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On the topology of the Newton boundary III

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§ 1. Introduction.

The purpose of this paper is to prove the following theorem: the Milnor fibration of an analytic function $f(z)$ is uniquely determined by the Newton boundary $\Gamma(f)$ if f is non-degenerate. We have proved the assertion in [**4**] for the case that the origin is an isolated critical point of f and in [**5**] for a weighted homogeneous polynomial. However the proof for the general case involves several essential arguments. For instance, we shall show in the process of the proof that the stable radius of the Milnor fibration of f is obtained by the Newton boundary $\Gamma(f)$. (Theorem 1, § 1).

We use the following notations. The other notations and terminology are the same as in [**4**] and [**5**].

$$S_r=\{z\in \boldsymbol{C}^n;\ \|z\|=r\}\,,\quad B_r=\{z\in \boldsymbol{C}^n;\ \|z\|\leqq r\}\,,\quad \mathrm{Int}(B_r)=\{z\in \boldsymbol{C}^n;\ \|z\|<r\}$$

and $S_r^1=\{u\in \boldsymbol{C};\ |u|=r\}$.

§ 2. A stable radius.

Let $f(z)$ be an analytic function of n variables which is defined in the neighborhood of the origin and assume that $f(\vec{0})=0$. Recall that a stable radius of the Milnor fibration of f is a positive number ε which satisfies the following condition.

(T) For any r, $0<r\leqq\varepsilon$, there exists a positive number $d(r)$ such that for any non-zero u, $|u|\leqq d(r)$, the hypersurface $f^{-1}(u)$ is non-singular in B_ε and it meets transversely with the spheres S_h for any h, $r\leqq h\leqq\varepsilon$.

The existence of a stable radius ε is proved by Hamm-Lê. (Lemme (2.1.4) of [**1**]). For any $0<r\leqq\varepsilon$, and $0<d\leqq d(r)$, let $E(r, d)$ be $f^{-1}(S_d^1)\cap\mathrm{Int}(B_r)$. The restriction of f to $E(r, d)$ gives a locally trivial fibration over S_d^1, say $\xi(r, d; f)$ and the isomorphism class of $\xi(r, d; f)$ does not depend on the particular choice

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of r and d. We call it the Milnor fibration of f at the origin.

In the rest of this section, we assume that f is a non-degenerate function in the sense of the Newton boundary (see [4] for the definition) and we investigate a sufficient condition for a given ε to be a stable radius of the Milnor fibration of f. This condition will be characterized in terms of the Newton boundary and we need some notations to state the following key lemma (Lemma 1).

Let $A(f)$ be the set of the non-empty subset I of $\{1, \cdots, n\}$ such that $\Gamma(f)\cap \boldsymbol{R}^I$ is non-empty. The restriction f^I of f to $\boldsymbol{C}^I$ is non-trivial if and only if $I\in A(f)$. Here $\boldsymbol{R}^I=\{(x_1, \cdots, x_n);\ x_i=0 \text{ for } i\notin I\}$. $\boldsymbol{C}^I$ is defined similarly. Let $F(I)$ be the set of closed faces Δ of $\Gamma_+(f^I)$ with respect to the canonical polyhedral decomposition of $\Gamma_+(f^I)$. Here we consider also the non-compact faces of $\Gamma_+(f^I)$. In particular, $\Gamma_+(f^I)$ is an element of $F(I)$. For each $I\in A(f)$ and $\Delta\in F(I)$, let $B(\Delta, I)$ be the set of subsets J of I such that there exist positive integers a_j $(j\in J)$ and non-negative integer d^* with the property that $\sum_{j\in J} a_j x_j=d^*$ for any $x\in\Delta$. The empty set $\emptyset$ is considered to be in $B(\Delta, I)$. Note that for any $I\in A(f)$, $\Delta\in F(I)$ and $J\in B(\Delta, I)$, $f_\Delta(z)$ is a weighted homogeneous polynomial of z_j $(j\in J)$ if we fix the other variables. It satisfies the functional equation $f_\Delta(t*z)=t^{d^*}f_\Delta(z)$ where $t*z$ is defined by $(t*z)_i=z_i$ for $i\in I-J$ and $t^{a_i}\cdot z_i$ for $i\in J$. Note that $A(f)$, $F(I)$ and $B(\Delta, I)$ are finite sets.

