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Integral representation of an analytic functional

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1. Introduction.

An analytic functional is a continuous linear functional on the space of all holomorphic functions in some set in the complex n dimensional space $\boldsymbol{C}^n$. For an open set U in $\boldsymbol{C}^n$, we denote by $\mathcal{O}(U)$ the space of all holomorphic functions in U equipped with the compact convergence topology. It is a Fréchet space. When K is a compact set in $\boldsymbol{C}^n$, $\mathcal{O}(K)$ is the space of all functions holomorphic in some open neighborhood U of K equipped with the inductive limit topology of $\mathcal{O}(U)$ for all such U. It is a DF space and its topological dual space $\mathcal{O}'(K)$ is a Fréchet space. When $n=1$, $\mathcal{O}'(K)$ is determined by S. e. Silva, G. Köthe and A. Grothendieck. It is known as the following isomorphism:

$$\mathcal{O}'(K)\cong\mathcal{O}(V-K)/\mathcal{O}(V),$$

where V is an open neighborhood of K. The duality is explicitly given by

$$\langle f, g\rangle=\int_{\partial U}f(z)g(z)dz$$

where $f\in\mathcal{O}(K)$, $g\in\mathcal{O}(V-K)$ and U $(K\subset U\Subset V)$ is taken so that $f\in\mathcal{O}(\bar{U})$ and ∂U is smooth. This duality formula is independent of the choice of the open set U and the function g in the class $[g]$ in $\mathcal{O}(V-K)/\mathcal{O}(V)$. When $n>1$, this isomorphism is extended by A. Martineau and R. Harvey (cf. H. Komatsu [6]) as the form

$$\mathcal{O}'(K)\cong H^{n-1}(V-K, \mathcal{O})$$

under the conditions $H^j(K, \mathcal{O})=0$ $(j\geqq 1)$ where $\mathcal{O}$ is the sheaf of germs of holomorphic functions and V is a Stein neighborhood of K. The proof of this duality depends on the Serre duality theorem and is given by the functional analytic method. The purpose of this paper is to give a new proof of this duality theorem establishing the direct duality formula between these two spaces $\mathcal{O}(K)$ and $H^{n-1}(V-K, \mathcal{O})$. We will interpret the cohomology space $H^{n-1}(V-K, \mathcal{O})$ as the Dolbeault cohomology space and establish the duality through the formula:

$$\langle f, g\rangle=\int_{\partial U}f(z)g(z)\wedge dz_1\wedge dz_2\wedge\cdots\wedge dz_n,$$

where $f\in\mathcal{O}(\bar{U})$, g is a form of type $(0, n-1)$ infinitely differentiable in $V-K$ and satisfies the equation $\bar{\partial}g=0$, $K\subset U\Subset V$ and ∂U is smooth. Our method of proof is the analogy of the case $n=1$, where the Cauchy integral kernel $1/(z-w)$ is essential. The corresponding integral kernel for $n>1$ is given by E. Ramirez [11] and G. M. Henkin [4] as a special case of the Cauchy-Fantappié formula. We state here the outline of our proofs. Let T be any analytic functional on K. First we construct a $(0, n-1)$ form $f_T(x)$ in some neighborhood of the boundary ∂G $(K\subset G\subset V)$ such that

$$\langle T, h\rangle=\int_{\partial G} h(x)f_T(x)\wedge dx_1\wedge\cdots\wedge dx_n$$

for all functions h holomorphic on $\bar{G}$. In this step, the Ramirez-Henkin integral kernel is essential. Secondly we modify $f_T(x)$ to extend the domain of existence. We use here the vanishing theorems $H^p(V-K, \mathcal{O})=0$ $(p=1, \cdots, n-2)$ if $n\geqq 3$ and Hartogs' theorem if $n=2$. Lastly we show that any $(0, n-1)$ form f in $V-K$ which is orthogonal to all holomorphic functions h on K is $\bar{\partial}$-exact.

The first five sections are preliminary, where we recall the known results which will be essential in our paper. Some important theorems, in Sections 4 and 5, will be presented and proved in a simplified form. The precise statements of the results of Sections 2 and 3 can be found in I. Lieb [9] and H. Grauert-I. Lieb [3]. As for Section 4, we refer the reader to the articles, G. Scheja [12], A. Friedman [2], M. Morimoto [10] and M. Kashiwara-T. Kawai-T. Kimura [5]. The original result in Section 5 is due to A. Dautov [1]. The author wishes to express his thanks to Professor M. Morimoto who kindly read the original manuscript, and also to the referee for many valuable comments.

