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Uniform ascent and descent of bounded operators

By Sandy GRABINER[(1)]

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1. Introduction.

In this paper we study the structure and perturbation theory of certain classes of bounded operators on a Banach space. These classes contain the semi-Fredholm operators, and also most of the generalizations of Fredholm operators which appear in the literature. For a bounded operator T, our study focuses on the sequences of ranges, $\{R(T^n)\}$, and of null-spaces, $\{N(T^n)\}$, and on the analogous sequences for small or compact perturbations of T. We are particularly interested in the spaces

$$(1.1)\qquad R(T^\infty)=\bigcap_n R(T^n) \quad \text{and} \quad N(T^\infty)=\bigcup_n N(T^n),$$

and the analogous spaces for perturbations of T. The results we prove will be similar to some results which have been useful in spectral theory [**21**], [**12**], in the structure theory of Banach algebras [**10**], [**23**], and in the study of automatic continuity [**13**], [**22**].

If T is a bounded linear operator on the Banach space X, then, for each nonnegative integer n, T induces a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to the space $R(T^{n+1})/R(T^{n+2})$. We will let $k_n(T)$ be the dimension of the null space of the induced map and let

$$(1.2)\qquad k(T)=\sum_0^\infty k_n(T).$$

The following definition describes the classes of operators we will study.

DEFINITION (1.3). If there is a nonnegative integer d for which $k_n(T)=0$ for $n\geqq d$ (i.e., if the induced maps are isomorphisms for $n\geqq d$), we say that T has *eventual uniform descent*; and, more precisely, that T has *uniform descent for* $n\geqq d$. If $k(T)$ is finite, we say that T has *almost uniform descent.*

We will see, in Lemma (2.3), that $k_n(T)$ is also the dimension of the cokernel of the map induced by T from $N(T^{n+2})/N(T^{n+1})$ to $N(T^{n+1})/N(T^n)$. Thus, some

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cases in which T has eventual uniform descent are when it has finite ascent, descent, nullity, or defect; or, more generally, if some $R(T^n)/R(T^{n+1})$ or $N(T^{n+1})/N(T^n)$ is finite-dimensional. If T has finite nullity or defect, it will even have almost uniform descent; but notice that if T has finite ascent and infinite nullity, or finite descent and infinite defect, then it will have eventual uniform descent but not almost uniform descent. It turns out that T has almost uniform descent with $k(T)=k$ precisely when T has Kaashoek's property $P(I, k)$ (compare [**14**, pp. 452-453] with Theorem (3.7) (b), below). Also, T has uniform descent for $n \geqq 0$ (that is, $k(T)=0$) precisely when, in Kato's notation, $\nu(T:I) =\infty$ (compare [**17**, pp. 289-290] with Theorem (3.1)(b), below).

Suppose that T has uniform descent for $n \geqq d$. Our main goal is to describe the structure of T, and of bounded operators V which are small or compact commuting perturbations of T. To do this we will need to assume that T has an additional property which we will call topological uniform descent for $n \geqq d$ (see Definition (2.5) and Theorem (3.2) for precise characterizations of this property). The most important cases in which T has topological uniform descent for $n \geqq d$ are when there is an $n>d$ for which $R(T^n)$ is closed or for which $R(T^n)/R(T^{n+1})$ is finite-dimensional. If T has almost uniform descent, then it has eventual topological uniform descent precisely when it has closed range (see Theorem (3.8)). The strongest conclusions about V occur in the case that $V-T$ is "small" in norm and invertible and that V commutes with T (which of course includes the important case that $V=T-\lambda I$ for λ small but nonzero). For these V we show, in Theorem (4.7), that:

(i) V has closed range and has uniform descent for $n \geqq 0$;

(ii) $\dim(R(V^n)/R(V^{n+1}))=\dim(R(T^d)/R(T^{d+1}))$ and $\dim(N(V^{n+1})/N(V^n)) =\dim(N(T^{d+1})/N(T^d))$, for $n \geqq 0$;

(iii) $R(V^\infty)=R(T^\infty)+N(T^\infty)$ and $cl(N(V^\infty))=cl[N(T^\infty)\cap R(T^\infty)]$.

Notice that one special consequence of (ii) is that if $T-\lambda I$ has eventual topological uniform descent and if λ is in the boundary of the spectrum of T, then λ is a pole of T. This generalizes useful characterizations of poles in [**21**] and [**9**].

In the special cases that T is semi-Fredholm or that T has almost uniform descent and closed range, conclusions roughly equivalent to (i) and (ii) can be found in [**7**, Theorem 1, p. 102], [**17**, Theorem 5, p. 315], [**14**, Theorem 4.1 and 4.2, pp. 460-461], [**16**] and [**5**]. The formulas in (iii) seem to be new, even for Fredholm operators, but the fact that $R(V^\infty)$ and $cl(N(V^\infty))$ depend only on T is given in [**7**] for semi-Fredholm T. Also, the special cases of (ii) described in Corollary (4.8) are mostly new.

If T has almost uniform descent and closed range, we can prove slightly weaker versions of (i), (ii) and (iii) without assuming that $V-T$ is invertible.

This is done for $V-T$ "small" in Theorem (4.10) and for $V-T$ compact in Theorem (5.9). When T has only eventual topological uniform descent and $V-T$ is not invertible, then (i) need not be true, and the identities in (ii) and (iii) are replaced by inequalities and set inclusions (see Theorem (4.10) for small perturbations and Theorem (5.9) for compact perturbations). All the results with weaker versions of (i), (ii) and (iii), described above, seem to be new, even for semi-Fredholm operators.

Our perturbation results are proved in three steps. Suppose that T has topological uniform descent for $n \geqq d$. First we show, in Theorem (3.4), that the restrictions of T to $R(T^{\infty})$ and $R(T^{d}) \cap cl[N(T^{\infty})]$ are onto, and that the maps induced by T on $R(T^{d})/R(T^{\infty})$ and $X/cl[N(T^{\infty})]$ are bounded below. Using this result we can then describe, in Lemmas (4.2) and (5.2), the restrictions of perturbations of T to $R(T^{d})$, or what amounts to the same thing: small and compact commuting perturbations of operators with closed range and uniform descent for $n \geqq 0$. The final perturbation results, described above, are then obtained by using the map $T^{d}: X \rightarrow R(T^{d})$ and the inclusion map of $R(T^{d})$ into X to determine properties of the perturbed operators from properties of their restrictions to $R(T^{d})$.

2. Preliminary lemmas.

In this section, we collect some technical lemmas which we will need repeatedly in the sequel. We start with two lemmas which collect, for easy reference, some relatively standard results.

LEMMA (2.1). *Suppose that U, V, and W are subspaces of the vector space X, that E is a subspace of the vector space Y, and that $T: X \rightarrow Y$ is linear. Then*

(a) $[U+V] \cap W = U + (V \cap W)$, *if* $U \subseteqq W$.
(b) *The identity induces a linear isomorphism from* $U/(U \cap V)$ *onto* $(U+V)/V$.
(c) $T^{-1}(T(U)) = N(T) + U$.
(d) $T(U \cap T^{-1}(E)) = T(U) \cap E$.