EXAMPLE. Let $f(z_1, z_2)=z_1^5+z_1^3z_2+z_1z_2^3$. Let $I=\{1, 2\}$ and let Δ_k be as in Figure 1. Then we can take as J, $\emptyset$ for Δ_1 and $\emptyset$ or $\{1\}$ for Δ_2 and $\emptyset$ or $\{1, 2\}$ for Δ_3.

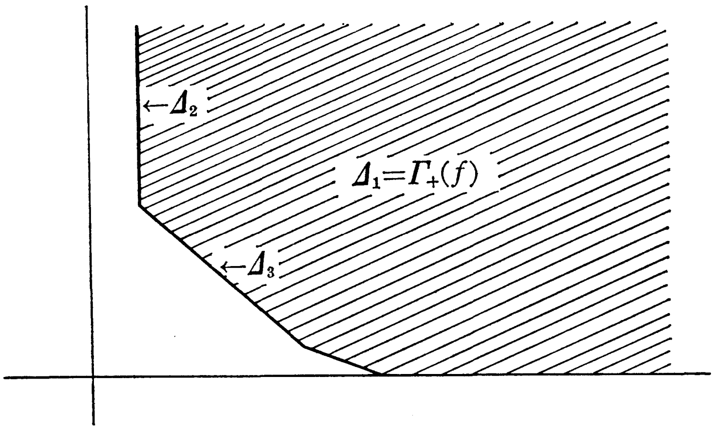

Figure 1.

LEMMA 1. *We can find a positive number ε which satisfies the following condition: For any $I\in A(f)$, $\Delta\in F(I)$ and $J\in B(\Delta, I)$ and any complex number λ, the equations $f_\Delta(u)=0$ and $\frac{\partial f_\Delta}{\partial z_i}(u)=\lambda\bar{u}_i$ for $i\in I-J$ and $=0$ for $i\in J$ have no common solution in $(\boldsymbol{C}^*)^I\cap\{u\in\boldsymbol{C}^I;\ \sum_{i\in I-J}|u_i|^2\leqq\varepsilon^2\}$.*

We prove this lemma at the end of the section and first explain two special cases to give some geometric ideas.

Case I. Suppose $J=\emptyset$. The lemma says that the hypersurface $\{z\in(\boldsymbol{C}^*)^I;\ f_\Delta^I(z)=0\}$ meets transversely with the sphere $\{z\in\boldsymbol{C}^I;\ \|z\|=r\}$ for any r, $r\leqq\varepsilon$.

Case II. Suppose that $I=J$. Then Δ must be a compact face of $\Gamma_+(f^I)$ because of the positivity of a_j $(j\in J)$. The assertion is that the hypersurface $\{z\in(\boldsymbol{C}^*)^I;\ f_\Delta(z)=0\}$ is non-singular, which is nothing but the non-degeneracy condition of f.

Assuming Lemma 1 we will first prove the following important theorem.

THEOREM 1. *Let f be an analytic function with a non-degenerate Newton boundary and take ε as in Lemma* 1. *Then ε is a stable radius of the Milnor fibration of f.*

PROOF. There exists a positive number δ such that $f^{-1}(\eta)\cap\mathrm{Int}(B_\varepsilon)$ is non-singular for any $\eta\in\boldsymbol{C}$, $0<|\eta|\leqq\delta$, because in the interior of a compact set the number of critical values of an analytic function defined in a neighborhood of this compact set is finite. Thus if the assertion is not true, we can apply the curve selection lemma (§3, [3] or [6]) to get a real analytic curve $z(s)$ $(0\leqq s\leqq 1)$ and a family of complex numbers $\lambda(s)$ $(0<s<1)$ such that

(I) $\dfrac{\partial f}{\partial z_i}(z(s))=\lambda(s)\overline{z_i(s)}$ for $i=1,\cdots,n$ and $s\neq 0$.

(II) $f(z(s))$ is not constantly zero and $f(z(0))=0$.

(III) There exists a positive number r, $0<r\leqq\varepsilon$, so that $r\leqq\|z(s)\|\leqq\varepsilon$.

