL)$, where

$$m(x, y; z) = |(HxK \cap zLy^{-1}K)/K| .$$

We shall first show the following:

(2) $\overline{\lambda S} = \bar{\lambda}\bar{S}$ for S of type s, that is, $S \in \mathcal{S}$.

In fact, let $\alpha \in \boldsymbol{a}(H)$, $\lambda = (HxK)$ and $S = (HyH^y)$. Then by the axioms for G-functors

$$(\alpha\bar{\lambda})\bar{S} = \alpha^x{}_{H^x \cap K}{}^{K^y} = \alpha^{xy}{}_{H^{xy} \cap K^y}{}^{K^y} = \alpha \cdot \overline{(HxyK^y)} .$$

On the other hand, (1) implies that

$$\lambda S = (HxK)\cdot(KyK^y) = (HxyK^y) ,$$

and so $(\alpha\lambda)S = \alpha(\lambda S)$, proving (2). Next we have the following:

(3) $\overline{\lambda T} = \bar{\lambda}\bar{T}$ for T of type t, that is, $T \in \mathcal{S}$.

In fact, let $\alpha \in \boldsymbol{a}(H)$, $\lambda = (HxK)$ and $T = T_K^L = (K1L)$ with $K \leq L$, and set $D = H^x \cap K$, $E = H^x \cap L$. Then by the cohomologicality of the G-functor $\boldsymbol{a}$, we have that

$$\begin{aligned}(\alpha\bar{\lambda})\bar{T} &= (\alpha^x{}_D{}^K)^L = \alpha^x{}_D{}^L \\ &= ((\alpha^x{}_E)_D{}^E)^L = |E : D|\, \alpha^x{}_E{}^L \\ &= |E : D|\, \alpha \cdot \overline{(HxL)} .\end{aligned}$$

On the other hand, by (1),

$$\begin{aligned}\lambda T &= (HxK)\cdot(K1L) = |(HxK \cap xL)/K|(HxL) \\ &= |E : D|(HxL) ,\end{aligned}$$

and so $(\alpha\lambda)T=\alpha(\lambda T)$, proving (3). By the similar way, we can prove the dual statements as follows:

(2′) $\overline{S\mu}=\bar{S}\,\bar{\mu}$ for S of type s.

(3′) $\overline{R\mu}=\bar{R}\,\bar{\mu}$ for R of type r.

Furthermore Mackey axiom (G. 5) for G-functors and Mackey decomposition ((H. 5) in Lemma 3.4) for the Hecke category yield the following:

(4) $\overline{TR}=\bar{T}\,\bar{R}$ for T of type t and R of type r.

Finally we will show the following:

(5) $\overline{\lambda R'}=\bar{\lambda}\,\overline{R'}$ for R' of type r, that is, $R'\in\mathcal{I}$.

To prove this, present λ as $\lambda=SRT$, where S, R, T are morphisms of type s, r, t, respectively, by Lemma 3.3. Then

$$\begin{aligned}\overline{\lambda R'}&=\overline{(SR)(TR')}=\overline{SR}\,\overline{TR'} && \text{by (2′), (3′)}\\ &=(\overline{SR}\,\bar{T})\,\overline{R'} && \text{by (4)}\\ &=(\overline{SRT})\,\overline{R'} && \text{by (3)}\\ &=\bar{\lambda}\,\overline{R'}\,,\end{aligned}$$

proving (5). Hence we see that $\mathcal{I}$ contains morphisms of type r, s, t. Since any morphisms in $\mathcal{H}_{kG}$ is presented as the compositions of morphisms of type r, s, t by Lemma 3.3, all morphisms of $\mathcal{H}_{kG}$ is contained in the subcategory $\mathcal{I}$, and hence $\mathcal{I}=\mathcal{H}_{kG}$, as required. The theorem is proved.