PROOF. (a) is just the modular law [**20**, p. 12] and (b) is one of the classical isomorphism theorems [**20**, Proposition 3, p. 20]. (c) and (d) are easy direct calculations.

Whenever quotient spaces are linearly isomorphic under an isomorphism induced by the identity, as in Lemma (2.1) (b), we will say that these quotient spaces are *naturally isomorphic*.

LEMMA (2.2). *Suppose that T is a bounded linear operator with closed range from the Banach space X to the Banach space Y. If E and F are linear subspaces of X and Y, respectively, and if $E \supseteqq N(T)$, and $F \subseteqq R(T)$, then*

(a) $T(cl(E))=cl(T(E))$.
(b) $T^{-1}(cl(F))=cl(T^{-1}(F))$.

PROOF. The theorem follows immediately from the facts that T induces a linear homeomorphism from $X/N(T)$ onto $R(T)$, and that a subspace $Z\supseteq N(T)$ is closed in X if and only if $Z/N(T)$ is closed in $X/N(T)$.

As we indicated in the introduction, we will mainly be studying bounded operators, T, for which the sequences of maps induced by T from $R(T^n)/R(T^{n+1})$ to $R(T^{n+1})/R(T^{n+2})$, and from $N(T^{n+2})/N(T^{n+1})$ to $N(T^{n+1})/N(T^n)$, have some nice properties. The next two lemmas provide the basic techniques for this study by focusing, respectively, on the algebraic and topological properties of the induced maps for a fixed n.

LEMMA (2.3). *Suppose that T is a linear transformation on X and that n is a nonnegative integer.*

(a) *The map $T^*: R(T^n)/R(T^{n+1})\to R(T^{n+1})/R(T^{n+2})$ induced by T is onto, and its null space is naturally isomorphic to $(N(T)\cap R(T^n))/(N(T)\cap R(T^{n+1}))$.*
(b) *The map $T': N(T^{n+2})/N(T^{n+1})\to N(T^{n+1})/N(T^n)$ induced by T is one-to-one, and its cokernel is naturally isomorphic to $(R(T)+N(T^{n+1}))/(R(T)+N(T^n))$.*
(c) *T^n induces a linear isomorphism from the cokernel of T' onto the null space of T^*.*

PROOF. It is clear that T^* is onto. Using Lemma (2.1) we see that

$$\begin{aligned} N(T^*)&=([R(T^{n+1})+N(T)]\cap R(T^n))/R(T^{n+1}) \\ &=(R(T^{n+1})+[N(T)\cap R(T^n)])/R(T^{n+1}), \end{aligned}$$

which is naturally isomorphic to $(N(T)\cap R(T^n))/(N(T)\cap R(T^{n+1}))$. This proves part (a), and we omit the similar proof of part (b).

Another application of Lemma (2.1) shows that T^n induces an isomorphism from the cokernel of T', which is $N(T^{n+1})/([N(T^{n+1})\cap R(T)]+N(T^n))$, onto $(N(T)\cap R(T^n))/(N(T)\cap R(T^{n+1}))$, which we showed above to be naturally isomorphic to $N(T^*)$. This completes the proof.

To study the topological properties of the maps induced by T^n, we will always assume that $R(T^n)$ is given the unique *operator range topology* under which it becomes a Banach space continuously imbedded in X. Then all restrictions of bounded linear operators to maps between operator ranges will be continuous, by the closed graph theorem (for a discussion of operator ranges and their topologies, see [**9**, pp. 1433-1444] or [**3**, pp. 255-257]).

LEMMA (2.4). *Suppose that T is a bounded operator on the Banach space X and that n is a nonnegative integer. If the map induced by T from $R(T^n)/R(T^{n+1})$ to $R(T^{n+1})/R(T^{n+2})$ has finite-dimensional null space, then the* *following are equivalent:*

(a) $R(T^{n+1})$ *is closed in the operator range topology on* $R(T^n)$.
(b) $R(T^{n+2})$ *is closed in the operator range topology on* $R(T^{n+1})$.
(c) $R(T^{n+2})$ *is closed in the operator range topology on* $R(T^n)$.

PROOF. Since $R(T^{n+1})$ is continuously embedded in $R(T^n)$, it is enough to prove the equivalence of (a) and (b). Let $\hat{T}$ be the bounded operator induced by T from $R(T^n)$ onto $R(T^{n+1})$. By Lemma (2.2), $R(T^{n+2})$ is closed in the topology of $R(T^{n+1})$ if and only if $\hat{T}^{-1}(R(T^{n+2}))$ is closed in the topology of $R(T^n)$. But, by hypothesis, $R(T^{n+1})$ is an operator range of finite codimension in $\hat{T}^{-1}(R(T^{n+2}))$. Hence, using the fact that operator ranges of finite codimension in a Banach space are closed [**2**, Corollary (3.2.5), p. 37], we see that $R(T^{n+1})$ is closed in the topology of $R(T^n)$ if and only if $\hat{T}^{-1}(R(T^{n+2}))$ is closed in this topology.

We can now define the additional topological condition that we will usually assume about operators with eventual uniform descent.

DEFINITION (2.5). Suppose that T is a bounded operator on the Banach space X and that there is a nonnegative integer d for which T has uniform descent for $n \geqq d$. If $R(T^n)$ is closed in the operator range topology of $R(T^d)$ for $n \geqq d$, then we say that T has *eventual topological uniform descent*, and, more precisely, that T has *topological uniform descent for* $n \geqq d$.

3. Characterization and structure theorems.

In this section we give several characterizations of eventual uniform descent, eventual topological uniform descent, and almost uniform descent. We also describe the structure of maps which are induced by operators with eventual topological uniform descent, and we include some technical identities which we will need in later sections.

Condition (a) in Theorem (3.1) below just repeats our definition of uniform descent for $n \geqq d$ from Definition (1.3).

THEOREM (3.1.) *If T is a linear transformation on X and d is a fixed non-negative integer, then the following are all equivalent.*

(a) *The maps induced by T from $R(T^n)/R(T^{n+1})$ to $R(T^{n+1})/R(T^{n+2})$ are isomorphisms for each $n \geqq d$.*
(b) *The sequence of subspaces $\{R(T^n) \cap N(T)\}$ is constant for $n \geqq d$.*
(c) $R(T^d) \cap N(T) = R(T^\infty) \cap N(T)$.
(d) *The maps induced by T from $N(T^{n+2})/N(T^{n+1})$ to $N(T^{n+1})/N(T^n)$ are isomorphisms for each $n \geqq d$.*
(e) *The sequence $\{N(T^n) + R(T)\}$ is constant for $n \geqq d$.*
(f) $N(T^d) + R(T) = N(T^\infty) + R(T)$.

PROOF. The equivalence of (a), (b), (d) and (e) follows from Lemma (2.3); and it is clear that (c) implies (b) and that (f) implies (e). If (b) holds, then $R(T^n)\supseteq R(T^d)\cap N(T)$ for all $n\geqq d$, so that $R(T^\infty)\supseteq R(T^d)\cap N(T)$, which clearly implies (c). We omit the similar proof that (e) implies (f).