By (II), $\lambda(s)$ is not constantly zero and we can express $\lambda(s)$ in a Laurent series: $\lambda(s)=\lambda s^c+\cdots$, $\lambda\neq 0$. Let $I=\{i;\ z_i(s)\not\equiv 0\}$. By (II), $I\in A(f)$. For each $i\in I$, let $z_i(s)=u_i s^{a_i}+\cdots$ where $u_i\neq 0$ and a_i is a non-negative integer. We define an element $\Delta\in F(I)$ to be the face of $\Gamma_+(f^I)$ on which the linear function $\sum_{i\in I}a_i x_i$ on $\Gamma_+(f^I)$ obtains its minimal value, say d. We define $J\in B(\Delta, I)$ by $\{i\in I;\ a_i>0\}$. Assertion: $f_\Delta(\ddot{u})=0$. Here $\ddot{u}$ is the point of $(\boldsymbol{C}^*)^I$ whose i-th coordinate is u_i. To see the assertion, note that $f_\Delta(\ddot{u})$ is the leading coefficient of the Taylor expansion of $f(z(s))$. If J is empty, then $d=0$ and obviously $f_\Delta(u)=0$ because it is the constant term of $f(z(s))$. The case $d=0$ is also trivial by the same reason. Thus we can suppose that J is a proper subset of I and d is positive. We use the equality: $f_\Delta(\ddot{u})=\dfrac{1}{d}\sum_{j\in J}a_j u_j\dfrac{\partial f_\Delta}{\partial z_j}(\ddot{u})$. If the differential $\dfrac{\partial f_\Delta}{\partial z_j}(\ddot{u})$ is zero for any $j\in J$, there is nothing to prove. Suppose that $\dfrac{\partial f_\Delta}{\partial z_k}(u)\neq 0$ for some $k\in J$. We compare the leading coefficients of (I)$'$: $z_i(s)\dfrac{\partial f}{\partial z_i}(z(s))=\lambda(s)|z_i(s)|^2$. The left side begins by the term s^d and the right side begins by the term s^c. Thus we obtain:

(IV) $d=c$ and $\frac{\partial f_\Delta}{\partial z_i}(u)=\lambda\bar{u}_i$ for $i\in I-J$ and $=0$ for $i\in J$.

This is a contradiction to $\frac{\partial f_\Delta}{\partial z_k}(u)\neq 0$. This completes the proof of the assertion. Now we use the assertion and Lemma 1, to find some $p\in I$ such that $\frac{\partial f_\Delta}{\partial z_p}(u)\neq 0$. We compare again the leading terms of (I)′ to obtain the same conclusion (IV). However this is a contradiction to Lemma 1 because we know that $\sum_{i\in I-J}|u_i|^2\leqq\varepsilon^2$ by (III). This completes the proof of Theorem 1 modulo Lemma 1.

Now we prove Lemma 1.

PROOF OF LEMMA 1. As we have explained before, the case $J=I$ reduces to the non-degeneracy assumption. Suppose that the assertion fails for some I, Δ and J which is a proper subset of I. We have a sequence $(v(m), \mu(m))$ for $m\in \boldsymbol{N}$ in $(\boldsymbol{C}^*)^I\times\boldsymbol{C}$ such that

(i) $f_\Delta(v(m))=0$ and $\frac{\partial f_\Delta}{\partial z_i}(v(m))=\mu(m)\overline{v_i(m)}$ for $i\in I-J$ and $=0$ for $i\in J$.

(ii) The sum $\sum_{i\in I-J}|v_i(m)|^2$ converges to zero when $m\to\infty$.

Let a_j $(j\in J)$ and d^* be the integers in the definition of "$J\in B(\Delta, I)$" and let $t*z$ be defined by $(t*z)_i=z_i$ for $i\in I-J$ and $t^{a_i}\cdot z_i$ for $i\in J$. We have the equality: $f_\Delta(t*z)=t^{d^*}f_\Delta(z)$ for any $z\in\boldsymbol{C}^I$. Take a sequence $c(m)$ in $\boldsymbol{C}^*$ which converges to zero rapidly enough so that $c(m)*v(m)$ converges to the origin. Let $v(m)'=c(m)*v(m)$ and $\mu(m)'=c(m)^{d^*}\mu(m)$. Taking the differential of $f_\Delta(t*z)=t^{d^*}f_\Delta(z)$ with respect to z_i and putting $t=c(m)$ and $z=v(m)$, we can easily see the following: The new sequence $(v(m)', \mu(m)')$ also satisfies the conditions (i) and (ii). We apply the Curve Selection Lemma (§3, [3]) in this situation in order to find a real analytic curve $u(s)$ $(0\leqq s\leqq 1)$ in $\dot{\boldsymbol{C}^I}$ and a family of complex numbers $\lambda(s)$ $(0<s\leqq 1)$ such that

(I) $f_\Delta(u(s))=0$ and $\frac{\partial f_\Delta}{\partial z_i}(u(s))=\lambda(s)\overline{u_i(s)}$ for $i\in I-J$ and 0 for $i\in J$.

