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Complex crystallographic groups II

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§ 0. Introduction.

Let $\boldsymbol{E}(n)$ be the complex motion group acting on the n-dimensional complex euclidean space $X\cong\boldsymbol{C}^{n}$. A complex crystallographic group is, by definition, a discrete subgroup of $\boldsymbol{E}(n)$ with compact quotient. In a previous paper [6], we studied general properties of the quotient varieties and determined all the two dimensional crystallographic reflection groups.

In this paper, we treat two dimensional complex crystallographic group Γ such that the quotient variety $M=X/\Gamma$ is biholomorphic to the two dimensional projective space $\boldsymbol{P}^{2}$. We list up all such groups (Theorem 1). Generators and fundamental relations are obtained (Theorem 2). Let ϕ denote the natural mapping: $X\rightarrow M$. The coordinate representation of ϕ, the branching locus D and the ramification indices of ϕ on D are determined (Theorem 3). We explicitly give the representation $h:\pi_{1}(M-D)\rightarrow\Gamma$ and the kernel of h (Theorem 4).

§ 1. Notations and definitions.

The unitary group of size 2 is denoted by $U(2)$. For $A\in U(2)$ and $a\in\boldsymbol{C}^{2}$, $(A|a)\in\boldsymbol{E}(2)$ denotes the transformation: $x\rightarrow Ax+a$. For a two dimensional complex crystallographic group Γ,

$$L:=\{a\,;\,(1|a)\in\Gamma\}$$

and

$$G:=\{A\,;\,(A|a)\in\Gamma\}$$

are called the lattice and the point group of Γ, respectively. If Γ has the representation $\{(A|a)\,;\,A\in G,\ a\in L\}$, then we call Γ the semidirect product $G\ltimes L$ of the lattice and the point group.

DEFINITION. Imprimitive reflection group $G(m,p,2)\subset U(2)$ is the group generated by

$$\begin{pmatrix} & 1\\ 1 & \end{pmatrix},\ \begin{pmatrix} & \theta\\ \theta^{-1} & \end{pmatrix}\ \text{and}\ \begin{pmatrix}\theta^{p} & \\ & 1\end{pmatrix},\ \theta=\exp\frac{2\pi\sqrt{-1}}{m}.$$

DEFINITION. An element $g\in\boldsymbol{E}(2)$ is called a reflection if g is of finite order, $g\neq$identity and keeps a line $H(g)\subset X$ pointwise fixed.

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§ 2. Generators and relations of Γ.

Let Γ be a two dimensional complex crystallographic group. If the quotient variety $M=X/\Gamma$ is biholomorphic to $\boldsymbol{P}^2$, then Γ is generated by finitely many reflections ([**6**, Corollary 3.2.2]). Every crystallographic reflection group and the quotient varieties are known ([**4**], [**6**, Theorem 5.1]). Combining them, we have the following theorem.

THEOREM 1. *Every two dimensional complex crystallographic group with $M\cong\boldsymbol{P}^2$ is conjugate, in the affine transformation group, to one of the following six groups,*

$$(2,1)_0 := G(2,1,2)\ltimes L^2(\tau),$$

$$(3,1)_0 := G(3,1,2)\ltimes L^2(\zeta),$$

$$(4,1)_0 := G(4,1,2)\ltimes L^2(i),$$

$$(6,1)_0 := G(6,1,2)\ltimes L^2(\zeta),$$

$$(4,2)_1 := G(4,2,2)\ltimes \left\{L^2(i)+\boldsymbol{Z}\frac{1+i}{2}\begin{pmatrix}1\\1\end{pmatrix}\right\},$$

$$(3,3)_0 := G(3,3,2)\ltimes \left\{L(\tau)\begin{pmatrix}-1\\1\end{pmatrix}+L(\tau)\begin{pmatrix}\zeta^2\\\zeta\end{pmatrix}\right\},$$

where $L(\tau)=\boldsymbol{Z}+\tau\boldsymbol{Z}$, $L^2(\tau)=L(\tau)\begin{pmatrix}1\\0\end{pmatrix}+L(\tau)\begin{pmatrix}0\\1\end{pmatrix}$, $\operatorname{Im}\tau>0$, $i=\sqrt{-1}$, $\zeta=\exp\frac{2\pi i}{6}$.