COROLLARY 4.2. *Let* $\boldsymbol{a}$ *be a cohomological G-functor and H a subgroup of G. Then* $\boldsymbol{a}(H)$ *is a* $k[H\backslash G/H]$*-module by*

$$\alpha\cdot(HxH)=\alpha^{x}{}_{H^x\cap H}{}^{H} \qquad for \quad \alpha\in\boldsymbol{a}(H)\,, \quad x\in G\,.$$

PROOF. This is an obvious conclusion of Theorem 4.1.

We can now prove Theorem 4.3 which is the main theorem of this paper. Let $\mathcal{M}_k(G)^c$ be the category of cohomological G-functors (Definition 2.4). Let $\mathrm{Add}_k(\mathcal{H}_{kG}, \mathcal{M}_k)$ be the category of k-additive functors from $\mathcal{H}_{kG}$ to $\mathcal{M}_k$ (the category of k-modules).

THEOREM 4.3. $\mathcal{M}_k(G)^c\cong\mathrm{Add}_k(\mathcal{H}_{kG}, \mathcal{M}_k)$.

PROOF. We shall first define an additive functor

$$\Psi:\mathcal{M}_k(G)^c\longrightarrow\mathrm{Add}_k(\mathcal{H}_{kG}, \mathcal{M}_k)\,.$$

For any cohomological G-functor $\boldsymbol{a}$, define $\Psi(\boldsymbol{a}):\mathcal{H}_{kG}\to\mathcal{M}_k$ by

$$\begin{aligned}\Psi(\boldsymbol{a}):&\ k[G/H]\longmapsto\boldsymbol{a}(H)\,,\\ :&\ k[H\backslash G/K]\longrightarrow\mathrm{Hom}_k(\boldsymbol{a}(H), \boldsymbol{a}(K)):\lambda\longmapsto\bar{\lambda}\,,\end{aligned}$$

where

$$\overline{(HxK)}:\boldsymbol{a}(H)\longrightarrow\boldsymbol{a}(K):\alpha\longmapsto\alpha^{x}{}_{H^x\cap K}{}^{K}\,.$$

By Theorem 4.1, $\Psi(\boldsymbol{a})$ is actually a k-additive functor. Let $\theta : \boldsymbol{a} \to \boldsymbol{b}$ be a morphism between cohomological G-functor. We define a natural transformation $\Psi(\theta) : \Psi(\boldsymbol{a}) \to \Psi(\boldsymbol{b})$ by $\Psi(\theta)(k[G/H]) := \theta(H) : \boldsymbol{a}(H) \to \boldsymbol{b}(H)$ for each H. Clearly θ is compatible with Hecke operations, and so $\Psi(\theta)$ is a natural transformation. Thus Ψ is a k-additive functor.

We shall next define a k-additive functor

$$\Phi : \mathrm{Add}_k(\mathcal{H}_{kG}, \mathcal{M}_k) \longrightarrow \mathcal{M}_k(G)^c .$$

Let $A : \mathcal{H}_{kG} \to \mathcal{M}_k$ be a k-additive functor. Define a cohomological G-functor $\boldsymbol{a} = (\boldsymbol{a}, \tau, \rho, \sigma)$ as follows:

$$\boldsymbol{a}(H) := A(k[G/H]) , \quad \tau_H^K := A(T_H^K) , \quad \rho_H^K := A(R_H^K) , \quad \sigma_H^g := A(S_H^g) .$$

By Lemma 3.4, $\boldsymbol{a} = (\boldsymbol{a}, \tau, \rho, \sigma)$ satisfies the axioms for G-functors. Let $\Theta : A \to B$ be a morphism in $\mathrm{Add}_k(\mathcal{H}_{kG}, \mathcal{M}_k)$ and set $\boldsymbol{a} := \Phi(A)$, $\boldsymbol{b} := \Phi(B)$. Then a morphism $\Phi(\Theta) : \boldsymbol{a} \to \boldsymbol{b}$ is defined by $\Theta(k[G/H]) : \boldsymbol{a}(H) \to \boldsymbol{b}(H)$. Thus we have a k-additive functor Φ. Clearly Φ is the inverse of Ψ. The theorem is proved.