2. The Bochner-Martinelli integral formula.

Following W. Koppelman [7] we use the determinant of differential forms. Let $x=(x_1, x_2, \cdots, x_n)$ and $y=(y_1, y_2, \cdots, y_n)$ be two points in $\boldsymbol{C}^n$. Set $B_{\bar{x}}(x, y)$ as follows:

$$(1)\qquad B_{\bar{x}}(x, y)=\begin{vmatrix} \dfrac{\bar{x}_1-\bar{y}_1}{|x-y|^2} & \bar{\partial}_x\left(\dfrac{\bar{x}_1-\bar{y}_1}{|x-y|^2}\right) & \cdots\cdots & \bar{\partial}_x\left(\dfrac{\bar{x}_1-\bar{y}_1}{|x-y|^2}\right) \\ \dfrac{\bar{x}_2-\bar{y}_2}{|x-y|^2} & \bar{\partial}_x\left(\dfrac{\bar{x}_2-\bar{y}_2}{|x-y|^2}\right) & \cdots\cdots & \bar{\partial}_x\left(\dfrac{\bar{x}_2-\bar{y}_2}{|x-y|^2}\right) \\ \vdots & \vdots & & \vdots \\ \dfrac{\bar{x}_n-\bar{y}_n}{|x-y|^2} & \bar{\partial}_x\left(\dfrac{\bar{x}_n-\bar{y}_n}{|x-y|^2}\right) & \cdots\cdots & \bar{\partial}_x\left(\dfrac{\bar{x}_n-\bar{y}_n}{|x-y|^2}\right) \end{vmatrix}$$

$$=\sum_{\sigma\in S_n}\operatorname{sgn}\sigma\frac{\bar{x}_{\sigma_1}-\bar{y}_{\sigma_1}}{|x-y|^2}\bar{\partial}_x\left(\frac{\bar{x}_{\sigma_2}-\bar{y}_{\sigma_2}}{|x-y|^2}\right)\wedge\cdots\wedge\bar{\partial}_x\left(\frac{\bar{x}_{\sigma_n}-\bar{y}_{\sigma_n}}{|x-y|^2}\right)$$

where S_n is the symmetric group of dimension n. $B_{\bar{x}}(x, y)$ is defined in $\boldsymbol{C}^n\times\boldsymbol{C}^n$

except on the diagonal set $\Delta=\{(x, y)\mid x=y\}$, and is a form of type $(0, n-1)$ with respect to x. This is called the Bochner-Martinelli kernel. In I. Lieb [**9**], a more general type of the Bochner-Martinelli kernel is given. By a simple calculation we have

$$B_{\bar{x}}(x, y)=(n-1)!\frac{1}{|x-y|^{2n}}\sum_{j=1}^{n}(-1)^{j+1}(\bar{x}_j-\bar{y}_j)\bigwedge_{k\neq j}d\bar{x}_k\,. \tag{2}$$

Let G be a bounded set in $\boldsymbol{C}^n$ with the smooth boundary ∂G. The orientation in $\boldsymbol{C}^n$ is taken so that

$$x_1', x_1'', \cdots, x_n', x_n'' \quad \text{or} \quad y_1', y_1'', \cdots, y_n', y_n'',$$

are the positively oriented coordinate system of $\boldsymbol{R}^{2n}=\boldsymbol{C}^n$, where $x_j=x_j'+\sqrt{-1}\,x_j''$ and $y_j=y_j'+\sqrt{-1}\,y_j''$. On ∂G the natural orientation is induced. Then we know the next theorems.

THEOREM 1 (I. Lieb [**9**, Satz 9] and W. Koppelman [**7**]). *Let $f(z)$ be an infinitely differentiable function in some open neighborhood of $\bar{G}$. Then for any y in G,*

$$f(y)=\frac{(-1)^{n(n-1)/2}}{(2\pi i)^n}\left\{\int_{\partial G}f(x)B_{\bar{x}}(x, y)\bigwedge_{j=1}^{n}dx_j-\int_G\bar{\partial}_x f(x)\wedge B_{\bar{x}}(x, y)\bigwedge_{j=1}^{n}dx_j\right\}. \tag{3}$$

THEOREM 2 ([**9**, Satz 9] and [**8**]). *Let $g(z)$ be an infinitely differentiable form of type $(0, n)$ in some open neighborhood of $\bar{G}$. Then for any x in G,*

$$g(x)=\frac{(-1)^{n(n-1)/2}}{(2\pi i)^n}\bar{\partial}_x\left\{\int_G g(y)B_{\bar{x}}(x, y)\wedge\bigwedge_{j=1}^{n}dy_j\right\}. \tag{4}$$

The proofs of these theorems are given in [**9**] in a more general context.

3. The Ramirez-Henkin integral formula.

In this section, we suppose that $G\Subset\boldsymbol{C}^n$ is a strongly pseudoconvex domain with the smooth boundary ∂G. Then E. Ramirez [**11**], G. M. Henkin [**4**] and H. Grauert-I. Lieb [**3**] show that there is a function

$$g(x, y)=\sum_{j=1}^{n}(x_j-y_j)g_j(x, y) \tag{5}$$

such that

(i) g is defined and infinitely differentiable in some open neighborhood $U\times V$ of $\partial G\times\bar{G}\ni(x, y)$,

(ii) if x is fixed in $U-\bar{G}$, then $g\neq 0$ in some neighborhood of $\bar{G}$,

(iii) if x is fixed in $U-\bar{G}$, then g_j $(j=1, 2, \cdots, n)$ are holomorphic with respect to y on $\bar{G}$.