The groups in Theorem 1 are generated by reflections. We shall give their fundamental relations.

THEOREM 2. *The groups in Theorem 1 have the following generators and the defining relations:*

Γ	generators	relations
$(2,1)_0$	$A_1=\left(\begin{array}{cc\|c}-1&0&0\\0&1&0\end{array}\right)$, $A_2=\left(\begin{array}{cc\|c}-1&0&1\\0&1&0\end{array}\right)$, $A_3=\left(\begin{array}{cc\|c}-1&0&\tau\\0&1&0\end{array}\right)$, $B=\left(\begin{array}{cc\|c}0&1&0\\1&0&0\end{array}\right)$,	$A_\nu^2=(A_1A_2A_3)^2=B^2=1$, $\nu=1,2,3$. $[BA_\nu B, A_\mu]=1$, $1\leqq\nu\leqq\mu\leqq3$,
$(3,1)_0$	$A_1=\left(\begin{array}{cc\|c}\zeta^2&0&1\\0&1&0\end{array}\right)$, $A_2=\left(\begin{array}{cc\|c}\zeta^2&0&\zeta\\0&1&0\end{array}\right)$, $B=\left(\begin{array}{cc\|c}0&1&0\\1&0&0\end{array}\right)$.	$A_1^3=A_2^3=(A_1A_2)^3=B^2=1$, $[BA_\nu B, A_\mu]=1$, $1\leqq\nu\leqq\mu\leqq2$.
$(4,1)_0$	$A_1=\left(\begin{array}{cc\|c}i&0&1\\0&1&0\end{array}\right)$, $A_2=\left(\begin{array}{cc\|c}-1&0&i\\0&1&0\end{array}\right)$, $B=\left(\begin{array}{cc\|c}0&1&0\\1&0&0\end{array}\right)$.	$A_1^4=A_2^2=(A_1A_2)^4=B^2=1$, $[BA_\nu B, A_\mu]=1$, $1\leqq\nu\leqq\mu\leqq2$.

$(6, 1)_0$	$A_1=\left(\begin{array}{cc\|c}\zeta^4 & 0 & \zeta\\ 0 & 1 & 0\end{array}\right)$, $A_2=\left(\begin{array}{cc\|c}-1 & 0 & 1\\ 0 & 1 & 0\end{array}\right)$, $B=\left(\begin{array}{cc\|c}0 & 1 & 0\\ 1 & 0 & 0\end{array}\right)$.	$A_1^3=A_2^2=(A_1A_2)^6=B^2=1$, $[BA_\nu B, A_\mu]=1$, $1\leqq\nu\leqq\mu\leqq 2$.
$(4, 2)_1$	$R_1=\left(\begin{array}{cc\|c}-1 & 0 & 0\\ 0 & 1 & 0\end{array}\right)$, $R_2=\left(\begin{array}{cc\|c}0 & 1 & 0\\ 1 & 0 & 0\end{array}\right)$, $R_3=\left(\begin{array}{cc\|c}0 & -i & 0\\ i & 0 & 0\end{array}\right)$, $R_4=\left(\begin{array}{cc\|c}1 & 0 & 0\\ 0 & -1 & i\end{array}\right)$, $R_5=\left(\begin{array}{cc\|c}0 & -1 & \frac{1+i}{2}\begin{pmatrix}1\\1\end{pmatrix}\\ -1 & 0 & \end{array}\right)$	$R_\nu^2=(R_1R_2R_3R_4R_5)^2=1$, $\nu=1, 2, \cdots, 5$. $R_1R_2R_3=R_2R_3R_1=R_3R_1R_2$, $R_1R_4=R_4R_1$, $R_2R_5=R_5R_2$.
$(3, 3)_0$	$A_1=\left(\begin{array}{cc\|c}0 & 1 & 0\\ 1 & 0 & 0\end{array}\right)$, $A_2=\left(\begin{array}{cc\|c}0 & 1 & 1\\ 1 & 0 & -1\end{array}\right)$, $A_3=\left(\begin{array}{cc\|c}0 & 1 & \tau\\ 1 & 0 & -\tau\end{array}\right)$, $B=\left(\begin{array}{cc\|c}0 & \zeta^4 & 0\\ \zeta^2 & 0 & 0\end{array}\right)$.	$A_\nu^2=B^2=(A_1A_2A_3)^2=1$, $\nu=1, 2, 3$. $BA_\nu B=A_\nu BA_\nu$, $\nu=1, 2, 3$. $(BA_1B)A_\nu(BA_1B)=A_\nu(BA_1B)A_\nu$, $\nu=2, 3$. $(A_1A_2A_3)B(A_1A_2A_3)=B(A_1A_2A_3)B$.