EXAMPLE 4.1. Let D be a subgroup of G. Set $A(k[G/H]) := k[D\backslash G/H]$. Then multiplications on the right make A into a functor $A : \mathcal{H}_{kG} \to \mathcal{M}_k$. By Theorem 4.3, there is a cohomological G-functor $(\boldsymbol{a}, \tau, \rho, \sigma)$ which corresponds to A. We have that $\boldsymbol{a}(H) = k[D\backslash G/H]$ and

$$\tau_H^K : (DxH) \longmapsto (DxK) ,$$

$$\rho_H^K : (DxK) \longmapsto \sum_{u \in K/H} (DxuH) ,$$

$$\sigma_H^g : (DxH) \longmapsto (DxgH^g) .$$

Now we can construct a cohomological G-functor $\boldsymbol{c}_{k[D\backslash G]}$ from the permutation module $k[D\backslash G]$ (Example 2.1). Then $\boldsymbol{a}$ is isomorphic to $\boldsymbol{c}_{k[D\backslash G]}$. The isomorphism of $\boldsymbol{a}(H)$ to $\boldsymbol{c}_{k[D\backslash G]}(H)$ is given by

$$(DxH) \longmapsto \sum_{u \in D\backslash DxH} (Du) .$$

EXAMPLE 4.2. Let k be a field of characteristic $p > 0$ and P a subgroup of G of index prime to p. Let A be a k-additive functor from $\mathcal{H}_{kG}$ to $\mathcal{M}_k$. Then $A(k[G/G])$ is isomorphic to the common eigenvector space of

$$\{A(\lambda) \in \mathrm{End}_k(A(k[G/P])) \mid \lambda \in k[P\backslash G/P]\}$$

with respect to the algebra homomorphism

$$\mathrm{ind} : k[P\backslash G/P] \longrightarrow k : (PgP) \longmapsto |P : P \cap P^g| ,$$

that is,

$$(1) \qquad A(k[G/G]) \cong \{v \in A(k[G/P]) \mid v \cdot A(\lambda) = \mathrm{ind}(\lambda) \cdot v \text{ for all } \lambda\} .$$

Let $\boldsymbol{a}$ be the cohomological G-functor corresponding to A. Then (1) is rewritten as

(2) $$\boldsymbol{a}(G)\cong \mathrm{Im}\,\rho_P^G=\{\alpha\in\boldsymbol{a}(P) \mid \alpha^g{}_{P^g\cap P}{}^P=\alpha_{P^g\cap P}{}^P \text{ for all } g\}.$$

This isomorphism is a kind of the "stable element theorem" (or the "focal subgroup theorem" for finite groups). The proof of (2) is found in [**5**, Theorem 3.2].

In general it is impossible that defining suitable τ, ρ, σ, we make a G-functor from the correspondence $H\mapsto k[H\backslash G/H]$. But by this idea we have a "ring" that all cohomological G-functors are "modules" over. The finite group G acts on the k-algebra kG by conjugation, and so we have a cohomological G-functor $\boldsymbol{c}_{kG}=(\boldsymbol{c}_{kG}, \tilde{\tau}, \tilde{\rho}, \tilde{\sigma})$ as in Example 2.1 with

$$\boldsymbol{c}_{kG}(H)=\{\gamma\in kG \mid \gamma^h=\gamma \text{ for all } h \text{ in } H\}.$$

