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Transcendence bases for field extensions

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Introduction. Let K be a field of characteristic $p>0$ and let L/K be a finitely generated field extension. For a transcendence basis T of L/K, let S_T denote the separable algebraic closure of $K(T)$ in L. Consider the minimums of the following degrees taken over all transcendence bases T of L/K: (1) $[L:S_T]$, (2) $[S_T:K(T)]$, (3) $[L:K(T)]$. A transcendence basis T which yields the minimum value in (1), (2), (3) is called an S-basis, I-basis and A-basis, respectively. Considerable information is known concerning S-bases. For example, if T is an S-basis of L/K, then $\log_p[L:S_T]$ is Weil's order of inseparability [**19**]. An intermediate field D of L/K is called distinguished when D/K is separable and $L\subseteqq K^{p^{-n}}(D)$ where n is the inseparability exponent of L/K [8]. In [**11**], it is shown that the distinguished subfields of L/K are those which are separable over K and over which L is of minimal degree. Hence if T is an S-basis of L/K, S_T is a distinguished subfield. Distinguished subfields have been studied anew in [3], [**4**], [**5**], [**6**], [7], [**14**], [**15**] and [**16**]. It is shown in [4] for example that if F is an intermediate field of L/K and T, X are S-bases of L/F, F/K respectively, then $[L:S_T]\geqq[F:S_X]$. A recent paper [**13**] gives a generalization of Luroth's theorem by showing that if K is infinite and F is an intermediate field of transcendence degree one over K, then $[L:K(T)]\geqq[F:K(X)]$ where T, X are A-basis of L/K, F/K respectively.

In this paper we show that if K is infinite and F is an intermediate field of transcendence degree one over K, then $[S_T:K(T)]\geqq[S_X:K(X)]$ where T, X are I-bases of L/K, F/K respectively. Along the way we determine properties of S-bases, I-bases, and A-bases. We connect these results to the theory of unirational varieties (one whose function field is a subfield of a pure transcendental extension), the theory of generalized primitive elements [**2**] and purely inseparable K-forms [**10**].

Transcendence Bases. If T is an S-basis of L/K, then $\log_p[L:S_T]$ is called the order of inseparability of L/K, $\mathrm{inor}(L/K)$ [**11**]. If T is an I-basis of L/K, then $[S_T:K(T)]$ is called the order of separability of L/K, $\mathrm{os}(L/K)$. If T is an A-basis of L/K, then $[L:K(T)]$ is called the irrationality of L/K, $\mathrm{irr}(L/K)$ [**12**]. We let tr.deg.(L/K) denote the transcendence degree of L/K.

PROPOSITION 1. *Let F be an intermediate field of L/K such that L/F is purely inseparable. Then every I-basis of F/K is one of L/K and* $\mathrm{os}(F/K)=\mathrm{os}(L/K)$.

PROOF. If T is an I-basis of L/K, then $\mathrm{os}(L/K)=[L:K(T^{p^m})]_s \geqq \mathrm{os}(F/K)$ where m is such that $T^{p^m} \subseteqq F$. Let X be an I-basis of F/K. Then $\mathrm{os}(F/K)=[F:K(X)]_s \geqq \mathrm{os}(L/K)$ since L/F is purely inseparable. Hence $[F:K(X)]_s=\mathrm{os}(L/K)$ so X is an I-basis of L/K.

This result corresponds to the fact that if L/F were separable, $\mathrm{inor}(L/K)=\mathrm{inor}(F/K)$ [**4**]. The effect on the order of separability under a separable algebraic extension is variable. It is clear that it can increase, and in view of the existence of unirational varieties in characteristic 0 it can also decrease.

THEOREM 2. *Let F be an intermediate field of L/K and suppose K is infinite. If* $\mathrm{tr.deg.}(F/K) \leqq 1$, *then* $\mathrm{os}(F/K) \leqq \mathrm{os}(L/K)$.

PROOF. If F/K is algebraic, then the result is immediate. Suppose $\mathrm{tr.deg.}(F/K)=1$. Let T be an I-basis of L/K. Then there exists e large enough so that $K(F^{p^e}) \subseteqq S_T$. By Proposition 1, $\mathrm{os}(L/K)=\mathrm{os}(S_T/K)$ and $\mathrm{os}(F/K)=\mathrm{os}(K(F^{p^e})/K)$. On the other hand, $\mathrm{os}(S_T/K)=\mathrm{irr}(S_T/K)$ by definitions and Proposition 1 and $\mathrm{irr}(S_T/K) \geqq \mathrm{irr}(K(F^{p^e})/K)$ by [**13**, Theorem 2]. Since we have $\mathrm{irr}(K(F^{p^e})/K) \geqq \mathrm{os}(K(F^{p^e})/K)$, we get our assertion.