REMARK 1. $A_1A_2A_3\in(2, 1)_0$, $A_1A_2\in(3, 1)_0$, $A_1A_2\in(4, 1)_0$, $A_1A_2\in(6, 1)_0$, $R_1R_2R_3R_4R_5\in(4, 2)_1$ and $A_1A_2A_3\in(3, 3)_0$ are reflections.

REMARK 2. To keep symmetry the above relations include unnecessary ones, e.g. the relations $A_\nu^2=(A_1A_2A_3)^2=1$ ($\nu=1, 2, 3$) for $(3, 3)_0$.

PROOF. We shall prove the theorem for the groups $(2, 1)_0$ and $(3, 3)_0$. The remaining groups can be treated similarly.

i) Let $\Gamma=G(2, 1, 2)\ltimes L^2(\tau)$. Since the group $G(2, 1, 2)$ is generated by two reflections $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, we have $\langle A_1, B\rangle=\left\{\gamma\in\Gamma;\ \gamma=\left(g\,\middle|\begin{matrix}0\\0\end{matrix}\right),\ g\in G(2, 1, 2)\right\}$. Note that

$$A_2A_1=\left(\begin{array}{cc|c}1 & 0 & 1\\ 0 & 1 & 0\end{array}\right),\qquad A_3A_1=\left(\begin{array}{cc|c}1 & 0 & \tau\\ 0 & 1 & 0\end{array}\right),$$

$$BA_2A_1B=\left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 1\end{array}\right),\qquad BA_3A_1B=\left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & \tau\end{array}\right).$$

Thus Γ is generated by A_ν, $\nu=1, 2, 3$ and B.

Next we shall determine the defining relations by the method of coset enumeration. Put

$$\gamma=\langle a_1, a_2, a_3, b\,|\, a_\nu^2=b^2=(a_1a_2a_3)^2=1,\ [ba_\nu b, a_\mu]=1,\ 1\leqq\nu\leqq\mu\leqq 3\rangle$$

and

$$\gamma^*=\langle a_\mu a_\nu,\ ba_\mu a_\nu b\,;\ 1\leqq\nu\leqq\mu\leqq 3\rangle\,.$$

Step 1. γ^* is a free abelian group with the basis a_2a_1, a_3a_1, ba_2a_1b, ba_3a_1b

and $\gamma^* \lhd \gamma$.

PROOF. Easy.

Step 2. $\gamma/\gamma^* \cong G(2, 1, 2)$.

PROOF. $\gamma/\gamma^* = \langle a_1, b \mid a_1^2 = b^2 = 1, [ba_1b, a_1] = 1 \rangle \cong G(2, 1, 2)$.