Furthermore, since the multiplication in kG is compatible with G-conjugation, $\boldsymbol{c}_{kG}$ becomes a "ring". Clearly the set of the class sums $[x^H]:=\sum_{y\in x^H} y\in kG$ of the H-conjugate classes $x^H=\{x^h \mid h\in H\}$ in G is the basis of $\boldsymbol{c}_{kG}(H)$. For $H\leq K\leq G$ and $g\in G$, we have the following:

$$\tilde{\tau}_H^K:\ \begin{array}{l} c_{kG}(H)\longrightarrow c_{kG}(K) \\ [x^H]\longmapsto |C_K(x):C_H(x)|[x^K], \end{array}$$

$$\tilde{\rho}_H^K:\ \begin{array}{l} c_{kG}(K)\longrightarrow c_{kG}(H) \\ [x^K]\longmapsto [x^K]=\sum_u [x^{uH}], \quad \text{where } u \text{ runs over } C_K(x)\backslash K/H, \end{array}$$

$$\tilde{\sigma}_H^g:\ \begin{array}{l} c_{kG}(H)\longrightarrow c_{kG}(H^g) \\ [x^H]\longmapsto [x^H]^g=[x^{gH^g}]. \end{array}$$

THEOREM 4.4. *Let $\boldsymbol{a}$ be a cohomological G-functor. Then $\boldsymbol{a}$ is a "c_{kG}-module" by*

$$\alpha\cdot[x^H]:=\alpha^x{}_{C_H(x)}{}^H=|H\cap H^x : C_H(x)|\,\alpha^x{}_{H^x\cap H}{}^H,$$

where α is an element of $\boldsymbol{a}(H)$ and x^H is the H-conjugate class in G.

PROOF. Let H be a subgroup of G. Let

$$\theta_H:\ \begin{array}{l} \boldsymbol{c}_{kG}(H)\longrightarrow k[H\backslash G/H] \\ [x^H]\longmapsto |H\cap H^x : C_H(x)|(HxH) \end{array}$$

be the k-linear map. We will first show that

(1) θ_H is a k-algebra homomorphism.

To prove this we first define a k-linear map

$$\phi:\boldsymbol{c}_{kG}(H)\longrightarrow \mathrm{End}_{kG}(k[G/H])$$

by

$$\psi(\gamma): gH \longmapsto g\gamma H, \qquad \gamma\in \boldsymbol{c}_{kG}(H).$$

Then ψ is a k-algebra homomorphism. If $x\in G$ and $C=C_H(x)$, then

$$(2) \qquad \psi([x^H]): gH \longmapsto \sum_{h\in C\backslash H} gx^h H = \sum_{h\in H/C} ghxH.$$

By Lemma 2.1, we have the algebra isomorphism

$$\phi: k[H\backslash G/H] \longrightarrow \mathrm{End}_{kG}(k[G/H])$$

such that

$$(3) \qquad \phi(HxH): gH \longmapsto \sum_u guxH,$$

where u runs over $H/H\cap xHx^{-1}$. By (2) and (3),

$$\psi([x^H]) = |H\cap H^x : C_H(x)|\,\phi(HxH).$$

Thus $\theta_H=\psi\phi^{-1}$ is an algebra homomorphism, proving (1). Now, since $\boldsymbol{a}(H)$ is a $k[H\backslash G/H]$-module by Corollary 4.2, $\boldsymbol{a}(H)$ is also a $\boldsymbol{c}_{kG}(H)$-module by

$$(4) \qquad \alpha\cdot[x^H] := |H\cap H^x : C_H(x)|\,\alpha^x{}_{H^x\cap H}{}^H = \alpha^x{}_{C_H(x)}{}^H.$$

We must show that the k-bilinear maps

$$\boldsymbol{a}(H)\times \boldsymbol{c}_{kG}(H) \longrightarrow \boldsymbol{a}(H): (\alpha, [x^H]) \longmapsto \alpha\cdot[x^H]$$

satisfy the axioms for pairing (Definition 2.3). Let $H\leq K\leq G$, $g\in G$, $\alpha\in\boldsymbol{a}(H)$, $\beta\in\boldsymbol{a}(K)$, $\lambda\in\boldsymbol{c}_{kG}(H)$, $\mu\in\boldsymbol{c}_{kG}(K)$. Then axioms that we must verify are

(P.1) $(\beta\cdot\mu)_H=\beta_H\cdot\mu_H$,

(P.2) $(\alpha\cdot\lambda)^g=\alpha^g\cdot\lambda^g$,

(P.3) $\alpha^K\cdot\mu=(\alpha\cdot\mu_H)^K$,

(P.4) $\beta\cdot\lambda^K=(\beta_H\cdot\lambda)^K$.

