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Theorems of Bertini type for certain types of polarized manifolds

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Introduction.

In this paper we will study sufficient conditions for a given linear system Λ of Cartier divisors on a manifold M to have a non-singular member. We consider the problem over the complex number field and thus the trouble comes from the base points of Λ. Indeed, the famous theorem of Bertini says that a general member of Λ is smooth outside $\mathrm{Bs}\Lambda$.

Our main goal is the theorem (4.1) below, which gives a satisfactory answer in the case $\Delta(M, \Lambda)=2$ and $[\Lambda]$ is ample. This result was the starting point of the study of polarized manifolds of Δ-genus two in [**F1**], which will appear in a series of papers in English. Besides this, the author hopes that the technique will be useful in other contexts too. So, many lemmas are given in more general forms than necessary to prove (4.1). The most important tool for our method is the theory of intersection numbers of semipositive line bundles.

In §1 we review basic facts on semipositive line bundles and on Hironaka's elimination theory of base points of linear systems. In §2 we study the case in which there are only finite base points. In §3 we give several technical lemmas. In §4 we consider the case $\Delta=2$. Thus, the main features of [**F1**; §5-1] are retained in this paper, except a few generalizations given in §3.

Notation, convention and terminology.

Variety means an irreducible reduced projective scheme over the complex number field $\boldsymbol{C}$. *Manifold* is a non-singular variety. Tensor products of line bundles are denoted additively, while multiplicative notation is used for intersection products in Chow ring. Line bundles are confused with invertible sheaves. Thus, we employ the same notation as in [**F2**], [**F3**], which usually follows the standard one in algebraic geometry (see, e.g., [**Ha 2**]). We give here several samples.

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$[\Lambda]$: The line bundle associated with the linear system Λ.

$\mathrm{Bs}\Lambda$: The intersection of all the members of Λ.

$|L|$: The complete linear system associated with a line bundle L.

$\{Y\}$: The algebraic cycle (element in Chow ring) defined by a subspace Y.

Important convention: A pull-back of a line bundle L by a morphism is often denoted just by L, especially during calculations of intersection numbers (to avoid awkward notation).

§ 1. Preliminaries.

(1.1) DEFINITION. A line bundle L on a variety V is said to be *(numerically) semipositive* if $LC \geqq 0$ for any (integral) curve C in V. Obviously f^*L is semipositive for any morphism $f: X \to V$ if L is so.

(1.2) PROPOSITION. *Let $F_1, \cdots, F_n, F, H$ be semipositive line bundles on a variety V of dimension n. Then*

1) $F_1 \cdots F_n\{V\} \geqq 0$.

2) *$A+tF$ is ample for any ample line bundle A and for any $t \geqq 0$.*

3) *If $E=F-H$ is effective, then $F^a H^{n-a} \geqq F^b H^{n-b}$ for any $a \geqq b$.*

4) $(F_1 \cdots F_n\{V\})^2 \geqq (F_1^2 F_3 \cdots F_n\{V\})(F_2^2 F_3 \cdots F_n\{V\})$.

PROOF. 1) was proved by Kleiman [**K**]. One can refer also [**Ha 1**; p. 34]. 2) follows from 1) and Nakai's criterion (cf. [**N**] or [**Ha 1**]). To show 3), we may assume $a=b+1$. Then we have $F^a H^{n-a} - F^b H^{n-b} = F^b E H^{n-a} \geqq 0$ by 1). To prove 4), replacing F_i's by $tF_i + A$ and letting $t \to \infty$, we may assume that F_i's are ample (even very ample, by taking positive multiples of them). Taking hyperplane sections in $|F_i|$ $(i=3, \cdots, n)$, we reduce the problem to the case $n=2$. Then the index theorem implies our assertion.

(1.3) REMARK. Employing the definition of semipositivity as in [**F4**; Appendix B], we can prove analogues of 1) and 3) in the analytic category too.

(1.4) Let Λ be a linear system (of Cartier divisors) on a manifold M. Then, thanks to Hironaka [**Hi**], we can find a sequence $M' = M_r \to M_{r-1} \to \cdots \to M_1 \to M_0 = M$ of birational morphisms together with linear systems Λ_i on M_i such that:

0) $\Lambda_0 = \Lambda$.

1) $\pi_i: M_i \to M_{i-1}$ is the blowing-up of a submanifold C_i of M_{i-1} contained in $\mathrm{Bs}\Lambda_{i-1}$.