Condition (c) in Theorem (3.1) above has been recognized as an important property of semi-Fredholm operators, and certain other operators, by several authors [**8**, p. 1243] [**14**, p. 454] [**19**].

For the definition of topological uniform descent, which is referred to in the next theorem, see Definition (2.5), above.

THEOREM (3.2). *If T is a bounded operator with uniform descent for $n\geqq d$ on the Banach space X, the following are equivalent:*

(a) *T has topological uniform descent for $n\geqq d$.*
(b) *There is an $n\geqq d$ and a positive integer k for which $R(T^{n+k})$ is closed in the operator range topology on $R(T^n)$.*
(c) *For each $n\geqq d$ and each positive integer k, $R(T^{n+k})$ is closed in the operator range topology on $R(T^n)$.*
(d) *There is an $n\geqq d$ and a positive integer k for which $N(T^n)+R(T^k)$ is closed in X.*
(e) *For all $n\geqq d$ and for all positive integers k, and also for $k=\infty$, $N(T^n)+R(T^k)$ is closed in X.*

PROOF. The equivalence of (a), (b), and (c) is immediate from Lemma (2.4). For each fixed n and k, T^n induces a bounded operator from X to $R(T^n)$, with the operator range topology. Hence it follows from Lemma (2.2) that $R(T^{n+k})$ is closed in the operator range topology on $R(T^n)$ if and only if $T^{-n}(R(T^{n+k}))=N(T^n)+R(T^k)$ is closed in the original topology on X. This completes the proof.

Theorem (3.4) below is our major result on the structure of operators with eventual topological uniform descent. Like almost all of our deeper theorems about an operator T with topological uniform descent for $n\geqq d$ on a Banach space X, Theorem (3.4) will be proved by first considering the restriction of T to $R(T^d)$ and then pulling back the results to X. The following notation will be very convenient for dealing with the topology of $R(T^d)$.

DEFINITION (3.3). If E is a subspace of $R(T^d)$, then $cl_d(E)$ is the closure of E in the operator range topology on $R(T^d)$.

THEOREM (3.4). *If T is a bounded operator on X with topological uniform descent for $n\geqq d$, then:*

(a) *The restriction of T to $R(T^\infty)$ is onto.*
(b) *The map induced by T on $R(T^d)/R(T^\infty)$ is bounded below.*
(c) *The restriction of T to $R(T^d)\cap cl(N(T^\infty))$ is onto.*
(d) *The map induced by T on $X/cl(N(T^\infty))$ is bounded below.*

PROOF. Let $\hat{T}$ be the restriction of T to $R(T^d)$. Then $\hat{T}$ has topological uniform descent for $n \geqq 0$, and it also has closed range. It follows from Theorem (3.1)(b) that $N(\hat{T}) \subseteq R(\hat{T}^n)$ for all n, so that

$$\hat{T}^{-1}(R(\hat{T}^\infty)) = \cap \hat{T}^{-1}(R(\hat{T}^{n+1})) = \cap (R(\hat{T}^n) + N(\hat{T})) = R(\hat{T}^\infty)\,.$$

Hence a simple application of Lemma (2.1)(d) yields $\hat{T}(R(\hat{T}^\infty)) = R(\hat{T}^\infty)$.

Since $R(\hat{T}^\infty) = R(T^\infty)$, we now have that the restriction of T to $R(T^\infty)$ is onto. Since $\hat{T}^{-1}(R(\hat{T}^\infty)) = R(T^d) \cap T^{-1}(R(T^\infty))$, the map induced by T on $R(T^d)/R(T^\infty)$ is one-to-one; and since $\hat{T}$ has closed range, this induced map has closed range. This proves (a) and (b).

For any operator T, we have $T^{-1}(N(T^\infty)) = N(T^\infty)$. For $\hat{T}$, which has uniform descent for $n \geqq 0$, we also have $\hat{T}(N(\hat{T}^\infty)) = N(\hat{T}^\infty)$, since $N(\hat{T}^\infty) \subseteq R(\hat{T})$ by Theorem (3.1)(e). Hence a direct application of Lemma (2.2) yields $\hat{T}(cl_d N(\hat{T}^\infty)) = cl_d(N(\hat{T}^\infty))$ and $\hat{T}^{-1}(cl_d N(\hat{T}^\infty)) = cl_d(N(\hat{T}^\infty))$.

It is easy to see that $N(\hat{T}^\infty) = R(T^d) \cap N(T^\infty)$ and hence that $T^{-d}(N(\hat{T}^\infty)) = N(T^\infty)$. Applying Lemma (2.2)(b) to the map induced by T^d from X onto $R(T^d)$ then yields

$$cl(N(T^\infty)) = T^{-d} cl_d(N(\hat{T}^\infty))\,.$$

So we have

$$T^{-1}(cl(N(T^\infty))) = T^{-1}T^{-d}(cl_d(N(\hat{T}^\infty))) = T^{-d}\hat{T}^{-1} cl_d(N(\hat{T}^\infty))$$
$$= T^{-d} cl_d(N(\hat{T}^\infty)) = cl(N(T^\infty))\,.$$

Hence the map in (d) is one-to-one. Also $R(T) + N(T^\infty)$ is closed, by Theorems (3.1)(f) and (3.2)(e). Hence $R(T) + cl(N(T^\infty)) = R(T) + N(T^\infty)$ is also closed. This proves (d).

Since $cl_d(N(\hat{T}^\infty)) = cl_d(R(T^d) \cap N(T^\infty))$, we have

$$T cl_d(R(T^d) \cap N(T^\infty)) = cl_d(R(T^d) \cap N(T^\infty))\,.$$

Thus we can complete the proof of part (c), and of the theorem, by proving the formula

$$R(T^d) \cap cl(N(T^\infty)) = cl_d(R(T^d) \cap N(T^\infty))\,. \tag{3.5}$$

Since $T^{-d}(cl(N(T^\infty))) = cl(N(T^\infty))$, the left side of formula (3.5) is $T^d(cl[N(T^\infty)])$. An application of Lemma (2.2)(a) to the map induced by T^d from X onto $R(T^d)$ shows that the right side of formula (3.5) is also $T^d(cl[N(T^\infty)])$. This completes the proof of formula (3.5) and of the theorem.

The following lemma collects some identities involving ranges and null spaces of operators with eventual topological uniform descent. We will need these identities in the next two sections.

LEMMA (3.6). *If T is a bounded operator on X and has topological uniform* *descent for* $n \geqq d$, *then:*

(a) $R(T^{\infty})+N(T^{d})=R(T^{\infty})+N(T^{\infty})=R(T^{\infty})+cl(N(T^{\infty}))$

(b) $R(T^{d})\cap N(T^{\infty})=R(T^{\infty})\cap N(T^{\infty})$

(c) $R(T^{d})\cap cl(N(T^{\infty}))=R(T^{\infty})\cap cl(N(T^{\infty}))$

(d) $cl[R(T^{\infty})\cap N(T^{\infty})]=cl[R(T^{\infty})\cap cl(N(T^{\infty}))]$.