Using the function $g(x, y)$ we define the Ramirez-Henkin kernel $\Omega_{\bar{x}}(x, y)$ as follows:

$$(6)\qquad \Omega_{\bar{x}}(x, y)=\begin{vmatrix} \frac{g_1}{g} & \bar{\partial}_x\left(\frac{g_1}{g}\right) & \cdots\cdots & \bar{\partial}_x\left(\frac{g_1}{g}\right) \\ \frac{g_2}{g} & \bar{\partial}_x\left(\frac{g_2}{g}\right) & \cdots\cdots & \bar{\partial}_x\left(\frac{g_2}{g}\right) \\ \vdots & \vdots & & \vdots \\ \frac{g_n}{g} & \bar{\partial}_x\left(\frac{g_n}{g}\right) & \cdots\cdots & \bar{\partial}_x\left(\frac{g_n}{g}\right) \end{vmatrix}$$

$$=\sum_{\sigma\in S_n} \operatorname{sgn}\sigma \frac{g_{\sigma_1}}{g}\bar{\partial}_x\left(\frac{g_{\sigma_2}}{g}\right)\wedge\cdots\wedge\bar{\partial}_x\left(\frac{g_{\sigma_n}}{g}\right)$$

$$=(n-1)!\,\frac{1}{g^n}\sum_{j=1}^{n}(-1)^{j+1}g_j\bigwedge_{k\neq j}\bar{\partial}_x g_k\,.$$

$\Omega_{\bar{x}}(x, y)$ is defined in $U\times V-\{g=0\}$ and is a form of type $(0, n-1)$ with respect to x. For a fixed x in $U-\bar{G}$, every coefficient of $\Omega_{\bar{x}}(x, y)$ is holomorphic with respect to $y\in\bar{G}$. In I. Lieb [**9**], the next homotopy formula is given.

THEOREM 3 ([**8**], [**9**]). *There exists an infinitely differentiable form $A(x, y)$ in $U\times V-\{g=0\}$ of type $(0, n-2)$ with respect to x, which satisfies the homotopy relation between $B_{\bar{x}}(x, y)$ and $\Omega_{\bar{x}}(x, y)$;*

$$(7)\qquad B_{\bar{x}}(x, y)-\Omega_{\bar{x}}(x, y)=\bar{\partial}_x A(x, y)\,.$$

By this relation we have the following integral formula.

THEOREM 4 ([**3**], [**4**], [**9**] and [**11**]). *Let $f(z)$ be a holomorphic function in some open neighborhood of $\bar{G}$. Then for any y in G,*

$$(8)\qquad f(y)=\frac{(-1)^{n(n-1)/2}}{(2\pi i)^n}\int_{\partial G}f(x)\Omega_{\bar{x}}(x, y)\wedge\bigwedge_{j=1}^{n}dx_j\,.$$

4. Vanishing theorems for certain cohomology groups.

Suppose that $K=\bigcap_{j=1}^{\infty}K_j$ is a compact set, where $K_j\subset K_{j-1}$ and K_j is a bounded domain of holomorphy in $\boldsymbol{C}^n$ $(n\geqq 3)$. Then A. Friedman [**2**] shows that for any open set X in $\boldsymbol{C}^n$, containing K, the restriction map

$$(9)\qquad H^p(X, \mathcal{O})\longrightarrow H^p(X-K, \mathcal{O})\qquad (1\leqq p\leqq n-2)$$

is bijective. We remark that the above mapping is also bijective for $p=0$ by Hartogs' theorem. As a corollary to this result it derives that if X is a pseudoconvex domain then

$$(10)\qquad H^p(X-K, \mathcal{O})=0\qquad (1\leqq p\leqq n-2)\,.$$

Because of the generality of the open set X in (9), Friedman's proof is complicated. So we give here the outline of an elementary proof of (10).

By the excision theorem for the relative cohomology groups (cf. H. Komatsu [**6**]), we have the next lemma.

LEMMA 1. *Let K be a compact set and U, V be pseudoconvex neighborhoods of K. Then the following isomorphisms are valid.*

$$H^p(U-K, \mathcal{O}) \cong H^p(V-K, \mathcal{O}) \qquad p \geqq 1 .$$

Because of this lemma, it suffices to show (10) for a special X, for which the cohomology groups are easy to calculate.

PROPOSITION 1. *Let* $K=\{z \in \boldsymbol{C}^n \mid |z_j| \leqq 1\}$ *and* $X=\{z \in \boldsymbol{C}^n \mid |z_j| < 1+\varepsilon\}$ $(\varepsilon > 0)$. *Then* (10) *is true.*

The proof of this proposition is due to G. Scheja [**12**]. It depends on the Cauchy integral formula for one variable and is elementary. It can be found also in M. Morimoto [**10**] in which the modified Cauchy integral kernels due to A. Martineau are used. Thus we only refer to [**12**, Hilfssatz, p. 349], [**2**, Lemma, p. 505] and [**10**, Lemma 2, p. 131] and omit the details.