2) $\pi_i^* \Lambda_{i-1} = m_i E_i + \Lambda_i$ for some positive integer m_i, where E_i is the exceptional divisor on M_i lying over C_i, and $\mathrm{Bs}\Lambda_i \not\supset E_i$.

3) $\mathrm{Bs}\Lambda_r = \emptyset$.

Such a pair $(M_r, \Lambda_r) = (M', \Lambda')$ will be called a *Hironaka model* of (M, Λ).

(1.5) A Hironaka model is unique up to birational equivalence in the following sense. Let (M'', Λ'') be another Hironaka model of (M, Λ) and let $\pi': M' \to M$ and $\pi'': M'' \to M$ be natural morphisms respectively. Then there is a third manifold M^* and birational morphisms $p': M^* \to M'$ and $p'': M^* \to M''$ such that $\pi' \circ p' = \pi'' \circ p''$ and $(p')^* \Lambda' = (p'')^* \Lambda''$.

Let W be the image of the morphism $\rho: M' \to \boldsymbol{P}^N$ ($N = \dim \Lambda$) defined by Λ'. Let H be the hyperplane section bundle $\mathcal{O}_W(1)$. Then the polarized variety (W, H) is determined uniquely by (M, Λ). Indeed, we have $(W, H) \cong (\mathrm{Proj}(A), \mathcal{O}(1))$ where A is the graded subalgebra of $\bigoplus_{t \geq 0} H^0(M, tL)$ generated by the linear subspace of $H^0(M, L)$ with $L = [\Lambda]$ corresponding to Λ.

(1.6) In the situation as above, we employ the following notation.

X: A general fiber of $\rho: M' \to W$.

$\gamma := \dim X = n - \dim W$, where $n = \dim M$.

$\beta := \dim \mathrm{Bs} \Lambda$.

$w := \deg W$ (in $\boldsymbol{P}^N$).

π: The morphism $M' \to M$.

$L_i := [\Lambda_i]$ and $L := [\Lambda]$.

E_i^*: The total transform of E_i on M'.

E_i': The proper transform (=the prime component of E_i^* mapped onto E_i) of E_i on M'.

E: The fixed part of $\pi^* \Lambda$. So $\pi^* \Lambda = E + \Lambda'$ and $E = \sum_{i \geq 1}^{r} m_i E_i^*$.

$Y_i := \pi(E_i') = \pi(E_i^*)$. This is also the image of C_i in M.

$\nu_i := L^d \{Y_i\}$, where $L = [\Lambda]$ and $d = \dim Y_i$.

$L^s H^{n-s} := (\pi^* L)^s (\rho^* H)^{n-s} \{M'\}$. By (1.5), this intersection number is independent of the choice of a Hironaka model. Note also that $L - H = E$ in $\mathrm{Pic}(M')$. Any prime component of E is one of the E_i'.

(1.7) LEMMA. *$\beta + 1 \geqq \gamma$ unless $L^n = 0$.*

PROOF. $L^{\beta+1} E = 0$ in $\mathrm{Chow}(M')$, since $\dim Y_i \leqq \beta$ for any i. So $L^{\beta+1} H^{n-\beta-1} = L^{\beta+2} H^{n-\beta-2} = \cdots = L^n$. On the other hand $L^s H^{n-s} = 0$ for $s < \gamma$ since $\dim W = n - \gamma$. Hence $L^n = 0$ if $\beta + 1 < \gamma$.

§ 2. Isolated base points.

(2.1) Let Λ be a linear system on a variety V and set $L = [\Lambda]$. We define $d(V, \Lambda) = L^n \{V\}$ and $\Delta(V, \Lambda) = n + d(V, \Lambda) - (1 + \dim \Lambda)$. The latter will be called the *total deficiency* of Λ. Given a prepolarized variety (V, L), we define $d(V, L) = L^n \{V\} = d(V, |L|)$ and $\Delta(V, L) = \Delta(V, |L|)$. The latter will be called the *Δ-genus* of (V, L).

(2.2) THEOREM. *Let Λ be a linear system on a variety V consisting of ample divisors. Then* $\dim \mathrm{Bs} \Lambda < \Delta(V, \Lambda)$. *This means $\Delta(V, \Lambda) \geqq 0$ if* $\mathrm{Bs} \Lambda = \emptyset$.

For a proof, see [**F2**; Theorem 1.9].