The correspondence $a_\nu \to A_\nu$, $b \to B$ gives the homomorphism $\gamma \to \Gamma$. Thus we have $\gamma \cong \Gamma$.

ii) Let $\Gamma = G(3, 3, 2) \ltimes \left\{ L(\tau)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + L(\tau)\begin{pmatrix} \zeta^2 \\ \zeta \end{pmatrix} \right\}$. Analogous proof is available for the assertion that Γ is generated by A_ν ($\nu=1, 2, 3$) and B. Put

$$\begin{aligned} \gamma = \langle a_1, a_2, a_3, b \mid\ & a_\nu^2 = b^2 = (a_1a_2a_3)^2 = 1,\ ba_\nu b = a_\nu b a_\nu,\ \nu = 1, 2, 3, \\ & (ba_1b)a_\mu(ba_1b) = a_\mu(ba_1b)a_\mu,\ \mu = 2, 3, \\ & (a_1a_2a_3)b(a_1a_2a_3) = b(a_1a_2a_3)b \rangle \end{aligned}$$

and

$$\gamma^* = \langle a_2a_1, a_3a_1, ba_2a_1b, ba_3a_1b \rangle .$$

Step 1. γ^* is a free abelian group with basis a_2a_1, a_3a_1, ba_2a_1b, ba_3a_1b, and $\gamma^* \lhd \gamma$.

PROOF. For the commutativity of γ^*, we shall show that $[a_2a_1, ba_3a_1b] = 1$. (The remainings are easier.)

$$\begin{aligned} [a_2a_1, ba_3a_1b] &= a_2a_1ba_3a_1ba_1a_2ba_1a_3b \\ &= a_2a_1ba_3ba_1ba_2ba_1a_3b \\ &= a_2a_1ba_3bba_2ba_1ba_2ba_3b \\ &= a_2a_1ba_3a_2ba_1ba_2ba_3b \\ &= a_2a_1ba_3a_2a_1ba_1a_2ba_3b \\ &= a_2a_1a_3a_2a_1ba_3a_2a_1a_1a_2ba_3b \\ &= a_3ba_3ba_3b \\ &= a_3a_3ba_3a_3b = 1 . \end{aligned}$$

For proving the normality of γ^*, we shall show that the conjugates of ba_2a_1b and ba_3a_1b belong to γ^*. (Easily seen for a_2a_1 and a_3a_1.)

Conjugates of ba_2a_1b: Since $a_1ba_2a_1ba_1 = a_2ba_2a_1ba_2 = a_3ba_2a_1ba_3$, it suffices to show $a_3ba_2a_1ba_3 \in \gamma^*$.

$$\begin{aligned} a_3ba_2a_1ba_3 &= a_3ba_3a_3a_2a_1ba_3 \\ &= ba_3ba_3a_2a_1ba_3 \\ &= ba_3a_3a_2a_1ba_3a_2a_1a_3 \end{aligned}$$

$$=ba_2a_1ba_1a_2\in\gamma^*.$$

Conjugates of ba_3a_1b: Since $a_1ba_3a_1ba_1=a_2ba_3a_1ba_2=a_3ba_3a_1ba_3$, it suffices to show $a_1ba_3a_1ba_1\in\gamma^*$.

$$\begin{aligned} a_1ba_3a_1ba_1 &= bba_1ba_3a_1ba_1 \\ &= ba_3ba_1ba_3 \\ &= ba_3a_1ba_1a_3\in\gamma^*. \end{aligned}$$

Step 2. $\gamma/\gamma^*\cong G(3, 3, 2)$.

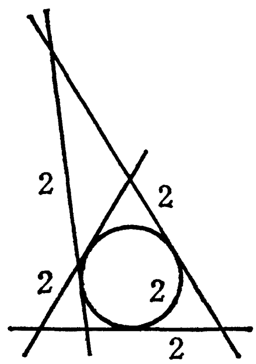
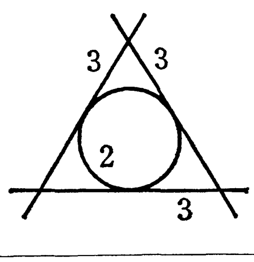
PROOF. $\gamma/\gamma^*=\langle a_1, b\,|\, a_1^2=b^2=1,\ a_1ba_1=ba_1b\rangle\cong G(3, 3, 2)$.