PROOF. It follows from Theorem (3.1)(c) that, for each $n \geqq d$, $R(T^{n})\cap N(T) \subseteq R(T^{\infty})$. Applying T^{-n} to both sides of this formula, and using Lemma (2.1)(c), yields $N(T^{n+1})\subseteq R(T^{\infty})+N(T^{n})$, or equivalently, $R(T^{\infty})+N(T^{n+1})=R(T^{\infty})+N(T^{n})$. An easy induction then yields $R(T^{\infty})+N(T^{d})=R(T^{\infty})+N(T^{\infty})$. Since $R(T^{\infty})+N(T^{d})$ is a closed subspace, by Theorem (3.2)(e), $R(T^{\infty})+N(T^{\infty})=R(T^{\infty})+cl(N(T^{\infty}))$. This proves (a). The proof of (b) follows similarly by applying T^{n} to both sides of the set inclusion: $N(T^{\infty})\subseteq R(T)+N(T^{n})$ for $n \geqq d$.

Using part (b), formula (3.5), and the fact that $R(T^{\infty})$ is closed in the operator range topology on $R(T^{d})$, we obtain:

$$R(T^{\infty})\cap cl(N(T^{\infty}))\subseteq R(T^{d})\cap cl(N(T^{\infty}))=cl_{d}(R(T^{\infty})\cap N(T^{\infty}))$$
$$\subseteq R(T^{\infty})\cap cl(N(T^{\infty}))\,.$$

This yields part (c) directly, and, by taking closures in the topology of X, also yields part (d) and completes the proof of the lemma.

For non-invertible perturbations, our best results will involve operators with almost uniform descent instead of the more general class of operators with eventual uniform descent. We will therefore give several characterizations of the quantity $k(T)$ and of almost uniform descent (recall the definitions from formula (1.2) and Definition (1.3)).

THEOREM (3.7). *If* T *is a linear transformation on the vector space* X, *then each of the following quantities are equal to each other and to* $k(T)$:

(a) $\sup\{\dim[N(T)/(N(T)\cap R(T^{n}))]\}$;

(b) $\dim[N(T)/(N(T)\cap R(T^{\infty}))]$;

(c) $\sup\{\dim[(R(T)+N(T^{n}))/R(T)]\}$;

(d) $\dim[(R(T)+N(T^{\infty}))/R(T)]$.

PROOF. It follows from Lemma (2.3)(a) that the dimension of $N(T)/[N(T)\cap R(T^{n})]$ equals $k_{0}(T)+k_{1}(T)+\cdots+k_{n-1}(T)$. The equality of $k(T)$ and the quantity in part (a) now follows from the definition of $k(T)$. A similar argument, starting from Lemma (2.3)(b) and (c), shows the equality of $k(T)$ and the quantity in part (c).

Since $N(T)\cap R(T^{n})\supseteq N(T)\cap R(T^{\infty})$, the quantities in (a) and (b) are equal when $k(T)=\infty$. Suppose $k(T)$ is finite; then it follows from (a) that there is an integer d for which

$$N(T)\cap R(T^{d})=N(T)\cap R(T^{n})\subseteq R(T^{\infty}), \qquad \text{for } n\geqq d.$$

The equality of the quantities in (a) and (b) for $k(T)$ finite is now clear. The equality of the quantities in (c) and (d) is proved similarly; so this completes the proof.

The following characterization of operators with almost uniform descent and closed range follows immediately from Lemma (2.4) and Definition (2.5).

THEOREM (3.8). *Suppose that T is a bounded operator on the Banach space X. If T has almost uniform descent, then the following are equivalent:*

(a) *$R(T^{k})$ is closed for some positive integer k.*
(b) *$R(T^{k})$ is closed for each positive integer k.*
(c) *T has eventual topological uniform descent.*

The next result describes $k(T)$ in terms of the nullity and defect of maps induced by T.

THEOREM (3.9). *If T is a bounded operator on the Banach space X, then:*

(a) *$k(T)$ is the dimension of the null space of the map induced by T on $X/R(T^{\infty})$.*
(b) *$k(T)$ is the codimension of the range of the restriction of T to $N(T^{\infty})$.*
(c) *When T has closed range, $k(T)$ is the codimension of the range of the restriction of T to $cl(N(T^{\infty}))$.*

PROOF. We will prove (c), omitting the similar but simpler proofs of (a) and (b). We first suppose that $k(T)$ is finite, so that T has eventual topological uniform descent. It follows, from Theorem (3.4)(d), that $T^{-1}(cl(N(T^{\infty})))=cl(N(T^{\infty}))$. Hence, using Lemmas (2.1) and (3.6)(a), we have that the cokernel of the restriction of T to $cl(N(T^{\infty}))$ is

$$\frac{cl(N(T^{\infty}))}{R(T)\cap cl(N(T^{\infty}))}\cong\frac{R(T)+cl(N(T^{\infty}))}{R(T)}=\frac{R(T)+N(T^{\infty})}{R(T)},$$

whose dimension is $k(T)$, by Theorem (3.7)(d).

When $k(T)$ is infinite, part (c) follows similarly from Theorem (3.7)(d) together with the fact that $T(cl[N(T^{\infty})])\subseteq R(T)\cap cl[N(T^{\infty})]$, so the proof is complete.

Our perturbation theorems involving operators with almost uniform descent will often not use $k(T)$, but rather the quantity

$$k(T^{\infty})=\sup_{n} k(T^{n}), \tag{3.10}$$

which we describe in the following lemma.

LEMMA (3.11). *Suppose that T is a bounded operator on X. Then*

(a) *T has almost uniform descent if and only if $k(T^\infty)$ is finite.*

(b) *$k(T^\infty)$ is the dimension of $(R(T^\infty)+N(T^\infty))/R(T^\infty)$.*

PROOF. First notice that it follows from Theorem (3.9)(a) or (b), or even from the definition of $k(T)$, that $\{k(T^n)\}$ is a nondecreasing sequence and that $k(T^n) \leqq n k(T)$.

Now suppose that T has almost uniform descent, so that there is an integer d for which T has uniform descent for $n \geqq d$. Then it follows from Theorems (3.1)(c) and (3.7)(d), together with Lemma (2.1)(b), that $k(T^n)$ is the dimension of $(R(T^n)+N(T^\infty))/R(T^n)=(R(T^\infty)+N(T^\infty))/R(T^\infty)$, for all $n \geqq d$. This proves the theorem for T with almost uniform descent.

Suppose that T does not have almost uniform descent, so that $k(T)=\infty$. The theorem then follows from the fact that both $k(T^\infty)$ and the dimension of $(R(T^\infty)+N(T^\infty))/R(T^\infty)$ must be greater than or equal to $k(T)$.

4. Small perturbations.

Suppose that T is a bounded operator on X and that T has topological uniform descent for $n \geqq d$. In this section we study bounded operators V which commute with T and for which $V-T$ is "sufficiently small," and in the next section we do the same for $V-T$ compact. In both cases our study proceeds in two steps. We first consider V in the special case that T has topological uniform descent for $n \geqq 0$, which in the general case gives a description of the restriction of V to $R(T^d)$. We then determine the properties of V from the properties of its restriction to $R(T^d)$. Though we say something about arbitrary commuting small or compact perturbations, the strongest results will require additional hypotheses on T, V, or $V-T$.