(2.3) COROLLARY. *$\Delta(V, L) \geqq 0$ for any polarized variety. Moreover,* $\mathrm{Bs}|L| = \emptyset$ *if $\Delta(V, L) = 0$.*

(2.4.) COROLLARY. *Let things be as in* (1.6) *and suppose that L is ample.*

Then $\Delta(M, \Lambda)\geqq\beta+1\geqq\gamma$.

(2.5) From now on, in this section, we consider the case $\beta=\dim \mathrm{Bs}\Lambda\leqq 0$ in the situation (1.6).

PROPOSITION. *Suppose that* $\beta\leqq 0$. *Then a general member of* Λ *is non-singular unless* $d(M, L)\geqq 2^n$.

PROOF. Thanks to Bertini's theorem, it suffices to show that, for any base point p of Λ, there exists a member D of Λ which is smooth at p. Changing the order of the blowing-ups if necessary, we may assume that p is the center of the first blowing-up $\pi_1: M_1\to M$. It is enough to show $m_1=1$. Assume that $m_1\geqq 2$. Then $L_1^n=L^n-m_1^n\geqq 0$ by (1.2), since $L_1=L-m_1E_1$ is semipositive on M_1. So $d(M, L)\geqq m_1^n\geqq 2^n$, as desired.

(2.6) PROPOSITION. *Let things be as above and suppose in addition that* $\gamma=0$. *Then a general member of* Λ *is non-singular unless* $\Delta(M, \Lambda)\geqq 2^n$.

PROOF. Similarly as above, it suffices to show $m_1=1$. If not, we have $L^n-L_1^n=m_1^n\geqq 2^n$. $L_1^n\geqq H^n$ by (1.2). We have also $0\leqq\Delta(W, H)=n+w-(1+N)$ $=\Delta(M, \Lambda)-L^n+w$. Hence $\Delta(M, \Lambda)\geqq L^n-w=L^n-H^n\geqq 2^n$, as required.

(2.7) PROPOSITION. *Let things be as in* (2.5) *and suppose this time that* $\gamma>0$. *Then a general member of* Λ *is non-singular if* $d(M, \Lambda)\geqq 2\Delta(M, \Lambda)-1$.

For a proof, see [**F3**; Proposition 3.5]. The hypothesis $\Lambda=|L|$ in [**F3**] can be removed easily.

(2.8) PROPOSITION. *Let things be as in* (2.5). *Then a general member of* Λ *is non-singular if* $\Delta(M, \Lambda)\leqq 2^{n-1}$.

PROOF. If $\gamma=0$, (2.6) applies. If $\gamma>0$, combine (2.5) and (2.7).

§ 3. Base loci of higher dimension.

Throughout in this section we study the situation as in (1.6) and employ the notation there. We set $d=L^n=d(M, \Lambda)$ and $\Delta=\Delta(M, \Lambda)$. So $N=\dim \Lambda$ $=n+d-\Delta-1$. Furthermore we let j be the least integer such that $\dim Y_j=\beta$.

(3.1) LEMMA. *For any component* Z *of* E, *the class* $L^\beta Z$ *in the Chow ring of* M' *is represented by a sum of subspaces on which the pull-backs of* L_j *are semipositive.*

PROOF. Let Y be the image of Z in M. We may assume $\dim Y=\beta$. If $Y\neq Y_j$, then $E_j^*=0$ on $L^\beta Z$ and the claim is obvious. If $Y=Y_j$, then $L^\beta Z$ is represented by several fibers of $Z\to Y$, each of which is mapped into a fiber P of $E_j\to C_j$ on M_j. $P\cong \boldsymbol{P}^{n-\beta-1}$ and the restriction of L_j to P is $\mathcal{O}(m_j)$. So our assertion is valid in this case too.

(3.2) LEMMA. $L^\beta E_j^*H^{n-\beta-1}\geqq m_j^{n-\beta-1}\nu_j=(L^n-L^\beta L_j^{n-\beta})/m_j\geqq L^\beta E_j'H^{n-\beta-1}$.