(II) $u(0)=0$ and $u(s)\in(\boldsymbol{C}^*)^I$ for $s\neq 0$.

For each $i\in I$, let $u_i(s)=w_i s^{b_i}+\cdots$ where w_i is a non-zero complex number and b_i is a positive integer for each $i\in I$. Let Δ' be the face of Δ on which the linear function $\sum_{i\in I}b_i x_i$ on Δ takes its minimal value, say d'. Note that d' is a positive integer. As b_i $(i\in I)$ are all positive, Δ' is a compact face of Δ. Namely Δ' is a face of $\Gamma(f)$. By the non-degeneracy condition of f, we can find some $p\in I$ so that $\frac{\partial f_{\Delta'}}{\partial z_p}(\vec{w})\neq 0$ where $\vec{w}$ is similarly defined. This implies that $\frac{\partial f_\Delta}{\partial z_p}(u(s))$ is not constantly zero and its leading term is $\frac{\partial f_{\Delta'}}{\partial z_p}(\vec{w})s^{d'-b_p}$. By (I), $\lambda(s)$ is non-zero and we can write $\lambda(s)$ in a Laurent series as $\lambda(s)=\lambda s^c+\cdots$

where λ is non-zero. As in the proof of Theorem 1, we compare the leading coefficients of

$$(\mathrm{I})': \quad u_i(s)\frac{\partial f_\Delta}{\partial z_i}(u(s))=\lambda(s)|u_i(s)|^2 \text{ for } i\in I-J \text{ and } 0 \text{ for } i\in J.$$

Let $b_0=\text{minimum}\{b_i\,;\,i\in I\}$ and let $J'=\{i\in I\,;\,b_i\neq b_0\}$. We obtain that

$$(\mathrm{III}) \quad d'=c+2b_0 \text{ and } w_i\cdot\frac{\partial f_{\Delta'}}{\partial z_i}(\bar{w})=\lambda|w_i|^2 \text{ for } i\in I-J\cup J' \text{ and } =0 \text{ for } i\in J\cup J'.$$

$J\cup J'$ is a proper subset of I because the above p is contained in the complement. Now we use (III) and the equality $d'f_{\Delta'}(\bar{w})=\sum_{i\in I} b_i w_i\frac{\partial f_{\Delta'}}{\partial z_i}(\bar{w})$ to conclude that $f_{\Delta'}(\bar{w})$ is non-zero which is impossible because it is the coefficient of $s^{d'}$ in the Taylor expansion of $f_\Delta(u(s))$. This completes the proof of Lemma 1.

§ 3. A uniformly stable family.

We consider a C^∞-family of analytic functions $f_t(z)$ $(-\alpha\leqq t\leqq\alpha)$. For the sake of convenience, we also write $f(z, t)$ for $f_t(z)$.

DEFINITION. We say that the family $\{f_t(z)\}$ $(-\alpha\leqq t\leqq\alpha)$ is a uniformly stable family if there exists a positive number ε such that (i) ε is a stable radius for the Milnor fibration of f_t for each t, $-\alpha\leqq t\leqq\alpha$, and (ii) for any positive number r, $0<r\leqq\varepsilon$, there exists a positive number $d(r)$ which satisfies the condition (T) in § 2 simultaneously for each f_t $(-\alpha\leqq t\leqq\alpha)$. We call ε a stable radius for $\{f_t\}$ $(-\alpha\leqq t\leqq\alpha)$. We denote the respective Milnor fibration of f_t by $\xi(t)$.