We start by defining what we mean by "sufficiently small." Our definition will naturally be in terms of the reduced minimum modulus of T, $\gamma(T)$, or of a map induced by T (see [**18**, p. 231], [**17**, pp. 271-272], or [**6**, Definition IV.1.2] for a definition of $\gamma(T)$; in particular recall that $\gamma(0)=\infty$).

DEFINITION (4.1). Suppose that T is a bounded operator on X with topological uniform descent for $n \geqq d$, and that V is a bounded operator which commutes with T. We say that $V-T$ is *sufficiently small* if the norm of the restriction of $V-T$ to $R(T^d)$ is less than the reduced minimum modulus of the restriction of T to $R(T^d)$.

If $R(T^d)$ is given the norm under which it is isometric to $X/N(T^d)$ [**9**, formula (3.1), p. 1433], or if $R(T^d)$ is closed in X and is given the restriction norm, it is easy to see that the norm of $V-T$ is no greater than the norm of its restriction to $R(T^d)$, so definition (4.1) is essentially a restriction on $\|V-T\|$.

The next lemma treats sufficiently small perturbations when T has closed range and uniform descent for $n\geqq 0$. Parts of the lemma can be extracted from [**18**, Theorem 3, pp. 297-298], [**14**, pp. 460-463], and [**4**, Satz 3, p. 62], at least when T is semi-Fredholm or when $V-T$ is invertible.

LEMMA (4.2). *Suppose that T is a bounded operator with closed range and with topological uniform descent for $n\geqq 0$ on the Banach space X, and suppose that V is a bounded operator which commutes with T. If $V-T$ is sufficiently small, then*

(a) *V has closed range and uniform descent for $n\geqq 0$.*

(b) $\dim(R(V^n)/R(V^{n+1}))=\dim(X/R(T))$ *for all* $n\geqq 0$.

(c) $\dim(N(V^{n+1})/N(V^n))=\dim(N(T))$ *for all* $n\geqq 0$.

(d) $R(V^\infty)=R(T^\infty)$.

(e) $cl(N(V^\infty))=cl(N(T^\infty))$.

PROOF. The proof is based on considering the maps induced by T and V on the Banach spaces $R(T^\infty)$, $X/R(T^\infty)$, $cl(N(T^\infty))$ and $X/cl(N(T^\infty))$. We start by letting Y be any of these four spaces. Let $\hat{T}$ and $\hat{V}$ be the maps induced by T and V on Y, and recall, from Theorem (3.4), that $\hat{T}$ is either bounded below or onto. It is clear that $\|\hat{V}-\hat{T}\|\leqq\|V-T\|$. Also, since both $R(T^\infty)$ and $cl(N(T^\infty))$ contain $N(T)$ and are contained in $R(T)$, by Theorem (3.1)(c) and (d), an easy calculation shows that $\gamma(\hat{T})\geqq\gamma(T)$. Hence if $\|V-T\|$ is sufficiently small, then $\|\hat{V}-\hat{T}\|<\gamma(\hat{T})$. Thus when $\hat{T}$ is bounded below or onto, so is $\hat{V}$, and

$$\gamma(\hat{V})\geqq\gamma(\hat{T})-\|\hat{V}-\hat{T}\|\,. \tag{4.3}$$

(The case where $\hat{T}$ is bounded below is a trivial calculation, and the case where $\hat{T}$ is onto can be obtained by taking adjoints.)

Specializing to $Y=R(T^\infty)$ and to $Y=X/cl(N(T^\infty))$, we see that $V(R(T^\infty))=R(T^\infty)$ and $V^{-1}(cl[N(T^\infty)])\subseteq cl[N(T^\infty)]$. So, using Lemma (3.6)(a), we obtain

$$N(V)\subseteq cl(N(V^\infty))\subseteq cl(N(T^\infty))\subseteq R(T^\infty)\subseteq R(V^\infty)\subseteq R(V)\,, \tag{4.4}$$

which, by Theorem (3.1), implies that V has uniform descent for $n\geqq 0$.

Specializing to $Y=X/R(T^\infty)$, we have that $\hat{T}$ and $\hat{V}$ are both bounded below and that $Y/R(\hat{T})$ and $Y/R(\hat{V})$ both have the same dimension [**6**, Corollary V.1.3, p. 111]. Thus it follows from formula (4.4) that $R(V)$ is closed and that $X/R(V)$ and $X/R(T)$ have the same dimension. Part (b) now follows from the fact that V has uniform descent for $n\geqq 0$.

We can also use the maps induced on $X/R(T^\infty)$ to prove that $R(V^\infty)=R(T^\infty)$. For each $0\leqq\lambda\leqq 1$, let $V_\lambda=T+\lambda(V-T)$; we need only show that $R(V_\lambda^\infty)$ is locally constant. Since each V_λ has closed range and uniform descent for $n\geqq 0$, it will be enough to show that if $\|V-T\|\leqq\gamma(T)/2$, then $R(V^\infty)=R(T^\infty)$. In fact, because of formula (4.4), we need only show $R(V^\infty)\subseteq R(T^\infty)$. As we observed above, $\|\hat{V}-\hat{T}\|\leqq\|V-T\|$ and $\gamma(T)\leqq\gamma(\hat{T})$, so it follows from formula (4.3) that $\|\hat{T}-\hat{V}\|$

$\leq\gamma(\hat{V})$. Hence we can apply formula (4.4), with V replaced by $\hat{T}$ and T replaced by $\hat{V}$, to obtain $R(\hat{V}^\infty)\subseteq R(\hat{T}^\infty)=\{0\}$. Hence $R(V^\infty)\subseteq R(T^\infty)$. This proves (d). We omit the proofs of the rest of the theorem, parts (c) and (e), since they follow from an argument very similar to the above, with the maps induced on $X/R(T^\infty)$ replaced by the maps induced on $cl(N(T^\infty))$.

When T has topological uniform descent for $n\geq d\neq 0$, the above lemma describes the restriction of the perturbed operator V to $R(T^d)$. The next lemma provides a tool for studying V in terms of its restriction to $R(T^d)$. We omit the proof, which is very similar to the proof of Lemma (2.3).

LEMMA (4.5). *If V and T are commuting linear transformations on the vector space X, and n and d are nonnegative integers, then:*

(a) *The map induced by T^d from $R(V^n)/R(V^{n+1})$ to $R(V^nT^d)/R(V^{n+1}T^d)$ is onto and has null space naturally isomorphic to $(R(V^n)\cap N(T^d))/(R(V^{n+1})\cap N(T^d))$.*

(b) *The map induced by the identity on X from $(N(V^{n+1})\cap R(T^d))/(N(V^n)\cap R(T^d))$ to $N(V^{n+1})/N(V^n)$ is one-to-one and has cokernel naturally isomorphic to $(N(V^{n+1})+R(T^d))/(N(V^n)+R(T^d))$.*

(e) *T^d induces an isomorphism from $N(T^{d+n+1})/N(T^{d+n})$ onto $(N(T^{n+1})\cap R(T^d))/(N(T^n)\cap R(T^d))$.*

The next theorem describes arbitrary small commuting perturbations of an operator with eventual topological uniform descent.