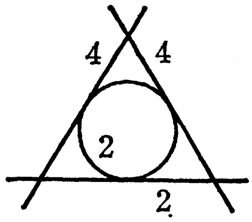
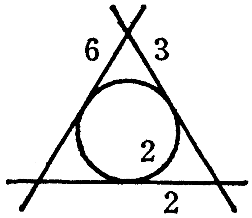
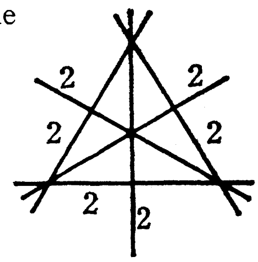
These lead to the conclusion that the homomorphism $\gamma\to\Gamma$ given by $a_\nu\to A_\nu$, $b\to B$ is bijective. Q.E.D.

§3. Coordinate representation and the branch locus D of ϕ.

Let x, y and $X:Y:Z$ be the coordinate of X and $M\cong \boldsymbol{P}^2$, respectively. We shall give the coordinate representation of the natural map $\phi: X\to M$ and determine the branch locus D of ϕ on M.

THEOREM 3. *For each group in Theorem* 1 *the coordinate representation of* ϕ, *the branch locus* D *and the ramification indices along the irreducible components of* D *are given by the following table.*

Γ	$\phi(x, y)=(X:Y:Z)$	$D\hookrightarrow M$	
$(2, 1)_0$	$\wp(x)+\wp(y):\wp(x)\wp(y):1$ $\wp(\cdot)=\wp(\cdot\,\vert\,\tau)\quad \mathrm{Im}\,\tau>0$	4 lines and a conic tangent to each line.	
$(3, 1)_0$	$\wp'(x)+\wp'(y):\wp'(x)\wp'(y):1$ $\wp'(\cdot)=\wp'(\cdot\,\vert\,\zeta)$	3 lines and a conic tangent to each line.	

$(4, 1)_0$	$\wp^2(x)+\wp^2(y):\wp^2(x)\wp^2(y):1$ $\wp(\cdot)=\wp(\cdot\,\vert i)$	ibid. 
$(6, 1)_0$	$\wp'^2(x)+\wp'^2(y):\wp'^2(x)\wp'^2(y):1$ $\wp'(\cdot)=\wp'(\cdot\,\vert\zeta)$	ibid. 
$(4, 2)_1$	$\left(\frac{\wp(x)\wp(y)+e_1^2}{\wp(x)\wp(y)-e_1^2}\right)^2:\left(\frac{\wp(x)+\wp(y)}{\wp(x)\wp(y)-e_1^2}\right)^2:1$ $\wp(\cdot)=\wp(\cdot\,\vert i)$, $e_1=\wp(1/2\,\vert i)$.	6 lines which have 3 double points and 4 triple points. 
$(3, 3)_0$	$\wp'(x_1)-\wp'(y_1):\wp(x_1)-\wp(y_1):$ $\wp'(x_1)\wp(y_1)-\wp(x_1)\wp'(y_1)$ $\wp(\cdot)=\wp(\cdot\,\vert\tau)\quad \mathrm{Im}\,\tau>0$, $x_1\begin{pmatrix}-1\\ 1\end{pmatrix}+y_1\begin{pmatrix}\zeta^2\\ \zeta\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$.	The dual curve of the non singular cubic with moduli τ. The ramification index=2.