LEMMA 2. *Suppose that f_t $(-\alpha\leqq t\leqq\alpha)$ has a uniform stable radius ε. Then $\xi(t)$ is isomorphic to $\xi(0)$.*

PROOF. We shall give a sketch of proof. Let $W=\{(z, t)\in\mathrm{Int}(B_\varepsilon)\times[-\alpha, \alpha]\,;\,|f(z, t)|=d(\varepsilon/2)\}$ and let $p: W\to[-\alpha, \alpha]$ be the projection. By (T), we can construct a smooth vector field X on W such that (i) $dp_{(z,t)}X(z, t)=(\partial/\partial t)_t$ where $(\partial/\partial t)_t$ is the unit vector of $T_t\boldsymbol{R}$. (ii) $X(z, t)$ is tangent to $S_{\|z\|}\times[-\alpha, \alpha]$ if $\|z\|\geqq\varepsilon/2$. (iii) $X(z, t)$ is tangent to the level hypersurface $W_{(z,t)}=\{(z', t')\in W\,;\,f(z', t')=f(z, t)\}$. Let $h(z, s)$ be the integral curve of X with the initial condition $h(z, 0)=(z, 0)$. By the above conditions (i), (ii) and (iii), $h(z, s)$ is defined for $-\alpha\leqq s\leqq\alpha$ and $p(h(z, s))=s$ and $f(h(z, s))$ is constant for any z fixed. The correspondence $z\mapsto h(z, t)$ gives the desired fiber-preserving diffeomorphism of $E(\varepsilon, d(\varepsilon/2)\,;\,f_0)$ and $E(\varepsilon, d(\varepsilon/2)\,;\,f_t)$ (=the total spaces of the Milnor fibrations of f_0 and f_t respectively).

Now we consider an analytic family $\{f_t\}$ $(-\alpha\leqq t\leqq\alpha)$ such that f_t is non-degenerate and has the same Newton boundary $\Gamma(f_0)$ for each $-\alpha\leqq t\leqq\alpha$.

LEMMA 3. *Let ε be a positive number satisfying the condition in Lemma 1 for f_0. Then there exists a positive number β, $\beta\leqq\alpha$, such that ε satisfies the same* *condition for each* f_t, $-\beta \leqq t \leqq \beta$.

PROOF. Assume that our assertion is not true. Then there exists a triple $(I, \Delta, J) \in (A(f), F(I), B(\Delta, I))$ and a sequence $(v(m), \mu(m), t(m))$ of $(\boldsymbol{C}^*)^I \times \boldsymbol{C} \times [-\alpha, \alpha]$ such that (i) $f_\Delta(v(m), t(m))=0$ and $\frac{\partial f_\Delta}{\partial z_i}(v(m), t(m))=\mu(m)\overline{v_i(m)}$ for $i \in I-J$ and $=0$ for $i \in J$. (ii) $\sum_{i \in I-J} |v_i(m)|^2 \leqq \varepsilon^2$ and $t(m)$ converges to 0. By the same discussion as in the proof of Lemma 1, we can assume that $v_j(m)$ converges to 0 for each $j \in J$. Now we apply the Curve Selection Lemma to get a real analytic curve $(u(s), q(s))$ $(0 \leqq s \leqq 1)$ in $\boldsymbol{C}^I \times [-\alpha, \alpha]$ such that

(I) $f_\Delta(u(s), q(s))=0$ and $\frac{\partial f_\Delta}{\partial z_i}(u(s), q(s))=\lambda(s)\overline{u_i(s)}$ for $i \in I-J$ and zero for $i \in J$ and some complex number $\lambda(s)$ for $0 \leqq s \leqq 1$.

(II) $q(0)=0$ and $u_j(0)=0$ for $j \in J$ and $u(s) \in (\boldsymbol{C}^*)^I$ for $s \neq 0$.

(III) $\sum_{i \in I-J} |u_i(s)|^2 \leqq \varepsilon^2$.

Let us consider the Taylor expansion of $u(s)$: $u_i(s)=w_i s^{b_i}+\cdots$ where w_i is a nonzero complex number and b_i is a non-negative integer for $i \in I$. Let Ξ be the face of Δ where the linear function $\sum_{i \in I} b_i x_i$ defined on Δ takes its minimal value, say d_1. Let $b_0=\text{minimum}\{b_i; i \in I\}$ and let J_1 be $\{i \in I; b_i \neq b_0\}$. Suppose first that b_0 is positive. Then Ξ is a compact face of $\Gamma(f_0)$ and by the non-degeneracy assumption of f_0, there exists a $p \in I$ such that $\frac{\partial f_\Xi}{\partial z_p}(\vec{w}, 0) \neq 0$. In particular, this implies that $\lambda(s)$ is a non-zero complex number and we can express it in a Laurent series: $\lambda(s)=\lambda s^c+\cdots$ where λ is a non-zero complex number. We compare the leading terms of

(I)$'$: $u_i(s)\frac{\partial f_\Delta}{\partial z_i}(u(s), q(s))=\lambda(s)|u_i(s)|^2$ for $i \in I-J$ and 0 for $i \in J$.