PROOF. By (3.1) and (1.2) we get $L^\beta L_j^{n-\beta}\geqq L^\beta L_jH^{n-\beta-1}$ and $L^\beta L_j^{n-\beta-1}E_j'\geqq$ $L^\beta H^{n-\beta-1}E_j'$ because E_i' is effective on E_j' for any $i\neq j$. The first inequality yields $d-m_j^{n-\beta}\nu_j\geqq d-m_jL^\beta H^{n-\beta-1}E_j^*$, and hence $L^\beta E_j^*H^{n-\beta-1}\geqq m_j^{n-\beta-1}\nu_j$. We have $L^{\beta}L_j^{n-\beta}=L^{\beta}(L-m_jE_j)^{n-\beta}\{M_j\}=L^n-m_j^{n-\beta}\nu_j$, since $C_j\to Y_j$ is birational by the choice of j. We have also $L^n-L^{\beta}L_j^{n-\beta}=L^{\beta+1}L_j^{n-\beta-1}-L^{\beta}L_j^{n-\beta}=m_jL^{\beta}E_j^*L_j^{n-\beta-1}=m_jL^{\beta}E_j'L_j^{n-\beta-1}$, since any component of $E_j^*-E_j'$ is mapped onto a subspace of M_j of codimension $\geqq 2$. Combining them we get the desired inequalities.

(3.3) COROLLARY. *If in addition L is ample, then* $L^{\beta+1}H^{n-\beta-1}>L^{\beta}H^{n-\beta}$.

(3.4) LEMMA. *Suppose that* $\varDelta=\beta+1=\gamma$, $L^{\gamma}X>0$ *and* $\nu_j>0$. *Then* $d=w=1$.

PROOF. $d=L^n=L^{\beta+1}H^{n-\gamma}=wL^{\gamma}X$. Hence $d\geqq w$, since $L^{\gamma}X$ is a positive integer. On the other hand we have $0\leqq\varDelta(W, H)=n-\gamma+w-(n+d-\varDelta)=w-d$ by assumption. Combining them we infer that we must have the equalities. So $d=w$ and $L^{\gamma}X=1$.

By (3.2) we have $0<m_j^{n-\gamma}\nu_j\leqq L^{\beta}EH^{n-\beta-1}=wL^{\beta}EX$. So $L^{\beta}EX\geqq 1$. Hence $w\leqq L^{\beta}EH^{n-\gamma}=L^{\gamma}H^{n-\gamma}-L^{\beta}H^{n-\beta}=d$. Again we must have the equality here, so $L^{\beta}EX=1$. On the other hand, we have $L^{\beta}EX\geqq m_jL^{\beta}E_j^*X$ by (1.2) and $L^{\beta}E_j^*X>0$ by (3.2). Hence $m_j=1=L^{\beta}E_j^*X$. This implies $\nu_j=1$ too, because $L^{\beta}E_j^*$ is divisible by ν_j in the Chow ring of M_j. Moreover, $L^{\beta}(E-E_j^*)X=0$ implies $L^{\beta}E_i^*X=0$ for any $i\neq j$, and hence $L^{\beta}E_j'H^{n-\gamma}=L^{\beta}E_j^*H^{n-\gamma}=w$. Now we get $w\leqq m_j^{n-\gamma}\nu_j=1$ from (3.2). So $d=w=1$.

REMARK. As a matter of fact, if $\beta+1=\gamma$, $L^{\gamma}X=1$ and if $\nu_j>0$, then one can show $d=w=1$ by the same argument as above.

(3.5) THEOREM. *Let (M, L) be a polarized manifold. Then* $\varDelta(M, L)=1+\dim \mathrm{Bs}|L|=\gamma$ *if and only if* $L^n=\deg W=1$, *where W and γ are as in* (1.6).

PROOF. The "only if" part follows from (3.4). So we prove the "if" part. $d=w=1$ implies $h^0(M, L)=n+1-\varDelta$ and $W\cong \boldsymbol{P}^{n-\varDelta}$. Hence we have the equalities in (2.4). QED.

(3.6) Now we return to the situation (1.6) and this time we assume $\beta=\gamma=\varDelta(M, \varLambda)-1$ until the end of this section.

LEMMA. *Suppose that* $L^{\gamma}X>0$ *and* $\nu_j>0$. *Then* $d-w=L^{\beta}EH^{n-\beta-1}=1=L^{\beta}X$ *and* $\varDelta(W, H)=0$.

PROOF. We have $d=L^n=L^{\beta+1}H^{n-\beta-1}>L^{\beta}H^{n-\beta}=wL^{\beta}X$ and $L^{\beta}X\geqq 1$ by (3.2) and assumption. So $d>w$. On the other hand $0\leqq\varDelta(W, H)\leqq n-\gamma+w-(n+d-\varDelta)=w-d+1$, so $d\leqq w+1$. Therefore we must have the equalities and $d-w=1=L^{\beta}X=L^{\beta}EH^{n-\beta-1}$.