Now we extend this proposition to the case where K is a compact analytic polyhedron in some pseudoconvex domain X. That is; K is given by

$$K=\{z \in X \mid |f_j(z)| \leqq 1,\ j=1, 2, f_2(z), \cdots, N\} ,$$

where f_j are holomorphic in X. We take a positive constant ε so small that $U=\{z \in X \mid |f_j(z)| < 1+\varepsilon,\ j=1, 2, \cdots, N\}$ is relatively compact in X. Consider the so-called Oka-mapping F from U into $\boldsymbol{C}^N$ which is defined by

$$F: U \ni z \longmapsto (f_1(z), \cdots, f_N(z)) \in \boldsymbol{C}^N .$$

This mapping F may be assumed to be one to one and closed if we take sufficiently many f_j $(N \geqq n)$. We denote by $\mathcal{O}_N$ the sheaf of germs of holomorphic functions in $\boldsymbol{C}^N$, by $F_*\mathcal{O}$ the direct image of the sheaf $\mathcal{O}$ on U by the map F, by $\hat{U}$ the open polydisc of the radius $1+\varepsilon$ and by $\hat{K}$ the closed unit polydisc in $\boldsymbol{C}^N$. The following two lemmas are given in M. Kashiwara-T. Kawai-T. Kimura [**5**].

LEMMA 2 ([**5**, Lemma 1 in p. 56]). *The next sequence is exact.*

$$0 \longleftarrow F_*\mathcal{O} \longleftarrow \mathcal{O}_N \longleftarrow \mathcal{O}_N^{N-n} \longleftarrow \mathcal{O}_N^{\binom{N-n}{2}} \longleftarrow \cdots \longleftarrow \mathcal{O}_N^{\binom{N-n}{N-n}} \longleftarrow 0 .$$

The mappings in the above lemma are defined as follows. First we may assume that $f_j(z)=z_j$ $(1 \leqq j \leqq n)$ in the mapping F. Secondly we change the coordinates $(y_1, y_2, \cdots, y_N)$ in $\boldsymbol{C}^N$ to the coordinates $(w_1, w_2, \cdots, w_N)$ such that $w_j=y_j$ $(1 \leqq j \leqq n)$ and $w_j=y_j-f_j(y_1, \cdots, y_n)$ $(n+1 \leqq j \leqq N)$. Under the coordinates $(w_1, \cdots, w_N)$, $F(U) \subset \{(w_1, \cdots, w_N) \in \boldsymbol{C}^N \mid w_{n+1}= \cdots =w_N=0\}$. Now the mapping $(\mathcal{O}_N)_w \to (F_*\mathcal{O})_w$ for $w \in F(U)$ is defined by the restriction $\phi(w_1, \cdots, w_N) \in (\mathcal{O}_N)_w \mapsto \phi(w_1, \cdots, w_n, 0, \cdots, 0) \in (F_*\mathcal{O})_w$. In general $\mathcal{O}_N^{\binom{N-n}{k}} \ni \phi$ can be expressed as

$\{\phi_{i_1,\cdots,i_k}\}_{n+1\leq i_1,\cdots,i_k\leq N}$ where $\phi_{i_1,\cdots,i_k}\in\mathcal{O}_N$ and $\phi_{i_1,\cdots,i_k}$ are alternative with respect to the indices $(i_1, \cdots, i_k)$. Then the mapping $\delta:\mathcal{O}_N^{\binom{N-n}{k}}\to\mathcal{O}_N^{\binom{N-n}{k-1}}$ is defined by $\{(\delta\phi)_{i_1,\cdots,i_{k-1}}\}=\left\{\sum_{j=n+1}^{N}\phi_{i_1,\cdots,i_{k-1},j}w_j\right\}$ for $\phi=\{\phi_{i_1,\cdots,i_k}\}$ in $\mathcal{O}_N^{\binom{N-n}{k}}$.

LEMMA 3 ([**5**, Lemma 2 in p. 58]).

$$H^p(\hat{U}-\hat{K},\ F_*\mathcal{O})=0 \qquad (p=1, 2, \cdots, n-2).$$

Lemma 3 results from Proposition 1 and Lemma 2. Since the Oka-mapping F is purely 0 codimensional with respect to the sheaf $F_*\mathcal{O}$ ([**5**, p. 56]), we have the isomorphism

$$H^p(U-K,\ \mathcal{O})=H^p(\hat{U}-\hat{K},\ F_*\mathcal{O}).$$

By this isomorphism and Lemmas 1 and 3, we have the next proposition.

PROPOSITION 2. *Let K be a compact analytic polyhedron in a pseudoconvex domain X. Then*

$$H^p(X-K,\ \mathcal{O})=0, \qquad p=1, 2, \cdots, n-2.$$

In the remaining part of this section, we will show that this proposition is also valid for a holomorphically convex compact set K in X.