We get: (IV) $d_1=c+2b_0$ and $\frac{\partial f_\Xi}{\partial z_i}(\vec{w}, 0)=\lambda \overline{w}_i$ for $i \in I-J \cup J_1$ and 0 for $i \in J \cup J_1$. $J \cup J_1$ is a proper subset of I because p is an element of the complement. By the same discussion as before, this implies $f_\Xi(\vec{w}, 0) \neq 0$. However this is impossible because it is the leading coefficient of $f_\Delta(u(s), q(s))$. Thus b_0 must be zero. Then it is clear that $\Xi \in F(I)$ and $J_1 \in B(\Xi, I)$ and $J_1 \supset J$. By Lemma 1, we can find some $p \in I-J_1$ such that $\frac{\partial f_\Xi}{\partial z_p}(\vec{w}, 0) \neq 0$. Comparing the leading terms of (I)$'$, we obtain: (IV)$'$ $d_1=c$ and $\frac{\partial f_\Xi}{\partial z_i}(\vec{w}, 0)=\lambda \overline{w}_i$ for $i \in I-J_1$ and 0 for $i \in J_1$. Here we have used the same Laurent expansion of $\lambda(s)$. Moreover $\sum_{i \in I-J_1} |w_i|^2 = \|u(0)\|^2 \leqq \varepsilon^2$. This is a contradiction to the assumption on ε and completes the proof.

THEOREM 2. *Under the same assumption as in Lemma* 3, *the restricted family* f_t, $(-\beta\leqq t\leqq\beta)$ *is a uniformly stable family with a uniform stable radius* ε.

PROOF. We must show that for any r, $0<r\leqq\varepsilon$, there exists a positive number $d(r)$ such that the condition (T) is satisfied simultaneously for any f_t, $-\beta\leqq t\leqq\beta$. Suppose that our assertion fails. Then we can find a sequence $(v(m), t(m), \mu(m))$ of $\boldsymbol{C}^n\times[-\beta, \beta]\times\boldsymbol{C}$ such that

(I) $f(v(m), t(m))\neq 0$ and converges to zero when $m\rightarrow\infty$.

(II) $\dfrac{\partial f}{\partial z_i}(v(m), t(m))=\mu(m)\overline{v_i(m)}$ for $i=1, \cdots, n$.

(III) $|t(m)|\leqq\beta$ and $\|v(m)\|\leqq\varepsilon$.

(IV) In the case that $\mu(m)$ is not constantly zero, $\|v(m)\|\geqq r$ for some positive number r.

In the case that (IV) occurs, $v(m)$ is a regular point of the hypersurface $\{z\in\boldsymbol{C}^n;\ f(z, t(m))=f(v(m), t(m))\}$ where the sphere $S_{\|v(m)\|}$ meets non-transversely with the hypersurface. We use the Curve Selection Lemma to obtain a real analytic curve $(z(s), q(s))$ in $\boldsymbol{C}^n\times[-\beta, \beta]$ and a family of complex numbers $\lambda(s)$ such that

(I) $f(z(s), q(s))\neq 0$ for $s\neq 0$ and $f(z(0), q(0))=0$.

(II) $\dfrac{\partial f}{\partial z_i}(z(s), q(s))=\lambda(s)\overline{z_i(s)}$ for $i=1, \cdots, n$.

(III) $|q(s)|\leqq\beta$ and $\|z(s)\|\leqq\varepsilon$.

(IV) If $\lambda(s)\not\equiv 0$, $\|z(s)\|\geqq r$.