Here the numeral beside the curve denotes the ramification index. By a suitable coordinate $(U:V:W)$ on M, the divisor D for $\Gamma=(2, 1)_0$ is represented by

$$D=\{V=0\}\cup\{W=0\}\cup\{U=V+W\}\cup\{eU=V+e^2W\}\cup\{U^2=4VW\}$$

where $e=\left(\wp\left(\frac{1}{2}\right)-\wp\left(\frac{\tau}{2}\right)\right)\Big/\left(\wp\left(\frac{1+\tau}{2}\right)-\wp\left(\frac{\tau}{2}\right)\right)$, $\wp(\cdot)=\wp(\cdot\,|\tau)$.

REMARK 1. Let $D=\bigcup_j D_j$ be the decomposition into irreducible components, d_j the degree of the divisor D_j and e_j the ramification index of ϕ along D_j. For each above groups, the following equality holds:

$$\sum_j d_j\left(1-\frac{1}{e_j}\right)=3.$$

It is interesting to compare this equality with the Sakai's inequality ([2]).

REMARK 2. The divisor D is nothing but the image of the union of the set $\{H(g) \mid g \in \Gamma,\ g$ is a reflection$\}$.

PROOF OF THEOREM 3. Put

$$\Gamma_0 = \{(1 \mid b) : b \in L^2(\tau)\},$$

$$\Gamma_1 = \left\{\left(\begin{matrix} \pm 1 & \\ & \pm 1 \end{matrix} \middle| b\right) ;\ b \in L^2(\tau)\right\}$$

and

$$E = \boldsymbol{C}/L(\tau).$$

Then we have

$$\Gamma_0 \triangleleft \Gamma_1 \triangleleft (2, 1)_0$$

and the following sequence of natural mappings:

$$\begin{array}{ccccccc} X & \longrightarrow & X/\Gamma_0 & \longrightarrow & X/\Gamma_1 & \longrightarrow & X/(2, 1)_0 \\ \wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert \\ \boldsymbol{C}^2 & & E \times E & & \boldsymbol{P}^1 \times \boldsymbol{P}^1 & & \boldsymbol{P}^2 \\ \cup\!\!\!\cup & & \cup\!\!\!\cup & & \cup\!\!\!\cup & & \cup\!\!\!\cup \end{array}$$

$$(x, y) \longmapsto (x, y) \bmod L^2(\tau) \mapsto (\wp(x), \wp(y)) \mapsto (\wp(x) + \wp(y) : \wp(x)\wp(y) : 1).$$

These prove the theorem for the group $(2, 1)_0$. Similar proofs are available for the groups $(3, 1)_0$, $(4, 1)_0$ and $(6, 1)_0$. We omit them.

The group $(2, 1)_0$ is a normal subgroup of $(4, 2)_1$ with moduli $\tau = i$. With respect to the homogeneous coordinate $(U : V : W)$:

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 0 & 1 & e_1^2 \\ 1 & 0 & 0 \\ 0 & 1 & -e_1^2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \qquad e_1 = \wp\left(\frac{1}{2} \middle| i\right),$$

on $X/(2, 1)_0$, the quotient group $(4, 2)_1/(2, 1)_0$ is represented by

$$\left\{\begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & 1 \end{pmatrix}\right\}.$$

This with the preceding proof implies the assertion for the group $(4, 2)_1$.

To prove the theorem for the group $(3, 3)_0$, we use the coordinate (x_1, y_1) on X which are related to (x, y) by

$$x_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + y_1 \begin{pmatrix} \zeta^2 \\ \zeta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

By this coordinate, $(3, 3)_0$ is represented by

$$\left\langle\begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\rangle \ltimes L^2(\tau)$$

and the union of the set $\{H(g) \mid g \in (3, 3)_0,\ g \text{ is a reflection}\}$ coincides with the set of all lines in X defined by $x_1 \equiv y_1$, $2x_1 + y_1 \equiv 0$ and $x_1 + 2y_1 \equiv 0 \mod L(\tau)$. On the other hand, we embed the torus $E = \boldsymbol{C}/L(\tau)$ into $Y \cong \boldsymbol{P}^2$ by