THEOREM 5. *Let K be a holomorphically convex compact set in a pseudoconvex domain X. Then*

$$H^p(X-K,\ \mathcal{O})=0, \qquad p=1, 2, \cdots, n-2.$$

PROOF. We take a sequence of analytic polyhedrons K_j in X such that

$$K\subset\cdots\Subset K_{j+1}\Subset K_j\Subset K_{j-1}\Subset\cdots\subset X,$$

and $K=\lim K_j$. Let $f(z)$ be a $\bar{\partial}$-closed (0, p) form infinitely differentiable in $X-K$. To simplify the notations, we consider the triple K_j $(j=1, 2, 3)$ such that

$$K_1\Supset K_2\Supset K_3\Supset K.$$

By Proposition 2, there exist smooth solutions g_2 and g_3 of $f=\bar{\partial}g$ in $X-K_2$ and $X-K_3$ respectively. Then $\bar{\partial}(g_2-g_3)=0$ in $X-K_2$. In the case $p=1$, g_2-g_3 is holomorphic in $X-K_2$. Thus g_2-g_3 can be extended holomorphically in X by Hartogs' theorem; $g_2-g_3=h(z)$ in $X-K_2$, where $h(z)$ is holomorphic in X. In this case we set $\tilde{g}_3=g_3+h(z)$ which coincides with g_2 in $X-K_2$ and satisfies $\bar{\partial}\tilde{g}_3(z)=f(z)$ in $X-K_3$. In the case $p\geqq 2$, $g_2-g_3=\bar{\partial}h_{2,3}(z)$ for some (0, $p-2$) form $h_{2,3}(z)$ in $X-K_2$ by Proposition 2. Now take a smooth function $s(z)$ which is equal to 1 near $X-K_1$ and 0 near K_2. Then

$$g_2-\bar{\partial}\{(1-s(z))h_{2,3}\}=g_3+\bar{\partial}(s(z)h_{2,3})$$

in $X-K_2$. The form in the left hand side of the above equality is smooth in

$X-K_2$ and equal to g_2 in $X-K_1$. On the other hand, the form in the right hand side is smooth in $X-K_3$, and both satisfy the equation $\bar{\partial}g=f$. Thus the solution g_2 of $\bar{\partial}g=f$ in $X-K_2$ can be prolonged to a solution of $\bar{\partial}g=f$ in $X-K_3$ without changing the values in $X-K_1$. Therefore in any case $(1\leqq p\leqq n-2)$, by repeating this argument we can construct a solution g of $\bar{\partial}g=f$ in $X-K$. This proves the theorem.

We call the "*three step method*" the argument in the above proof, which will be useful in the following section.

5. Infinitely differentiable forms orthogonal to holomorphic functions.

The problem of extending a smooth form defined in some neighborhood of the boundary of a bounded domain into its interior has been studied by many mathematicians. The next theorem is originally due to S. A. Dautov [1] in a more general situation. We give here a simplified proof under the restricted assumptions.

THEOREM 6 ([**1**]). *Let G be a strictly pseudoconvex bounded domain in C^n with the smooth boundary ∂G and $f(z)$ be a $\bar{\partial}$-closed* $(0, n-1)$ *form infinitely differentiable in some neighborhood of ∂G. Then the following conditions on the form $f(z)$ are equivalent,*

(i) $\int_{\partial G} g(z)f(z)\wedge dz_1\wedge \cdots \wedge dz_n=0$ *for all functions g holomorphic near $\bar{G}$,*

(ii) *there exists a $\bar{\partial}$-closed* $(0, n-1)$ *form $\tilde{f}(z)$ which is infinitely differentiable in some neighborhood V of $\bar{G}$ and coincides with $f(z)$ in $V-G$.*

PROOF. By Stokes' theorem, it is evident that (ii) implies (i). Thus we shall show that (i) implies (ii). Let $\Omega_{\bar{x}}(x, y)$ be the Ramirez-Henkin integral kernel for the domain G. For some neighborhood U of ∂G, $\Omega_{\bar{x}}(x, y)$ is holomorphic with respect to $y\in\bar{G}$ if x is fixed in $U-\bar{G}$. Suppose $f(z)$ be a smooth $\bar{\partial}$-closed $(0, n-1)$ form in $V-K$, where $K\Subset G\Subset V$, K is compact and $V-K\subset U$. $f(z)$ is assumed to satisfy the condition (i). We may assume that V is a domain of holomorphy. Take a function $s(z)$ infinitely differentiable in C^n such that $s(z)$ is equal to 0 in some open neighborhood of K and is equal to 1 in some open neighborhood of C^n-G. We remark that $\bar{\partial}(sf)=0$ in a neighborhood of $V-G$ in V. Set $h(z)$ as follows:

$$h(x)=\int_V \bar{\partial}_y(s(y)f(y))\wedge B_{\bar{x}}(x, y)\wedge dy_1\wedge \cdots \wedge dy_n \tag{11}$$

for x in V. The integral is taken with respect to the variables $(y_1, y_2, \cdots, y_n)$. Then by the generalized Bochner-Martinelli integral formula (Theorem 2), we have

$$\bar{\partial}(s(x)f(x))=c\,\bar{\partial}h(x) \qquad (x\in V),$$

where $c=(-1)^{n(n-1)/2}(2\pi i)^{-n}$. Since V is a domain of holomorphy, we can find a smooth $(0, n-2)$ form $t(x)$ in V such that