An S-basis which yields the minimum of $[S_T:K(T)]$ over all S-bases T is called an S^*-basis of L/K and an I-basis which yields the minimum of $[L:S_T]$ over all I-bases T is called an I^*-basis of L/K. If F is an intermediate field of L/K, we let $\mathcal{T}_S^*(F/K)$, $\mathcal{T}_I^*(F/K)$, and $\mathcal{T}_A(F/K)$ denote the set of all S^*-bases, I^*-bases, and A-bases of F/K respectively. We write $\mathcal{T}_S^*=\mathcal{T}_S^*(L/K)$, $\mathcal{T}_I^*=\mathcal{T}_I^*(L/K)$, and $\mathcal{T}_A=\mathcal{T}_A(L/K)$.

PROPOSITION 3. (1) *Either* $\mathcal{T}_S^* \cap \mathcal{T}_I^*=\emptyset$ *or* $\mathcal{T}_S^*=\mathcal{T}_I^*$. (2) *If* $\mathcal{T}_S^*=\mathcal{T}_I^*$, *then* $\mathcal{T}_S^*=\mathcal{T}_I^*=\mathcal{T}_A$.

PROOF. (1) Suppose $T \in \mathcal{T}_S^* \cap \mathcal{T}_I^*$. Let $T_1 \in \mathcal{T}_S^*$ and $T_2 \in \mathcal{T}_I^*$. Since $T, T_1 \in \mathcal{T}_S^*$, $[S_T:K(T)]=[S_{T_1}:K(T_1)]$. Since $T, T_2 \in \mathcal{T}_I^*$, $[L:S_T]=[L:S_{T_2}]$. Thus $T_1 \in \mathcal{T}_I^*$ since $T \in \mathcal{T}_I^*$ and $T_2 \in \mathcal{T}_S^*$ since $T \in \mathcal{T}_S^*$.

(2) Let $T \in \mathcal{T}_S^*=\mathcal{T}_I^*$. Then $[L:S_T]$ and $[S_T:K(T)]$ are minimal. Hence $[L:K(T)]$ is minimal and so $T \in \mathcal{T}_A$. Let $T \in \mathcal{T}_A$. For $X \in \mathcal{T}_S^*=\mathcal{T}_I^*$, $[L:S_X] \leqq [L:S_T]$ and $[S_X:K(X)] \leqq [S_T:(T)]$. Since $T \in \mathcal{T}_A$, these inequalities must be equalities. Thus $T \in \mathcal{T}_S^* \cap \mathcal{T}_I^*$.

PROPOSITION 4. (1) $\mathrm{os}(L/K)=1$ *and* $\mathcal{T}_S^*=\mathcal{T}_I^*$ *if and only if L/K has a pure transcendental distinguished subfield.*

(2) $\mathrm{os}(L/K)=1$, $\mathrm{inor}(L/K)=0$, *and* $\mathcal{T}_S^*=\mathcal{T}_I^*$ *if and only if L/K is pure transcendental.*

PROOF. (1) Suppose $\mathrm{os}(L/K)=1$ and $\mathcal{T}_S^*=\mathcal{T}_I^*$. Let $T \in \mathcal{T}_S^*=\mathcal{T}_I^*$. Then $S_T=K(T)$ by the assumption that $\mathrm{os}(L/K)=1$. Conversely, suppose $D=K(T)$

where T is algebraically independent over K and D is distinguished. Then $\mathrm{os}(D/K)=1$ and so $\mathrm{os}(L/K)=1$ by Proposition 1. Since $T\in\mathfrak{T}_S^*\cap\mathfrak{T}_I^*$, $\mathfrak{T}_S^*=\mathfrak{T}_I^*$ by Proposition 2.

(2) The result here follows from (1) and the fact that L/K is separable if and only if $\mathrm{inor}(L/K)=0$.

If $\{x, y\}$ is algebraically independent over K, then any surface whose function field L satisfies $K(x^p, y^p)\subsetneqq L\subsetneqq K(x, y)$ is called a Zariski surface.

It follows from (2) of Proposition 4 that if L/K is the function field of a Zariski surface [**1**], then $\mathfrak{T}_S^*=\mathfrak{T}_I^*$ if and only if the surface is rational. We also note that if L/K is separable and $\mathrm{os}(L/K)=1=\mathrm{tr.deg.}(L/K)$, then an I-basis t of L/K is a generalized primitive element of L/K [**2**]. This follows since if F is an intermediate field of L/K such that L/F is separable algebraic, then $L=F(t)$ since $L/F(t)$ is purely inseparable.