THEOREM (4.6). *Suppose that T is a bounded operator with topological uniform descent for $m\geq d$ on the Banach space X, and that V is a bounded operator which commutes with T. If $V-T$ is sufficiently small, then:*

(a) $\dim(R(V^n)/R(V^{n+1}))\geq\dim(R(T^m)/R(T^{m+1}))$ *for all $n\geq 0$ and $m\geq d$.*

(b) $\dim(N(V^{n+1})/N(V^n))\geq\dim(N(T^{m+1})/N(T^m))$ *for all $n\geq 0$ and $m\geq d$.*

(c) $R(T^\infty)\subseteq R(V^\infty)\subseteq R(T^\infty)+N(T^\infty)$.

(d) $R(T^\infty)\cap cl(N(T^\infty))\subseteq cl(N(V^\infty))\subseteq cl(N(T^\infty))$.

PROOF. Recall that $\dim(R(T^m)/R(T^{m+1}))$ and $\dim(N(T^{m+1})/N(T^m))$ are constant for $m\geq d$; since T has uniform descent for $m\geq d$. Hence parts (a) and (b) follow directly from Lemmas (4.2) and (4.5).

For the proofs of (c) and (d), we let $\hat{V}$ be the restriction of V to $R(T^d)$. It follows from Lemma (4.2)(d) that $R(\hat{V}^\infty)=R(T^\infty)$. The first inclusion in part (c) is now clear, since $R(\hat{V}^\infty)\subseteq R(V^\infty)$. For the second inclusion, we apply T^{-d} to both sides of the identity $R(T^\infty)=\bigcap_n R(T^dV^n)$, and use Lemma (2.1)(c) and Theorem (3.4)(a), to obtain

$$R(T^\infty)+N(T^d)=\bigcap_n(R(V^n)+N(T^d))\supseteq R(V^\infty).$$

Part (c) now follows directly from Lemma (3.6)(a).

To prove (d), we first express Lemma (4.2)(e) in the notation of Definition (3.3) and obtain

$$cl_d(R(T^d)\cap N(T^\infty))=cl_d(N(\hat{V}^\infty))\subseteq cl(N(V^\infty)).$$

The first inclusion in part (d) now follows from formula (3.5) and Lemma (3.6)(c).

Finally, as we observed in the proof of Lemma (4.2), the map induced by V on $R(T^d)/cl_d(R(T^d)\cap N(T^\infty))$ is one-to-one. Hence, exactly as in the proof of Theorem (3.4)(d), we can conclude that $V^{-1}(cl[N(T^\infty)])\subseteq cl[N(T^\infty)]$. This clearly implies that $cl(N(V^\infty))\subseteq cl(N(T^\infty))$, and this completes the proof of the theorem.

For semi-Fredholm operators, Gol'dman and Kračkovskii [8, Theorem 3, p. 1244] prove the inclusions $R(T^\infty)\subseteq R(V^\infty)$ and $R(T^\infty)\cap cl(N(T^\infty))\subseteq cl(N(V^\infty))$. I believe that the other conclusions of Theorem (4.6), above, are new even for semi-Fredholm operators.

In order to replace the inequalities and set inclusions in Theorem (4.6) with equalities, we need additional hypotheses on V or T. For a spectral analysis of T, the most important case is when $V=T-\lambda I$. Since the proof is no harder under the assumption that $V-T$ is invertible, rather than a scalar multiple of the identity, we use this assumption as the added hypothesis on V.

THEOREM (4.7). *Suppose that T is a bounded operator with topological uniform descent for $n\geqq d$ on the Banach space X, and that V is a bounded operator that commutes with T. If $V-T$ is sufficiently small and invertible, then:*

(a) *V has closed range and uniform descent for $n\geqq 0$.*
(b) $\dim(R(V^n)/R(V^{n+1}))=\dim(R(T^d)/R(T^{d+1}))$ *for all* $n\geqq 0$.
(c) $\dim(N(T^{n+1})/N(T^n))=\dim(N(T^{d+1})/N(T^d))$ *for all* $n\geqq 0$.
(d) $R(V^\infty)=R(T^\infty)+N(T^\infty)$.
(e) $cl[N(V^\infty)]=cl[R(T^\infty)\cap N(T^\infty)]$.

PROOF. The proof will follow easily from our previous results, once we verify the formulas:

$$N(V^\infty)\subseteq R(T^\infty) \quad \text{and} \quad N(T^\infty)\subseteq R(V^\infty).$$

These formulas are a consequence of the fact that $V-T=U$ is invertible. Since V and T commute, it follows from the binomial theorem that for each fixed k, there is a bounded operator S, depending on k, for which $U^{-k}V^k=I-TS$. Hence if $x\in N(V^k)$, then $x=T^nS^nx\in R(T^n)$ for all n. Since k is arbitrary, we have $N(V^\infty)\subseteq R(T^\infty)$, and by interchanging the roles of V and T we also have $N(T^\infty)\subseteq R(V^\infty)$.

Parts (b) and (c) now follow immediately from Lemma (4.2)(b) and (c) and Theorem (4.6)(a) and (b). Also, it follows from Lemma (4.2)(a) that $R(VT^d)$ is closed in the topology of $R(T^d)$, so that

$$T^{-d}(R(VT^d))=R(V)+N(T^d)=R(V)$$

is closed in X. The inclusion $N(V^\infty)\subseteq R(T^\infty)$, together with Theorem (4.6)(c), yields $N(V^\infty)\subseteq R(V^\infty)$, which implies that V has uniform descent for $n\geqq 0$, by Theorem (3.1).

Part (b) follows easily from $N(T^\infty)\subseteq R(V^\infty)$ together with Theorem (4.6)(d). Finally, from Theorem (4.6)(d) we obtain $R(T^\infty)\cap cl(N(T^\infty))\subseteq cl(N(V^\infty))$ and $N(V^\infty)\subseteq R(T^\infty)\cap cl(N(T^\infty))$. Taking closures, and applying Lemma (3.6)(e), yields part (e), and completes the proof of the theorem.

The next corollary is a straightforward application of Theorem (4.7)(a) and (b) to special classes of operators with eventual topological uniform descent.

COROLLARY (4.8). *Suppose that T is a bounded operator with topological uniform descent for $n\geqq d$, and that V is a bounded operator commuting with T. If $V-T$ is invertible and sufficiently small, then:*

(a) *V has infinite ascent or descent if and only if T does.*
(b) *V cannot have finite non-zero ascent or descent.*
(c) *V is onto if and only if T has finite descent.*
(d) *V is one-to-one (or bounded below) if and only if T has finite ascent.*
(e) *V is invertible if and only if 0 is a pole of T.*
(f) *V is upper semi-Fredholm if and only if some $N(T^{n+1})/N(T^n)$ is finite-dimensional.*
(g) *V is lower semi-Fredholm if and only if some $R(T^n)/R(T^{n+1})$ is finite-dimensional.*

The following corollary gives a condition under which λ is a pole of T for λ in the boundary of the spectrum of T. The corollary is really just a special case of Corollary (4.8)(e), but we state it explicitly since similar characterizations of poles have proved useful in a number of contexts [**10**, p. 419], [**12**], [**21**].