(12) $$s(x)f(x)=ch(x)+\bar{\partial}t(x).$$

Then we apply the homotopy formula (Theorem 3) to (11) and obtain

(13) $$h(x)=\int_G \bar{\partial}_y(s(y)f(y))\wedge\bar{\partial}_x A(x, y)\wedge dy_1\wedge\cdots\wedge dy_n$$
$$+\int_G \bar{\partial}_y(s(y)f(y))\wedge\Omega_{\bar{x}}(x, y)\wedge dy_1\wedge\cdots\wedge dy_n$$

for $x\in U-\bar{G}$. The first integral term in the right hand side of (13) is well defined for x in some neighborhood of $U-G$ and the differentiation $\bar{\partial}_x$ and the integration are commutative, because $\bar{\partial}_y(s(y)f(y))=0$ near ∂G. The second integral term can be reduced to the following

(14) $$\int_{\partial G} s(y)f(y)\wedge\Omega_{\bar{x}}(x, y)\wedge dy_1\wedge\cdots\wedge dy_n,$$

because there is no singularity in $\bar{G}$ with respect to y and Stokes' formula is applicable. Then by the condition (i), (14) is equal to zero. Consequently there exists a smooth $(0, n-2)$ form $a(x)$ in some neighborhood of ∂G such that

$$h(x)=\bar{\partial}a(x)$$

for x in a neighborhood of $U-G$. Now we can make a suitable extension $\tilde{a}(x)$ of $a(x)$ such that $\tilde{a}(x)$ is a smooth $(0, n-2)$ form in the whole of $U\cup G$ and coincides with $a(x)$ in $U-G$. Set $\tilde{f}(x)$ for x in $V\subset U\cup G$ as follows;

$$\tilde{f}(x)=\bar{\partial}\{c\tilde{a}(x)+t(x)\}.$$

Then $\tilde{f}$ is infinitely differentiable and $\bar{\partial}$-closed in V. In $V-G$,

$$\begin{aligned}\tilde{f}(x)&=\bar{\partial}\{ca(x)+t(x)\}\\ &=ch(x)+\bar{\partial}t(x)\\ &=s(x)f(x) \qquad \text{by (12)}\\ &=f(x).\end{aligned}$$

Thus $\tilde{f}(x)$ is a desired extension of $f(x)$. This completes the proof.

Let V be a pseudoconvex domain not necessarily bounded in $\boldsymbol{C}^n$ and K be a holomorphically convex compact set in V. The next problem is to characterize the $\bar{\partial}$-closed $(0, n-1)$ form $f(z)$ in $V-K$ which is orthogonal to $\mathcal{O}(K)$. To solve this problem, the "three step method" in the preceding section is useful.

THEOREM 7. *We denote by V a pseudoconvex domain in $\boldsymbol{C}^n$ and by K a holomorphically convex compact set in V. Let $f(z)$ be a smooth $\bar{\partial}$-closed $(0, n-1)$ form in $V-K$. Then the followings are equivalent:*

(i) $\int_{\partial K} g(z)f(z)\wedge dz_1\wedge\cdots\wedge dz_n=0$ *for all functions g holomorphic near K,*

(ii) *there exists a* $(0, n-2)$ *form* $h(z)$ *infinitely differentiable in* $V-K$ *and satisfies* $f(z)=\bar{\partial}h(z)$.

We remark that the integration over ∂K must be interpreted as the integration over ∂U where $K\subset U\subset V$, $g\in\mathcal{O}(\bar{U})$ and ∂U is smooth. The condition (i) is independent of the choice of such U. Therefore we adopt the notation $\int_{\partial K}$ for the convenience.

PROOF. Since the integration of an exact form over a manifold without boundary is always zero, (ii) implies (i). Therefore we have only to show that (i) implies (ii). First we take a triple G_j $(j=1, 2, 3)$ of the bounded domains of holomorphy with the smooth boundaries such that

$$V\Supset G_1\Supset G_2\Supset G_3\Supset K.$$

By Theorem 6, we find $\bar{\partial}$-closed $(0, n-1)$ forms $f_j(z)$ $(j=1, 2, 3)$ which are infinitely differentiable in V and coincide with $f(z)$ near ∂G_j $(j=1, 2, 3)$ respectively. Set

$$f_{1,2}(z)=\begin{cases} f_2(z) & \text{in } G_2 \\ f(z) & \text{in } G_1-G_2 \\ f_1(z) & \text{in } V-G_1, \end{cases}$$

and

$$f_{2,3}(z)=\begin{cases} f_3(z) & \text{in } G_3 \\ f(z) & \text{in } G_2-G_3 \\ f_2(z) & \text{in } V-G_2. \end{cases}$$

Then $f_{1,2}$ and $f_{2,3}$ are smooth and $\bar{\partial}$-closed in V. Since V is pseudoconvex, there exist $h_{1,2}$ and $h_{2,3}$ such that

$$f_{1,2}(z)=\bar{\partial}h_{1,2}, \qquad f_{2,3}(z)=\bar{\partial}h_{2,3}(z).$$