Recall that L/K is called unirational if L is a subfield of a pure transcendental extension of K.

PROPOSITION 5. *If L/K is unirational and K is infinite then there is a separable algebraic extension L_1 of L such that* $\mathrm{os}(L_1/K)=1$.

PROOF. Let $K\subset L\subset K(x_1, x_2, \cdots, x_n)$ where $\{x_1, \cdots, x_n\}$ is algebraically independent over K. By [**17**, Lemma 1, p. 209] we may assume L/K has transcendence degree n. Let L_1 be the separable algebraic closure of L in $K(x_1, \cdots, x_n)$. Since $K(x_1, \cdots, x_n)$ is purely inseparable over L_1, $\mathrm{os}(L_1/K)=1$.

PROPOSITION 6. *$\mathfrak{T}_S^*=\mathfrak{T}_I^*$ if and only if $\mathfrak{T}_S^*(D/K)=\mathfrak{T}_I^*(D/K)$ for some distinguished subfield D of L/K. In either case, $\mathfrak{T}_S^*=\mathfrak{T}_I^*\supseteq\mathfrak{T}_S^*(D/K)=\mathfrak{T}_I^*(D/K)$.*

PROOF. Suppose $\mathfrak{T}_S^*=\mathfrak{T}_I^*$ and let $T\in\mathfrak{T}_S^*\cap\mathfrak{T}_I^*$. Then $D=S_T$ is a distinguished subfield of L/K and $T\in\mathfrak{T}_S^*(D/K)\cap\mathfrak{T}_I^*(D/K)$ since $T\in\mathfrak{T}_I^*$ and $\mathrm{os}(L/K)=\mathrm{os}(D/K)$. Hence $\mathfrak{T}_S^*(D/K)=\mathfrak{T}_I^*(D/K)$ by Proposition 3. Conversely, suppose $\mathfrak{T}_S^*(D/K)=\mathfrak{T}_I^*(D/K)$ for some D. Let $T\in\mathfrak{T}_S^*(D/K)=\mathfrak{T}_I^*(D/K)$. Then T is an S-basis and an I-basis of L/K by Proposition 1. Thus T is in both $\mathfrak{T}_S^*$ and $\mathfrak{T}_I^*$.

PROPOSITION 7. *If $\mathfrak{T}_S^*=\mathfrak{T}_I^*$, then $\mathfrak{T}_S^*(K(L^{p^i})/K)=\mathfrak{T}_I^*(K(L^{p^i})/K)$ for $i=1, 2, \cdots$.*

PROOF. We show $\mathfrak{T}_S^*(K(L^p)/K)=\mathfrak{T}_I^*(K(L^p)/K)$. Let $T\in\mathfrak{T}_S^*$ and $D=S_T$. Then $T\in\mathfrak{T}_S^*(D/K)=\mathfrak{T}_I^*(D/K)$ as in the proof of Proposition 6. Since $D/K(T)$ is separable algebraic, $K(D^p)/K(T^p)$ is separable algebraic and so T^p is an S-basis of $K(D^p/K)$. Since $[K(D^p):K(T^p)]=[D:K(T)]=\mathrm{os}(D/K)=\mathrm{os}(K(D^p)/K)$ by Proposition 1, T^p is an I-basis of $K(D^p)/K$. Thus $T^p\in\mathfrak{T}_S^*(K(D^p)/K)\cap\mathfrak{T}_I^*(K(D^p)/K)$. Hence $\mathfrak{T}_S^*(K(D^p)/K)=\mathfrak{T}_I^*(K(D^p)/K)$. Thus $\mathfrak{T}_S^*(K(L^p)/K)=\mathfrak{T}_I^*(K(L^p)/K)$ by Proposition 6.

PROPOSITION 8. *There exists a subfield L_1 of L/K with L purely inseparable over L_1 and such that $\mathfrak{T}_S^*(L_1/K)=\mathfrak{T}_I^*(L_1/K)$. In particular if* $\mathrm{tr.deg.}(L/K)=1$, *there is a non-negative integer r such that $\mathfrak{T}_S^*(K(L^{p^r})/K)=\mathfrak{T}_I^*(K(L^{p^r})/K)$.*

PROOF. Let T be an I^*-basis of L/K. Consider $L_1=S_T$. Then T is an I^*-

basis of S_T and an S-basis, hence an S^*-basis of S_T over K. If tr.deg.$(L/K)=1$, then $K(L_1^{p^m})=K(L^{p^r})$ for some m and r for since L_1/K is separably generated, $[K(L_1^{p^{e-1}}): K(L_1^{p^e})]=p$ and the only chain of subfields between L_1 and $K(L_1^{p^e})$ is $L_1 \supset \cdots \supset K(L_1^{p^{e-1}})$ for each e. Thus Proposition 7 shows $\mathcal{T}_S^*(K(L^{p^r})/K)=\mathcal{T}_I^*(K(L^{p^r})/K)$ for some r.