(3.7) COROLLARY. *Let things be as in the above lemma. Then* $L^{\beta}E_j^*H^{n-\beta-1}=L^{\beta}E_j'H^{n-\beta-1}=1=m_j=\nu_j$.

PROOF. $L^{\beta}E_j^*H^{n-\beta-1}>0$ by (3.2). $L^{\beta}(E-m_jE_j^*)H^{n-\beta-1}\geqq 0$ by (1.2). Hence $L^{\beta}EH^{n-\beta-1}=1$ implies $L^{\beta}E_j^*H^{n-\beta-1}=m_j=1$. Moreover $L^{\beta}E_i^*H^{n-\beta-1}=0$ for any $i\neq j$. So $L^{\beta}E_j'H^{n-\beta-1}=1=\nu_j$ by virtue of (3.2).

(3.8) COROLLARY. *Let things be as in* (3.7). *If* $\dim Y_k=\beta$ *for some* $k\neq j$, *then* $Y_k\neq Y_j$ *and* $L^{\beta}\{Y_k\}=0$.

PROOF. Clearly $k>j$ by the choice of j. Let Z be the image of C_k on M_j. Set $\alpha=0$ if $Z\not\subset E_j$, or $\alpha=\dim Z-\beta$ if $Z\subset E_j$. In the latter case α is the dimension of a general fiber of $Z\to C_j$. Now, by a similar argument as in (3.2) and (3.1), we infer $L^\beta L_j^\alpha L_k^{n-\beta-\alpha}\geqq L^\beta L_j^\alpha H^{n-\beta-\alpha}$. Since $L^\beta L_j^{n-\beta}=d-m_j^{n-\beta}\nu_j=d-1=w=L^\beta H^{n-\beta}$ by (3.6) and (3.7), we have $L^\beta L_j^\alpha H^{n-\beta-\alpha}=w$ by virtue of (1.2). On the other hand, we have $L^\beta L_j^\alpha L_k^{n-\beta-\alpha}=L^\beta L_j^{n-\beta}-m_k^{n-\beta-\alpha}L^\beta L_j^\alpha Z$. Thus we obtain $L^\beta L_j^\alpha Z=0$. Our assertion follows from this equality.

(3.9) COROLLARY. *Let things be as in* (3.7) *and suppose that* L *is ample. Then* BsΛ *has only one irreducible component* Y *of dimension* β. *Moreover* $L^\beta Y=1$ *and a general member of* Λ *is smooth at a general point of* Y.

PROOF. The first assertion follows from (3.8). The smoothness is a consequence of $m_j=1$.

(3.10) LEMMA. *Let things be as in* (3.9). *Then* $L^{\beta-1}E_j^*X=1$ *and* $L^{\beta-1}E_i^*X=0$ *for any* $i\neq j$.

PROOF. We have $L^{\beta-1}EX=L^{\beta-1}(L-H)X=L^\beta X=1$ by (3.6). Hence it suffices to show $L^{\beta-1}E_i^*X=0$ for $i\neq j$. Assume that $L^{\beta-1}E_k^*X>0$ for some $k\neq j$. Then, from $L^{\beta-1}E_i^*X\geqq 0$, we infer $m_k=L^{\beta-1}E_k^*X=1$ and $L^{\beta-1}E_i^*X=0$ for any $i\neq k$. Moreover, by (3.8), we infer that $\dim Y_k=\beta-1$ and $L^{\beta-1}Y_k=1$.

Now, changing the order of the blowing-ups to obtain a Hironaka model if necessary, we may assume that $k<j$. Moreover $Y_i\neq Y_k$ for any $i<k$. Indeed, if $Y_i=Y_k$, the image of C_k in M_i would be contained in E_i, and hence $E_i^*-E_k^*$ would be effective. This contradicts $L^{\beta-1}(E_i^*-E_k^*)X<0$. Thus we conclude that π_k is the first blowing-up whose center lies over Y_k.