Because $f_{1,2}(z)=f(z)=f_{2,3}(z)$ near ∂G_2, we have

$$\bar{\partial}(h_{1,2}(z)-h_{2,3}(z))=0 \qquad \text{near} \quad \partial G_2.$$

In the case $n=2$, $h_{1,2}(z)-h_{2,3}(z)$ is holomorphic near ∂G_2. Therefore it can be continued holomorphically in $\bar{G}_2$ by Hartogs' theorem; $h_{1,2}-h_{2,3}=\tilde{h}(z)$, where $\tilde{h}(z)$ is holomorphic in $\bar{G}_2$. In this case we set $h_{1,2,3}(z)$ as

$$h_{1,2,3}(z)=\begin{cases} h_{1,2}(z) & \text{in } V-G_2 \\ h_{2,3}(z)+\tilde{h}(z) & \text{in } G_2. \end{cases}$$

Then $h_{1,2,3}(z)$ satisfies the equation $f=\bar{\partial}h_{1,2,3}$ in G_1-G_3. In the case $n\geqq 3$, $h_{1,2}(z)$

$-h_{2,3}(z)$ is an exact form by the Friedman-Scheja theorem (Theorem 5); $h_{1,2}-h_{2,3}=\bar\partial\tilde{h}(z)$ for some $(0, n-2)$ form $\tilde{h}(z)$ near ∂G_2. By multiplying a suitable function with compact support, $\tilde{h}(z)$ may be assumed to be infinitely differentiable in V. In this case we set $h_{1,2,3}(z)$ as

$$h_{1,2,3}(z)=\begin{cases} h_{1,2}(z) & \text{in } V-G_2 \\ h_{2,3}(z)+\bar\partial\tilde{h}(z) & \text{in } G_2 . \end{cases}$$

Then $h_{1,2,3}(z)$ satisfies the equation $f=\bar\partial h_{1,2,3}$ in G_1-G_3. Therefore in any case $(n\geqq 2)$, by repeating this argument (three step method), we can continue a solution $h(z)$ of $\bar\partial h=f$ near some ∂G $(K\Subset G\Subset V)$ into the whole of $G-K$ without changing the value of the previous steps. By the similar method, a solution $h(z)$ of $\bar\partial h=f$ in $G-K$ can be extended to the whole of $V-K$. This shows that (i) implies (ii). This completes the proof.

6. Integral representation of an analytic functional.

Now we prove the duality theorem by the Ramirez-Henkin integral kernel.

THEOREM 8. *Let K be a compact set in $\boldsymbol{C}^n$ which possesses a sequence of pseudoconvex domains as a fundamental system of neighborhoods and V be a pseudoconvex domain such that $K\subset V$. Then we have*

$$\mathcal{O}'(K)\cong H^{n-1}(V-K, \mathcal{O}) .$$

The duality in this isomorphism is given as follows: if $f(z)$ is a $(0, n-1)$ form on $V-K$ and $g(z)$ is holomorphic on $\bar{U}$ $(K\subset U\subset V)$ with ∂U smooth, then

$$\langle f, g\rangle=\int_{\partial U} g(z)f(z)\wedge dz_1\wedge \cdots \wedge dz_n . \tag{15}$$

It is evident that this formula is independent of the choice of f in the class $[f]\in H^{n-1}(V-K, \mathcal{O})$ and the open set U.

The proof of this theorem is an adaptation from that of the cace $n=1$ (S.e. Silva-G. Köthe-A. Grothendieck) which is given for example in H. Komatsu [**6**].

PROOF. We denote by $Z^{(0,n-1)}(V-K)$ the linear topological space of all $(0, n-1)$ forms $f(z)$ which are infinitely differentiable in $V-K$ and satisfy $\bar\partial f=0$. It is evident that f defines a continuous linear functional on $\mathcal{O}(K)$ by the formula (15). Thus we obtain the linear mapping L which is easily seen to be continuous:

$$L: Z^{(0,n-1)}(V-K) \longrightarrow \mathcal{O}'(K) .$$

1. The surjectivity of L. Let G $(K\subset G\Subset V)$ be a strictly pseudoconvex domain with the smooth boundary ∂G and $\Omega_{\bar x}(x, y)$ be the Ramirez-Henkin integral kernel for G. Then for any $g\in\mathcal{O}(\bar{G})$,

$$g(y)=\frac{(-1)^{n(n-1)/2}}{(2\pi i)^n}\int_{\partial G} g(x)\Omega_{\bar{x}}(x, y)\wedge dx_1\wedge \cdots \wedge dx_n \qquad (y\in G).$$

If we interpret the surface integral in the sense of Riemann, the integral converges in $\mathcal{O}(K)$. Thus for any $T\in\mathcal{O}'(K)$,

$$\langle T(y), g(y)\rangle=\frac{(-1)^{n(n-1)/2}}{(2\pi i)^n}\int_{\partial G} g(x)\langle T(y), \Omega_{\bar{x}}(x, y)\rangle\wedge dx_1\wedge \cdots \wedge dx_n . \tag{16}$$