$$\eta : x \longmapsto (\wp(x) : \wp'(x) : 1)\,.$$

We shall denote the image by $C \hookrightarrow Y$. For two points $x_1, y_1 \in E$, we correspond the line $l_{x_1 y_1}$ joining $\eta(x_1)$ and $\eta(y_1)$. This correspondence induces the mapping

$$\phi : E \times E \longrightarrow \hat{Y} \cong \boldsymbol{P}^2$$
$$(x_1, y_1) \longmapsto l_{x_1 y_1}$$

where $\hat{Y}$ is the dual of Y. By virtue of Abel's theorem, ϕ is locally biholomorphic except for the divisors defined by $x_1 \equiv y_1$, $2x_1 + y_1 \equiv 0$ and $x_1 + 2y_1 \equiv 0 \mod L(\tau)$. Finally one sees that ϕ gives the natural mapping

$$X/\Gamma_0 \cong E \times E \longrightarrow X/(3, 3)_0 \cong \boldsymbol{P}^2\,.$$

The branch locus D is, by definition, coincides with the dual curve of C in $\hat{Y}$, and the equation of $l_{x_1 y_1}$ gives the desired coordinate representation of ϕ.

Q. E. D.

§4. Fundamental groups of the space of regular orbits.

Let Γ be the group in Theorem 1, D the branch locus given in Theorem 3, and $\pi_1(M-D)$ the fundamental group of $M-D$. In this section we shall define the homomorphism $h : \pi_1(M-D) \to \Gamma$ and determine the kernel of h.

We shall first represent the group $\pi_1(M-D)$ by giving a set of generators and relations:

Γ	Generators of $\pi_1(M-D)$	Relations
$(2, 1)_0$	a_1, a_2, a_3, b	$[ba_\nu b, a_\nu] = 1$, $\nu = 1, 2, 3$, $[ba_\nu b^{-1}, a_\mu] = 1$, $1 \leqq \nu \leqq \mu \leqq 3$.
$(3, 1)_0$ $(4, 1)_0$ $(6, 1)_0$	$a_1 a_2, b$	$[ba_\nu b, a_\nu] = 1$, $\nu = 1, 2$, $[ba_1 b^{-1}, a_2] = 1$.
$(4, 2)_1$	r_1, r_2, r_3, r_4, r_5	$r_1 r_2 r_3 = r_2 r_3 r_1 = r_3 r_1 r_2$, $r_3 r_4 r_5 = r_4 r_5 r_3 = r_5 r_3 r_4$, $r_1 r_4 = r_4 r_1$, $r_2 r_5 = r_5 r_2$.

$(3, 3)_0$	$g_{11}, g_{01}, g_{10}, g_2$	$g_{ij}g_2g_{ij}=g_2g_{ij}g_2$, $i, j=0, 1, 2$, *where* $g_{00}=g_{10}^{-1}g_{11}^{-1}g_{01}^{-1}g_2^2$, $g_{i2}=g_{i1}g_{i0}g_{i1}^{-1}$, $i=0, 1, 2$, $g_{20}=g_{10}g_{00}g_{10}^{-1}$, $g_{21}=g_{11}^{-1}g_{01}g_{11}$.

For the group $(3, 1)_0$, $(4, 2)_1$ and $(3, 3)_0$, we quoted Kaneko [1], Terada [5] and Zariski [8], respectively. For the group $(2, 1)_0$, let $D=C\cup D_1\cup D_2\cup D_3\cup D_4$ be the decomposition into irreducible components (C is the conic, D_j is the line). Let l be a generic line in M and put $\{q, q'\}:=C\cap l$, $\{p_j\}:=D_j\cap l$ $(i=1, 2, 3, 4)$. The generators a_j $(j=1, 2, 3)$ and b are defined to be the loops in $l-\{p_1, \cdots, p_4, q, q'\}$ starting from a point in l and turning around only the points p_j and q, respectively. The relations are obtained by the method used in [1].