Let $I=\{i;\ z_i(s)\not\equiv 0\}$ and we consider the Taylor expansions: $z_i(s)=w_i s^{a_i}+w_i' s^{a_i+1}+\cdots$ were $w_i\neq 0$ for $i\in I$. Let Δ be the face where the linear function $\sum_{i\in I} a_i x_i$ on $\Gamma_+(f_0^I)$ takes its minimal value, say d. By (I), $I\in A(f_0)$ and $\Delta\in F(I)$. Let $a_0=\text{minimum}\{a_i;\ i\in I\}$ and $J=\{i\in I;\ a_i\neq a_0\}$. Assume that $a_0>0$. Then Δ is a compact face of $\Gamma_+(f_0)$ and by the non-degeneracy assumption of $f_{q(0)}$, there exists some $p\in I$ such that $\dfrac{\partial f_\Delta}{\partial z_p}(\vec{w}, q(0))\neq 0$ which implies that $\dfrac{\partial f}{\partial z_p}(z(s), q(s))$ is not constantly zero. However this is a contradiction because $z(0)=0$ and (IV) says that $\lambda(s)\equiv 0$. Now assume that $a_0=0$. Then we have that $J\in B(\Delta, I)$. Note that $\sum_{i\in I-J}|w_i|^2=\|z(0)\|^2\leqq\varepsilon^2$ by (III) and $f_\Delta(\vec{w}, q(0))=0$. The last equality is obtained by the exact same argument as in the proof of Theorem 1, § 2. Let $\lambda(s)=\lambda s^c+\cdots$ be the Laurent expansion of $\lambda(s)$. We apply Lemma 1 for $f_{q(0)}$ to find $p\in I$ such that $\dfrac{\partial f_\Delta}{\partial z_p}(\vec{w}, q(0))\neq 0$. The proof is completely parallel to the proof of Theorem 1. Comparing the leading coefficients of (II), we get that $d=c$ and $\dfrac{\partial f_\Delta}{\partial z_i}(\vec{w}, q(0))=\lambda\overline{w}_i$ for $i\in I-J$ and $=0$ for $i\in J$. This is a contradiction to our assumption (Lemma 3).

By the compactness of the interval $[-\alpha, \alpha]$, we get:

COROLLARY 1. Let $\{f_t\}$, $(-\alpha \leqq t \leqq \alpha)$ be an analytic family of analytic functions with the same Newton boundary $\Gamma(f_0)$ and suppose that f_t is non-degenerate for each t. Then $\{f_t\}$, $(-\alpha \leqq t \leqq \alpha)$ is a uniformly stable family.

§ 4. The main Theorem.

Using the results of § 3 we prove the following theorem:

THEOREM 3. *Suppose that f and g are analytic functions with the same Newton boundary and that they are non-degenerate. Then their Milnor fibrations are isomorphic.*

PROOF. For any analytic function $h(z)=\sum_{\nu \in N^n} a_\nu z^\nu$, we define the principal part of h by the partial sum $\sum_{\nu \in \Gamma(h)} a_\nu z^\nu$ and we denote it by h^*.

ASSERTION 1. The Milnor fibrations of f and f^* (respectively g and g^*) are isomorphic.

To see this, we consider the analytic family $f_t=f^*+(1-t)(f-f^*)$ for $0 \leqq t \leqq 1$. Then $f_0=f$ and $f_1=f^*$. This family clearly satisfies the condition of Corollary 1, § 3.

ASSERTION 2. The Milnor fibrations of f^* and g^* are isomorphic.

We consider the integral points of (f), say $\nu_1, \cdots, \nu_N$ and for any $u \in \boldsymbol{C}^N$, let $h(z, u)=\sum_{k=1}^{N} u_k z^{\nu_k}$. Let U be the set of points u so that $h(z, u)$ is non-degenerate as an analytic function of z and $(h(z, u))=\Gamma(f)$. Then U is a Zariski open set by the appendix of [4]. In particular, U is arcwise-connected. Let $u(f^*)$ and $u(g^*)$ be the corresponding points of f^* and g^*. We can choose a finite sequence of points $u(1), \cdots, u(m)$ in U such that $u(1)=u(f^*)$ and $u(m)=u(g^*)$ and for each $i=1, \cdots, m-1$, the segment $L_i(t)$ defined by $L_i(t)=(1-t)u(i)+tu(i+1)$ is included in U. For each $i=1, \cdots, m-1$, we consider the analytic family which is defined by $h_{i,t}(z)=h(z, L_i(t))$. It clearly satisfies the condition of Corollary 1, § 3. Thus the Milnor fibrations of $h_{i,0}$ and $h_{i,1}$ are isomorphic. As $h_{1,0}=f^*$ and $h_{m-1,1}=g^*$ and $h_{i,1}=h_{i+1,0}$, Assertion 2 is immediate from Corollary 1, § 3. Now the proof of the theorem is completed by Assertion 1 and Assertion 2.

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