COROLLARY (4.9). *Suppose that T is a bounded linear operator and that λ belongs to the boundary of the spectrum of T. If $T-\lambda I$ has eventual topological uniform descent, then λ is a pole of T.*

Corollary (4.9) above generalizes characterizations of poles due to Lay [**21**, pp. 202-206] and to us [**9**, Theorem 5.4, p. 1439]. Lay [**21**, pp. 202-203] and Bart and Lay [**1**, p. 161] also have part of Corollary (4.8). They show that if T has finite descent, then V is onto; and that if T has ascent no more than d and if some $R(T^{d+n})$ is closed, then V is one-to-one. As far as I know, the other statements in Corollary (4.8) are new. We have already discussed, in Section 1, the relation between Theorem (4.7) and earlier results.

Of course Theorem (4.7) can be applied to a variety of operators not covered in Corollary (4.8). We illustrate how this is done with one simple example. Suppose that the restriction of T to $R(T^d)$ is a unilateral shift of multiplicity

$k \leqq \infty$ (in the simplest case T could be the direct sum of a shift and a nilpotent operator). Suppose that V commutes with T and that $V-T$ is sufficiently small and invertible. Then we can conclude from Theorem (4.7) that V is bounded below; that $R(V^n)/R(V^{n+1})$ has dimension k, for all $n \geqq 0$; and that $R(V^\infty)= N(T^d)$. If $V-T$ were not invertible, we could still conclude, from Theorem (4.6) in this case, that both $R(V^\infty)$ and $cl(N(V^\infty))$ are subspaces of $N(T^d)$, and that $\dim(R(V^n)/R(V^{n+1})) \geqq k$ for all $n \geqq 0$. Similar statements can be made for backward shifts, or for direct sums of forward and backward shifts.

The next result shows that when T has almost uniform descent we can obtain conclusions nearly as strong as those in Theorem (4.7) even without assuming that $V-T$ is invertible.

THEOREM (4.10). *Suppose that T is a bounded operator with almost uniform descent and closed range on the Banach space X, and that V is a bounded operator which commutes with T. If $V-T$ is sufficiently small, then:*

(a) *V has almost uniform descent and closed range.*
(b) *$R(V^n)/R(V^{n+1})$ and $R(T^m)/R(T^{m+1})$ have the same dimension for all sufficiently large m and n.*
(c) *$N(V^{n+1})/N(V^n)$ and $N(T^{m+1})/N(T^m)$ have the same dimension for all sufficiently large m and n.*
(d) *$R(T^\infty)$ is a subspace of $R(V^\infty)$ with codimension less than or equal to $k(T^\infty)$.*
(e) *$cl[N(V^\infty)]$ is a subspace of $cl[N(T^\infty)]$ with codimension less than or equal to $k(T^\infty)$.*
(f) *$k(V^\infty) \leqq k(T^\infty)$.*

PROOF. From Theorem (3.8), we know that there is an integer d for which T has topological uniform descent for $n \geqq d$. Also, from Lemmas (3.11)(b) and (3.6)(a), we have that $k(T^\infty)$ is the dimension of

$$(R(T^\infty)+N(T^\infty))/R(T^\infty) \cong cl[N(T^\infty)]/(R(T^\infty) \cap cl[N(T^\infty)]).$$

Parts (d) and (e) are now immediate from Theorem (4.6)(c) and (d). From the set inclusions

$$R(T^\infty) \subseteq R(V^\infty) \subseteq R(V^\infty)+N(V^\infty) \subseteq R(T^\infty)+cl(N(T^\infty))$$

together with Lemma (3.11), we see that $k(V^\infty) \leqq k(T^\infty)$ and therefore that V has almost uniform descent.

We now prove part (b) and that V has closed range. From Theorem (4.6) (b), we have that

$$R(T^\infty) \cap N(T^d) \subseteq R(V^n) \cap N(T^d) \subseteq N(T^d)$$

for all $n \geqq 0$. Since $k(T^{\infty})$ is the codimension of $R(T^{\infty}) \cap N(T^{d})$ in $N(T^{d})$, by Lemmas (3.11)(b) and (3.6)(a) again, it follows that the sequence of spaces $\{R(V^{n}) \cap N(T^{d})\}_{n}$ is eventually constant. Part (b) is now a direct application of Lemma (4.5)(a). Also, from Lemma (4.5)(a), it follows that the map induced by T^{d} from $X/R(V)$ to $R(T^{d})/R(T^{d}V)$ has finite-dimensional null space, so that $R(V)$ has finite codimension in $T^{-d}(R(T^{d}V))$. But $T^{-d}(R(T^{d}V))$ is closed, by Lemma (4.2)(a), so that $R(V)$ is closed by [**2**, Corollary (3.2.5), p. 37].

We omit the proof of part (c), which follows similarly from the set inclusions: $R(T^{d}) \subseteq N(V^{n}) + R(T^{d}) \subseteq R(T^{d}) + cl(N(T^{\infty}))$.

We remark that if we added the assumption $\|V-T\| < \gamma(T)$ to the hypothesis of Theorem (4.10), we could use Theorem (3.9) to conclude also that $k(V) \leqq k(T)$.

5. Compact perturbations.

Suppose that T is a bounded operator with topological uniform descent for $n \geqq d$. In this section we study bounded operators V which commute with T and for which $V-T$ is compact. Except in the case that T is semi-Fredholm and has finite ascent or descent, which we treated in [**11**] (compare also [**15**] and [**24**]), the results in this section are all new even for Fredholm operators.

Just as in our study of small perturbations, we start with the special case that $d=0$. We treat this case by applying results on perturbations of operators which are bounded below or onto to various maps induced by T. We start with a lemma which adapts from [**11**] the results that we need on compact perturbations of operators which are bounded below or onto.

LEMMA (5.1). *Suppose that T and V are commuting bounded operators on the Banach space X, and that $V-T$ is compact.*

(a) *If T is bounded below, then V has finite ascent and for each integer k, $T(N(V^{k}))=N(V^{k})$.*
(b) *If T is onto, then V has finite descent and for each integer k, $T^{-1}(R(V^{k}))=R(V^{k})$.*

PROOF. The statements about ascent and descent are special cases of [**11**, Theorem 2, p. 80]; and the statement about $R(V^{k})$ begins the proof of [**11**, Lemma 1, p. 79]. If T is bounded below, then $N(V^{k})$ is finite dimensional [**2**, Corollaries (1.3.7)(a) and (1.3.4), p. 9], and the restriction of T to $N(V^{k})$ is one-to-one. Hence T maps $N(V^{k})$ onto itself; so the proof is complete.

LEMMA (5.2). *Suppose that T is a bounded operator with topological uniform descent for $n \geqq 0$, and that V is a bounded operator that commutes with T. If $V-T$ is compact, then:*

(a) *V has eventual topological uniform descent.*
(b) $\dim(R(V^{n})/R(V^{n+1})) = \dim(R(T^{m})/R(T^{m+1}))$ *for all $m \geqq 0$ and all*

sufficiently large n.