L_k is semipositive on subspaces representing the cycle $L^{\beta-1}E_k'$, and L_k-H is effective there. Hence $L^{\beta-1}L_k^{n-\beta}E_k'\geqq L^{\beta-1}H^{n-\beta}E_k'$. The right side is equal to $wL^{\beta-1}E_k'X=w$ and the left side is $m_k^{n-\beta}\nu_k=1$, where $\nu_k=L^{\beta-1}C_k=L^{\beta-1}Y_k$. So $w=1$, hence $d=2$ by (3.6). Thus $\dim\Lambda=\dim W=n+d-\Delta-1=n-\gamma$ and $W\cong \boldsymbol{P}^{n-\beta}$. Hence Bs$\Lambda$ is the intersection of $n-\beta$ general members of it, and so any component of it is of dimension β. From this and (3.9) we get Bs$\Lambda=Y_j$ and in particular $Y_k\subset Y_j$.

Now we infer that BsΛ_k contains the proper transform of Y_j. So $E_k\cap$BsΛ_k meets every general fiber P of $E_k\to C_k$. The proper transform P' of $P\cong\boldsymbol{P}^{n-\beta}$ on M' gives a Hironaka model with respect to the restriction of Λ_k to P. Therefore we infer that $H^{n-\beta}\{P'\}=0$ since the above linear system consists of hyperplanes and has base points. This implies $L^{\beta-1}H^{n-\beta}E_k'=0$, contradicting $L^{\beta-1}E_k'X=L^{\beta-1}E_k^*X=1$. QED.

§4. The case $\Delta=2$.

The main purpose of this section is to prove the following

(4.1) THEOREM. *Let* (M, L) *be a polarized manifold such that* $n=\dim M\geqq 3$,

$d=d(M, L)\geqq 2$ and $\Delta(M, L)=2$. Then a general member of $|L|$ is non-singular.

Actually, we can show the assertion for a non-complete linear system Λ with $\Delta(M, \Lambda)=2$ too. In the proof below, we employ the notation as in (1.6).

(4.2) $\beta\leqq 1$ by (2.2). If $\beta\leqq 0$, then (2.8) applies. So it suffices to consider the case $\beta=1$. Then $\gamma\leqq 2$ by (1.7). $\gamma\neq 2$ by (3.5) because $d\geqq 2$. So $\gamma=1$ or 0. At first, until (4.7), we consider the case $\gamma=1$.

(4.3) CLAIM. *Λ has no isolated base point.*

Otherwise, we can change the order of blowing-ups so that there is an integer k such that $k>j$, $Y_i\subset Y_j$ for any $i<k$, $Y_i\cap Y_j=\emptyset$ for any $i\geqq k$ and Y_k is an isolated point of BsΛ. Then, by similar arguments as in (3.2), we get $L_kH^{n-1}\leqq L_k^n<L_j^n\leqq LL_j^{n-1}<L^n$, and $L_kH^{n-1}=L_jH^{n-1}=d-1$ by (3.10). This leads to a contradiction.

(4.4) Taking general members of Λ' successively we get a sequence $M'=S_n\supset S_{n-1}\supset\cdots\supset S_2$ of submanifolds of M' with $\dim S_i=i$. Let $D_i=\pi(S_i)\subset M$ for each i. Then D_i is a member of the restriction of Λ to D_{i+1}. D_i is non-singular outside BsΛ, which is the integral curve Y_j by (3.9) and (4.3). Moreover, by (3.7) and (3.8), we see that D_i is non-singular at a general point on Y_j. Thus, the singular locus of D_i is at most finite. Hence D_i is normal for every $i\geqq 2$ by Serre's criterion.

Let $\Sigma=\bigcup_{i\neq j}E_i'$ and set $\Sigma_k=\Sigma\cap S_k$ for $k=n,\cdots,2$. Note that $\pi(\Sigma)$ is a finite subset of M. From $m_j=1$, we infer that $S_k-\Sigma_k$ is mapped isomorphically onto D_k minus this subset for every $k\geqq 2$.

(4.5) CLAIM. *$Y_j=$BsΛ is a rational normal curve.*

PROOF. Let W_k be the image of S_k via ρ. They are successive hyperplane sections of W. So $W_2\cong \boldsymbol{P}^1$ since $\Delta(W, H)=0$ (cf. [**F2**]). By (3.10) we have $H\Sigma_2=H^{n-1}\Sigma=0$. Hence any component of Σ_2 is contained in a fiber of $\rho_2: S_2\to W_2$.