Using notation $R_H^K=(K1H)$, $S_H^g=(HgH^g)$, $T_H^K=(H1K)$ in Section 3, we can rewrite the above statement as follows:

(P.1′) $\beta\cdot\mu\cdot R_H^K=\beta\cdot R_H^K\cdot\mu_H$,

(P.2′) $\alpha\cdot\lambda\cdot S_H^g=\alpha\cdot S_H^g\cdot\lambda^g$,

(P.3′) $\alpha\cdot T_H^K\cdot\mu=\alpha\cdot\mu_H\cdot T_H^K$,

(P.4′) $\beta\cdot\lambda^K=\beta\cdot R_H^K\cdot\lambda\cdot T_H^K$.

To prove these statements, it will suffice to show that the following equalities in the Hecke category $\mathcal{H}_{kG}$ hold:

(Q.1) $\theta(\mu)\cdot R_H^K=R_H^K\cdot\theta(\mu_H)$,

(Q.2) $\theta(\lambda)\cdot S_H^g=S_H^g\cdot\theta(\lambda^g)$,

(Q.3) $T_H^K\cdot\theta(\mu)=\theta(\mu_H)\cdot T_H^K$,

(Q.4) $\theta(\lambda^K)=R_H^K\cdot\theta(\lambda)\cdot T_H^K$.

Thus it will suffice to show that (P.1)-(P.4) hold for the G-functor $H\mapsto k[G/H]$

defined in Example 4.1. But this G-functor is isomorphic to the G-functor $\boldsymbol{c}_V$, where $V=kG$ is the right regular kG-module. It is easily proved that $\boldsymbol{a}=\boldsymbol{c}_V$ satisfies (P.1)-(P.4). The resulting pairing $\boldsymbol{c}_V\times\boldsymbol{c}_{kG}\to\boldsymbol{c}_V$ is one induced from the G-invariant bilinear map $V\times kG\to V:(v, g)\mapsto vg$. We proved the theorem.

By this theorem, the set of indecomposable cohomological G-functors is divided to blocks.

5. Infinite groups.

We can extend many definitions and results in previous sections to the infinite-group-case. Let G be a finite or infinite group (or more generally a sub-semigroup of a group). Let $\mathcal{F}$ be a family of subgroups of G such that

(*) if $A, B\in\mathcal{F}$ and $g\in G$, then $A\cap B$, $A^g\in\mathcal{F}$.

Then a *G-functor* $(\boldsymbol{a}, \tau, \rho, \sigma)$ over k with respect to $\mathcal{F}$ consists of k-modules $\boldsymbol{a}(H)$ and k-maps τ_H^K, ρ_H^K, σ_H^g, where $H, K\in\mathcal{F}$, $g\in G$, and τ_H^K is defined only when $|K:H|$ is finite, and these families satisfy axioms as in Definition 3.1. Furthermore, we can similarly define cohomological G-functors, pairings, the category $\mathcal{M}_k(G;\mathcal{F})^c$ of cohomological G-functors with respect to $\mathcal{F}$, etc.

In the present case, Lemma 3.1 is generalized as follows:

LEMMA 3.1′. *Let H and K be subgroups of G. Then*

$$\mathrm{Hom}_{kG}(k[G/H], k[G/K])\cong k[H\backslash G/K]',$$

the free k-module with basis $\{HxK\in H\backslash G/K \mid |HxK/K|<\infty\}$.