Thus, for any L/K there is always a subfield L_1 as in Proposition 8 over which L is purely inseparable of minimal degree. If L is the function field of an irrational Zariski surface, then in view of Proposition 4, this minimal degree is p. If the Zariski surface is also $K3$, then any subfield L_1 over which L is purely inseparable and of dimension p has $\mathcal{T}_S^*(L_1/K)=\mathcal{T}_I^*(L_1/K)$ [**18**, Theorem 5, p. 1216]. We conjecture that this is true for all irrational Zariski surfaces.

PROPOSITION 9. *Suppose* tr.deg.$(L/K)=1$. *Then* os$(L/K)=1$ *if and only if* $K(L^{p^s})/K$ *is pure transcendental for some nonnegative integer* s.

PROOF. Suppose os$(L/K)=1$. By Proposition 8, there exists an r such that $\mathcal{T}_S^*(K(L^{p^r})/K)=\mathcal{T}_I^*(K(L^{p^r})/K)$. By Proposition 7, $\mathcal{T}_S^*(K(L^{p^s})/K)=\mathcal{T}_I^*(K(L^{p^s})/K)$ for $s \geqq r$. For large s, $K(L^{p^s})/K$ is separable and os$(K(L^{p^s})/K)=1$ by Proposition 1. Thus $K(L^{p^s})$ is pure transcendental over K by (2) of Proposition 4. The converse follows from Proposition 1.

PROPOSITION 10. *Let D be a distinguished subfield of L/K. Then D/K is a purely inseparable K-form of $K(t)$ where t is transcendental over K* [**10**] *if and only if* tr.deg.$(L/K)=1=$os(L/K). *If D is a purely inseparable K-form of $K(t)$, then $u=s=r$ where u is the height of D/K* [**10**, p. 12], *s is the smallest non-negative integer such that $K(D^{p^s})$ is pure transcendental over K, and r is the smallest non-negative integer such that $\mathcal{T}_S^*(K(D^{p^r})/K)=\mathcal{T}_I^*(K(D^{p^r})/K)$.*

PROOF. Suppose D/K is a purely inseparable K-form of $K(t)$. Then D is K-isomorphic to distinguished subfield of $K^{p^{-u}} \otimes_K K(t)$. Hence os$(D/K)=1$ so os$(L/K)=1$. Conversely, suppose tr.deg.$(L/K)=1=$os(L/K). Then the same is true for D/K and by Proposition 9, $K(D^{p^s})/K$ is pure transcendental for some (smallest) s. Thus $K(D^{p^s})=K(x)$ for some x transcendental over K and so $K^{p^{-s}} \otimes_K D=K^{p^{-s}} \otimes_K K(x^{p^{-s}})$. Hence D is a purely inseparable K-form of $K(t)$ and $u \leqq s$. Now suppose D is a purely inseparable K-form of $K(t)$. Since $K(D^{p^u})/K$ is pure transcendental, $s \leqq u$. By (2) of Proposition 4, $K(D^{p^r})/K$ is pure transcendental so $s \leqq r$. However if $K(D^{p^s})/K$ is pure transcendental, then clearly $\mathcal{T}_S^*(K(D^{p^s})/K)=\mathcal{T}_S^*(K(D^{p^s})/K)$, so $r \leqq s$.

In view of Proposition 10, if D/K is a separable extension such that tr.deg.$(D/K)=1=$os(D/K) and $D=K(x, y)$ where x is an I-basis of D/K, then results 1.5.1, 1.5.2, and 1.5.3 of [**10**] hold with D/K replacing K/k there.

COROLLARY 11. *Suppose K is infinite and let F be an intermediate field of L/K. If* tr.deg.$(L/K)=1=$os(L/K), *then there exists r such that $K(F^{p^r})/K$ is pure transcendental or $K(F^{p^r})=K$.*

PROOF. There exists r large enough such that $\mathcal{T}_S^*(K(F^{p^r})/K)=\mathcal{T}_I^*(K(F^{p^r})/K)$ and $K(F^{p^r})/K$ is separable. If tr.deg.$(F/K)=1$, then $K(F^{p^r})/K$ is pure transcendental since $\mathrm{os}(K(F^{p^r})/K)=1$. If F/K is algebraic, then F/K is purely inseparable since $\mathrm{os}(L/K)=1$.

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