Let X_1 and X_2 be general fibers of ρ_2. By the above observation we see $X_q\cap\Sigma_2=\emptyset$ for $q=1, 2$. So they are mapped isomorphically onto Cartier divisors F_1, F_2 on D_2. Since D_2 is normal and X_q's are linearly equivalent, $F_1=F_2$ in Pic(D_2). Thus Bs$|F|=\emptyset$ where $F=[F_q]$. We obtain $FY_j=EX=1$ by (3.10). Combining them we infer $Y_j\cong\boldsymbol{P}^1$.

(4.6) In order to finish the proof in case $\gamma=1$, it suffices to show that, for any point p on Bs$\Lambda\cong\boldsymbol{P}^1$, there exists a member D of Λ which is smooth at p.

We change the order of the blowing-ups so that the first center $C_1=Y_1$ is the given point p. It is easy to see that $(m_1-1)L+L_1=m_1(L-E_1)$ is semipositive. So $((m_1-1)L+L_1)^n\geqq((m_1-1)L+L_1)((m_1-1)L+H)^{n-1}$ by (1.2). The left hand side is $(m_1(L-E_1))^n=m_1^n(d-1)$. To calculate the right hand side R, we note $E_1L=0$ in the Chow ring. We have $E_1H^{n-1}=0$ by (3.10). So $R=m_1(L-E_1)((m_1-1)L+H)^{n-1}=m_1L((m_1-1)L+H)^{n-1}=m_1^nd-m_1(L^n-LH^{n-1})=$

$m_1^n d - m_1$ by (3.6). Thus the above inequality gives $m_1^n \leqq m_1$, hence $m_1 = 1$. This implies the smoothness of D at p.

(4.7) From now on, we consider the case $\gamma = 0$. So $\dim W = n$.

$L^n = L^2 H^{n-2} > LH^{n-1}$ by (3.3). Hence $LH^{n-1} > H^n$ by (1.2; 4). $H^n = w \deg(\rho) \geqq w$. On the other hand we have $0 \leqq \Delta(W, H) = n + w - (n + d - 2) = 2 + w - d$. Combining them we obtain $\Delta(W, H) = w + 2 - d = 0$, $\deg(\rho) = 1$ and $L^n - LH^{n-1} = LH^{n-1} - H^n = 1$. Hence $LEH^{n-2} = EH^{n-1} = 1$.

(4.8) Let j be the least integer such that $\dim Y_j = 1$. Similarly as in (3.7), we obtain $m_j = \nu_j = LE_j^* H^{n-2} = LE_j' H^{n-2} = 1$ from $LEH^{n-2} = 1$. So $LE_i^* H^{n-2} = LE_i' H^{n-2} = 0$ for any $i \neq j$. Using this similarly as in (3.8), we infer that Y_i is a point for $i \neq j$.

(4.9) CLAIM. *Λ has no isolated base point.*

Otherwise, we can take k as in (4.3) after changing the Hironaka model. Note that $\mathrm{Bs}\Lambda_{k-1}$ is a finite set and L_{k-1} is semipositive. One easily sees that L_k is semipositive too. By (3.3) we get $L^n = L^2 L_{k-1}^{n-2} > LL_j^{n-1} = LL_{k-1}^{n-1}$. So $LL_{k-1}^{n-1} > L_{k-1}^n$ by (1.2; 4). We have $L_{k-1}^n > L_k^n \geqq H^n$ too. Combining them we obtain $L^n - H^n \geqq 3$, contradicting $d - w = 2$ in (4.7).

(4.10) CLAIM. $E_j^* H^{n-1} = E_j' H^{n-1} = 1$ *and* $E_i^* H^{n-1} = E_i' H^{n-1} = 0$ *for any* $i \neq j$.

This is proved similarly as (3.10). Assume that $E_k^* H^{n-1} > 0$ for some $k \neq j$. Then $EH^{n-1} = 1$ implies that $m_k = 1$ and $H^{n-1} E_k^* = 1$. Moreover $E_i^* H^{n-1} = 0$ for any $i \neq k$. So $H^{n-1} E_k' = 1$.