If we set $f_T(x)$ as

$$f_T(x)=(-1)^{n(n-1)/2}(2\pi i)^{-n}\langle T(y), \Omega_{\bar{x}}(x, y)\rangle , \tag{17}$$

then $f_T(x)$ is defined and infinitely differentiable in some neighborhood U of the boundary ∂G. Since $\bar{\partial}_x\Omega_{\bar{x}}(x, y)=0$, we have $\bar{\partial}_x f_T(x)=0$ in U. We consider this $f_T(x)$ as a $\bar{\partial}$-closed $(0, n-1)$ form in the intersection $(G\cup U)\cap((V-G)\cup U)$. By the arguments analogous to the Cousin I problem, there exist $(0, n-1)$ forms $f'_T(x)$ and $f''_T(x)$ such that

$$f_T(x)=f'_T(x)-f''_T(x) \qquad x\in U \tag{18}$$

where $f'_T(x)$ and $f''_T(x)$ are $\bar{\partial}$-closed in $(V-G)\cup U$ and $G\cup U$ respectively. Stokes' theorem implies that

$$\int_{\partial G} g(x)f''_T(x)\wedge dx_1\wedge \cdots \wedge dx_n=0 .$$

Thus by (16), (17) and (18),

$$\langle T(y), g(y)\rangle=\int_{\partial G} g(x)f'_T(x)\wedge dx_1\wedge \cdots \wedge dx_n \tag{19}$$

for all $g\in\mathcal{O}(\bar{G})$. This means that $f'_T(x)$ is an integral kernel which corresponds to the functional T in $\bar{G}$. The next step is to extend this $f'_T(x)$ to the form on $V-K$ with the condition (19) for all $g\in\mathcal{O}(K)$. Take G_1 and G_2 so that

$$V\Supset G_1\Supset G_2\Supset K$$

and $f_T^{(1)}$ and $f_T^{(2)}$ are the corresponding integral kernels which are defined in some neighborhoods of $V-G_1$ and $V-G_2$ respectively. Then for any $g\in\mathcal{O}(\bar{G}_1)$,

$$\langle f_T^{(1)}, g\rangle=\langle f_T^{(2)}, g\rangle \ (=\langle T, g\rangle) .$$

Thus by the theorem of S. A. Dautov (Theorem 6), there exists a $\bar{\partial}$-closed $(0, n-1)$ form $h(x)$ on a neighborhood U_1 of $\bar{G}_1$ $(U_1\subset V)$ such that

$$f_T^{(1)}-f_T^{(2)}=h(x) \quad \text{in} \quad U_1-G_1 .$$

Thus we define $\tilde{f}_T^{(2)}$ as

$$\tilde{f}_T^{(2)}(x)=\begin{cases} f_T^{(1)}(x) & \text{in} \quad V-G_1 \\ f_T^{(2)}(x)+h(x) & \text{in a neighborhood of } G_1-G_2 , \end{cases}$$

then

$$\langle T, g\rangle=\int_{\partial G_2} g(z)\tilde{f}_T^{(2)}(z)\wedge dz_1\wedge \cdots \wedge dz_n$$

for all $g\in\mathcal{O}(\overline{G}_2)$. This means that $f_T^{(1)}$ can be prolonged to $V-G_2$ without changing the values in $V-G_1$. Repeating this step we find a smooth $\bar{\partial}$-closed $(0, n-1)$ form $f_T(x)$ on $V-K$ such that (19) holds for all $g\in\mathcal{O}(K)$. Therefore the mapping L is surjective.

2. Determination of the kernel of the mapping L. This problem has been answered by Theorem 7 which asserts that the kernel of the map L is equal to the space of all $\bar{\partial}$-exact $(0, n-1)$ forms in $V-K$.

Steps 1 and 2 result in the algebraic isomorphism:

$$\mathcal{O}'(K)\cong H^{n-1}(V-K, \mathcal{O}).$$

Since both spaces $Z^{(0,n-1)}(V-K)$ and $\mathcal{O}'(K)$ are Fréchet spaces, this isomorphism holds also topologically by the open mapping theorem. This completes the whole proof of Theorem 8.

Here we remark that the space of all $\bar{\partial}$-exact $(0, n-1)$ forms in $V-K$ is equal to the space of all $\bar{\partial}$-closed $(0, n-1)$ forms which can be "almost" prolongable in V as $\bar{\partial}$-closed forms. This means that the space is equal to

$$\{f(z)\in Z^{(0,n-1)}(V-K) \mid \text{for any open set } G\ (K\subset G\subset V), \text{ there exists an } \tilde{f}(z)\in Z^{(0,n-1)}(V) \text{ such that } f(z)=\tilde{f}(z) \text{ in } V-G\}.$$

Thus Theorem 8 can be considered as a natural extension of the S. e. Silva-G. Köthe-A. Grothendieck theorem for $n=1$.

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