By virtue of the results and Theorem 2, we establish

THEOREM 4. (i) *The following correspondence*

$$(2, 1)_0 : \quad a_\nu \longmapsto A \quad (\nu=1, 2, 3), \qquad b \longmapsto B$$

$$\left.\begin{matrix}(3, 1)_0\\(4, 1)_0\\(6, 1)_0\end{matrix}\right\}: \quad a_\nu \longmapsto A \quad (\nu=1, 2), \qquad b \longmapsto B$$

$$(4, 2)_1 : \quad r_\nu \longmapsto R_\nu \quad (\nu=1, 2, \cdots, 5)$$

$$(3, 3)_0 : \quad g_{11} \longmapsto A_1, \quad g_{01} \longmapsto A_2, \quad g_{10} \longmapsto A_3, \quad g_2 \longmapsto B$$

defines the surjective homomorphism $h:\pi_1(M-D)\to\Gamma$ *for each groups.*

(ii) *The kernel of the homomorphism* $h:\pi_1(M-D)\to\Gamma$ *is the smallest normal subgroup containing the following elements:*

$$(2, 1)_0 : \quad a_\nu^2 \ (\nu=1, 2, 3),\ (a_1a_2a_3)^2,\ b^2$$

$$(3, 1)_0 : \quad a_1^2,\ a_2^3,\ (a_1a_2)^3,\ b^2$$

$$(4, 1)_0 : \quad a_1^4,\ a_2^2,\ (a_1a_2)^4,\ b^2$$

$(6, 1)_0$: $a_1^3, a_2^2, (a_1a_2)^6, b^2$

$(4, 2)_1$: r_ν^2 $(\nu=1, 2, \cdots, 5)$, $(r_1\cdots r_5)^2$

$(3, 3)_0$: $g_{11}^2, g_{01}^2, g_{10}^2, g_2^2, (g_{11}g_{01}g_{10})^2$.

PROOF. We shall prove the theorem for the group $(3, 3)_0$, which is only the non obvious case. Let $R(a, b)$ denote the relation

$$aba=bab.$$

Recall that

$$\pi_1(M-D)=\langle g_{11}, g_{01}, g_{10}, g_2 \mid R(g_{ij}, g_2), i, j=0, 1, 2\rangle$$

$$\begin{aligned}\Gamma=\langle a_1, a_2, a_3, b \mid\ & a_j^2=b^2=(a_1a_2a_3)^2=1,\ j=1, 2, 3,\\ & R(a_j, b),\ j=1, 2, 3,\ R(a_1a_2a_3, b),\\ & R(a_1a_ja_1, b),\ j=2, 3\rangle.\end{aligned}$$

The correspondence h defined in the theorem gives

$$(*)\qquad \begin{aligned} R(g_{11}, g_2) &\longrightarrow R(a_1, b)\\ R(g_{01}, g_2) &\longrightarrow R(a_2, b)\\ R(g_{10}, g_2) &\longrightarrow R(a_3, b)\\ R(g_{21}, g_2) &\longrightarrow R(a_1a_2a_1, b)\\ R(g_{12}, g_2) &\longrightarrow R(a_1a_3a_1, b)\\ R(g_{20}, g_2) &\longrightarrow R(a_1a_2a_3, b)\end{aligned}$$

and

$$\begin{aligned} R(g_{02}, g_2) &\longrightarrow R(a_2a_3a_1, b)\\ R(g_{00}, g_2) &\longrightarrow R(a_3a_1a_2, b)\\ R(g_{22}, g_2) &\longrightarrow R(a_1a_3a_1a_2a_1, b).\end{aligned}$$

The three relations $R(a_2a_3a_1, b)$, $R(a_3a_1a_2, b)$ and $R(a_1a_3a_1a_2a_1, b)$ are derived from the defining relations of Γ. These are proved by substituting the matrices A_j and B in a_j and b, respectively. Thus h induces a homomorphism.

By the correspondence $(*)$ and the fact proved above imply the second assertion.

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