Y_k is a point by (4.8). Moreover $Y_i \not\supset Y_k$ for any $i < k$, because otherwise $E_i^* - E_k^*$ would be effective and $H^{n-1} E_i^* \geqq H^{n-1} E_k^*$. So, changing the order of blowing-ups if necessary, we may assume $k = 1$. Then we have $1 = H^{n-1} E_k' \leqq L_k^{n-1} E_k' = 1$, since $L_k - H$ is effective on E_k'. On M_k, the restriction of Λ_k to $E_k \cong \boldsymbol{P}^{n-1}$ is a linear system consisting of hyperplanes. Therefore, $L_k^{n-1} E_k' = H^{n-1} E_k'$ implies that $E_k \cap \mathrm{Bs}\Lambda_k = \emptyset$. On the other hand, Y_k is a point on Y_j by (4.9). So E_k meets the proper transform of Y_j on M_k, which is clearly a component of $\mathrm{Bs}\Lambda_k$. This contradiction proves our claim.

(4.11) CLAIM. $\mathrm{Bs}\Lambda = Y_j$ *and this is a rational normal curve.*

PROOF. Take S_i ($i = n, n-1, \cdots, 2$), D_i and Σ_i as in (4.4). Using (4.8), we infer that D_i's are non-singular outside a finite set. So they are normal as in (4.4). Moreover, $S_i - \Sigma_i$ is mapped isomorphically onto an open subset of D_i.

$H^{n-1} \Sigma = 0$ and $H^{n-1} E_j' = 1$ by (4.10). So the restrictions X_1 and X_2 of two general members of Λ' to S_2 satisfy $X_q \cap \Sigma_2 = \emptyset$, and $X_q Y = 1$ for $q = 1, 2$ where $Y = S_2 \cap E_j'$. Therefore they are mapped isomorphically onto Cartier divisors F_1, F_2 on D_2. $F_1 = F_2$ in $\mathrm{Pic}(D_2)$ since D_2 is normal. Thus, their restrictions to $Y_j = \pi(Y)$ give rise to a linear system Ξ with $\deg \Xi = 1$ and $\mathrm{Bs}\Xi = \emptyset$. Hence $Y_j \cong \boldsymbol{P}^1$. By virtue of (4.8) and (4.9), $\mathrm{Bs}\Lambda$ has no other component than Y_j.

(4.12) Now, we finish the proof of (4.1) by the same argument in (4.6).

Here we use (4.10) and (4.7) instead of (3.10) and (3.6). QED.

(4.13) REMARK. As a matter of fact, the case $\beta=1$ and $\gamma=0$ does not happen when $\Lambda=|L|$. We will show this by induction on $n=\dim M$.

Suppose that $n=2$. Write $|L|=F+\Lambda'$ where F is the fixed part. From the preceding argument we infer that F consists of one prime component $Y\cong \boldsymbol{P}^1$ and $\mathrm{Bs}\Lambda'=\emptyset$. Furthermore, Λ' defines a birational morphism onto $W\subset \boldsymbol{P}^N$ where W is a subvariety with $\deg W=w$ and $\Delta(W, H)=0$. Using [**F3**; Theorem 4.1; c)], we infer that a general member X of Λ' is a smooth rational curve. Moreover $XY=1$ as in (4.12). Since $X^2>0$, M is a rational surface by Noether's criterion. So, using the exact sequence $H^1(M, \mathcal{O})\to H^1(M, H)\to H^1(X, H_X)$, we obtain $H^1(M, H)=0$. The exact sequence $0\to\mathcal{O}_M[H]\to\mathcal{O}_M[L]\to\mathcal{O}_Y[L]\to 0$ yields $H^0(M, L)\to H^0(Y, L)\to H^1(M, H)=0$. Hence the restriction mapping $H^0(M, L)\to H^0(Y, L)$ is surjective. However, this is a zero mapping because $Y=\mathrm{Bs}|L|$. So $H^0(Y, L)=0$, contradicting $Y\cong\boldsymbol{P}^1$ and $LY>0$.

In case $n\geqq 3$, let D be a general member of $|L|$. This is smooth by (4.1). We have $\Delta(D, L_D)\leqq\Delta(M, L)$ by [**F2**; Proposition 1.5]. If $\Delta(D, L_D)\leqq 1$, then $h^1(D, \mathcal{O}_D)=0$ by [**F2**] and [**F5**]. So the inequality in the proposition cited above cannot be satisfied. Hence $\Delta(D, L_D)=2$ and $H^0(M, L)\to H^0(D, L_D)$ is surjective. Applying the induction hypothesis to (D, L_D) we prove the assertion for (M, L) too. QED.

More precise description of the structure of polarized manifolds with $\Delta=2$ and $\beta=1$ will be given in a future paper (or see [**F1**]).

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