By this lemma, we can define the Hecke category $\mathcal{H}'_{kG}(\mathcal{F})$ as in Definition 3.1. Thus the objects of it are $k[G/H]$, $H\in\mathcal{F}$, and the hom-sets are $k[H\backslash G/K]'$, $H, K\in\mathcal{F}$. Now the main theorem of this paper becomes as follows:

THEOREM 4.3′. $\mathcal{M}_k(G;\mathcal{F})^c\cong\mathrm{Add}_k(\mathcal{H}'_{kG}(\mathcal{F}), \mathcal{M}_k)$.

Unfortunately it seems that Theorem 4.4 is not extended to infinite groups.

REMARK. We can define the dense subcategory $\mathcal{H}''_{kG}(\mathcal{F})$ of $\mathcal{H}'_{kG}(\mathcal{F})$ of which hom-sets are $k[H\backslash G/K]''\subsetneq k[H\backslash G/K]$ generated by (HxK) with $|H\backslash HxK|<\infty$, $|HxK/K|<\infty$. If we redefine G-functors as ρ_H^K also exists only when $|K:H|<\infty$, then the analogy of Theorem 4.3′ holds for such G-functors and $\mathcal{H}''_{kG}(\mathcal{F})$.

EXAMPLE 5.1. Let G be the group $GL_2^+(\boldsymbol{R})$ (or a subsemigroup of $GL_2^+(\boldsymbol{R})$) and let $\mathcal{F}$ be the set of all Fuchsian groups of the first kind contained in G. For each Fuchsian group H of the first kind, we denoted by $A_n(H)$ (resp. $G_n(H)$, $S_n(H)$) the set of all automorphic forms (resp. integral forms, cusp forms) of weight n. Then A_n, G_n, and S_n, together with double coset actions defined by Shimura, become additive functors of $\mathcal{H}'_{kG}(\mathcal{F})$ to $\mathcal{M}_\Omega$, where Ω is the $\boldsymbol{C}$-algebra of all meromorphic functions on the upper half plane ([**4**, Propositions 3.37, 3.38]). Now, $GL_2^+(\boldsymbol{R})$ acts on Ω by $f^\sigma=f|[\sigma]_n$ ([**10**, Section 2.1]), we have a G-functor

$\omega_n = \boldsymbol{c}_\Omega$ with respect to $\mathfrak{F}$. Let $\boldsymbol{a}_n$, $\boldsymbol{g}_n$, $\boldsymbol{s}_n$ be the cohomological G-functors which correspond to the functors A_n, G_n, S_n, respectively. Then $\boldsymbol{s}_n \leq \boldsymbol{g}_n \leq \boldsymbol{a}_n \leq \omega_n$, where "$\leq$" means to be a subfunctor. Of course, $\boldsymbol{s}_n(H) = S_n(H)$, etc. In general, these G-functors are different. A pairing $\omega_m \times \omega_n \to \omega_{m+n}$ is defined by $\Omega_m(H) \times \Omega_n(H) \to \Omega_{m+n}(H) : (f, g) \mapsto f \cdot g$. This pairing induces pairings $\boldsymbol{a}_m \times \boldsymbol{a}_n \to \boldsymbol{a}_{m+n}$, $\boldsymbol{g}_m \times \boldsymbol{g}_n \to \boldsymbol{g}_{m+n}$, $\boldsymbol{g}_m \times \boldsymbol{s}_n \to \boldsymbol{s}_{m+n}$. Take the direct sums $\boldsymbol{a} = \sum_n \boldsymbol{a}_n$, $\boldsymbol{g} = \sum_n \boldsymbol{g}_n$, $\boldsymbol{s} = \sum_n \boldsymbol{s}_n$, where n is an integer. Then $\boldsymbol{a}$ and $\boldsymbol{g}$ are "rings" and $\boldsymbol{s}$ is an "ideal" of $\boldsymbol{g}$.

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