(c) $\dim(N(V^{n+1})/N(V^n))=\dim(N(T^{m+1})/N(T^m))$ *for all* $m\geqq 0$ *and all sufficiently large n.*

(d) $(R(T^\infty)+R(V^\infty))/R(V^\infty)$ *is finite-dimensional.*

PROOF. We start by examining the maps induced by V on $X/R(T^\infty)$, $R(T^\infty)$, $cl[N(T^\infty)]$, and $X/cl[N(T^\infty)]$. We have from Theorem (3.4)(b) that the operator induced by T on $X/R(T^\infty)$ is bounded below, and it is clear that this operator maps no subspace onto itself. Hence it follows from Lemma (5.1)(a) that the map induced by V on $X/R(T^\infty)$ is one-to-one. Therefore $V^{-1}(R(T^\infty))=R(T^\infty)$.

The restriction of T to $R(T^\infty)$ is onto, by Theorem (3.4)(a). So it follows from Lemma (5.1)(b) that there is an integer p for which the space $M=V^p(R(T^\infty))$ satisfies

$$M=V(M)=T^{-1}(M)=T(M)\,. \tag{5.3}$$

Also, since the restriction of V to $R(T^\infty)$ is a compact perturbation of an onto operator, we know [**2**, Corollary (1.3.7)(b), p. 9] that M has finite codimension in $R(T^\infty)$.

The restriction of T to $cl(N(T^\infty))$ is onto, by Theorem (3.4)(c). So if we let $D=V(cl[N(T^\infty)])$ we have, from Theorem (3.4)(d) and Lemma (5.1)(b), that

$$T^{-1}(D)=T^{-1}(D)\cap cl[N(T^\infty)]=D\,,$$

so that $N(T^\infty)\subseteq D\subseteq cl[N(T^\infty)]$. Since $D=V(cl[N(T^\infty)])$ is closed [**2**, Corollary (1.3.7)(b), p. 9], we have that $V(cl[N(T^\infty)])=cl[N(T^\infty)]$.

The map induced by T on $X/cl[N(T^\infty)]$ is bounded below, by Theorem (3.4)(d). Hence it follows from Lemma (5.1)(a) that there is an integer q for which the space $N=V^{-q}(cl[N(T^\infty)])$ satisfies

$$N=V^{-1}(N)=T(N)=T^{-1}(N)\,. \tag{5.4}$$

Also, since the map induced by V on $X/cl[N(T^\infty)]$ is a compact perturbation of a map which is bounded below, we know that $cl[N(T^\infty)]$ has finite codimension in N.

We can now prove the various parts of the theorem from the information obtained above about maps induced by T and V. From formula (5.3), we have that

$$M\subseteq R(V^\infty)\cap R(T^\infty)\subseteq R(V)\cap R(T^\infty)\subseteq R(T^\infty)\,.$$

Since M has finite codimension in $R(T^\infty)$, both part (d) and the formula

$$\dim(R(T^\infty)+R(V))/R(V)<\infty \tag{5.5}$$

now follow easily, using Lemma (2.1)(b).

We now prove that V has eventual uniform descent. Since $VV^{-1}(R(T^\infty))=$

$R(T^\infty)$, it follows that $N(V^n) \subseteq R(T^\infty)$ and hence that

$$(5.6) \qquad R(V) \subseteq N(V^n) + R(V) \subseteq R(T^\infty) + R(V)$$

for all n. Hence formula (5.5) implies that the sequence $\{N(V^n)+R(V)\}$ is eventually constant, so that V has eventual uniform descent, by Theorem (3.1)(e).

To finish the proof of (a) and to prove (b), we consider the maps induced by T and V on X/M. We denote these maps by $\hat{T}$ and $\hat{V}$, respectively. It follows from formula (5.3) that $\hat{T}$ is one-to-one and has range $R(T)/M$, which is closed. Hence $\hat{V}$ is upper semi-Fredholm and has the same index as $\hat{T}$ (see [**2**, Corollary (1.3.7) and Theorem (4.4.1), pp. 9 and 66]). Since $M \subseteq R(V^n)$ for all n, by formula (5.3), we can conclude that $R(V^n)$ is closed for all n. This completes the proof of (a).

We break the proof of (b) into two cases. First suppose that $X/R(T)$ is finite-dimensional. For all $n \geqq p$, we have $V^{-n}(M) = V^{-n}(V^n[R(T^\infty)]) = R(T^\infty) + N(V^n) = R(T^\infty)$. Thus $\hat{V}^n$ has index $\dim(R(T^\infty)/M) - \dim(X/R(V^n))$, which must equal the index of $\hat{T}^n$, which is $-n(\dim(X/R(T)))$. Part (b) now follows in the case that $X/R(T)$ is finite-dimensional.

Now suppose that $X/R(T)$ is infinite-dimensional (i.e. suppose that $\hat{T}$ has index $-\infty$), so that $X/R(V)$ is also infinite-dimensional. For each n, the map induced by V^n from $X/R(V)$ into $R(V^n)/R(V^{n+1})$ is onto. Since the null space, $(N(V^n)+R(V))/R(V)$, of this map is finite-dimensional, by formulas (5.5) and (5.6), it follows that $R(V^n)/R(V^{n+1})$ is infinite-dimensional for all n. This completes the proof of (b).

For the proof of (c), we consider the restrictions of T and V to N. Since the restriction of T is onto, by formula (5.4), the restriction of V is semi-Fredholm and its index is the dimension of $N(T) \subseteq N$. When the index is finite, the proof of (c) uses the formula $V^n(N) = cl[N(T^\infty)]$ for $n \geqq q$. We omit the details since the proof is similar to the finite index case of (b).

When the index of the restriction to N is infinite, both $N(V)$ and $N(T)$ are infinite-dimensional subspaces of N. Also, for each n the map induced by V^n from $N(V^{n+1})/N(V^n)$ to $N(V)$ is one-to-one, and, by Lemma (2.3)(c), has cokernel whose dimension is the same as the dimension of the null space of the map induced by V^n from $X/R(V)$ to $R(V^n)/R(V^{n+1})$. But we observed above that this null space is finite-dimensional. Hence each $N(V^{n+1})/N(V^n)$ is infinite-dimensional when $N(T)$ is infinite-dimensional. This completes the proof of part (c) and of the theorem.

The next theorem describes arbitrary compact commuting perturbations of an operator with eventual topological uniform descent.

THEOREM (5.7). *Suppose that T is a bounded operator on X with topological uniform descent for $n \geqq d$, and that V is a bounded operator that commutes with*

T. *If $V-T$ is compact, then:*

(a) $\dim(R(V^n)/R(V^{n+1}))\geqq\dim(R(T^m)/R(T^{m+1}))$ *for all $m\geqq d$ and all sufficiently large n.*

(b) $\dim(N(V^{n+1})/N(V^n))\geqq\dim(N(T^{m+1})/N(T^m))$ *for all $m\geqq d$ and all sufficiently large n.*

(c) $(R(T^\infty)+R(V^\infty))/R(V^\infty)$ *is finite-dimensional.*

(d) $(cl[N(V^\infty)]+cl[N(T^\infty)])/cl[N(T^\infty)]$ *is finite-dimensional.*

PROOF. Let $\hat{T}$ and $\hat{V}$ be the restrictions of T and V to $R(T^d)$. Then $\hat{T}$ has topological uniform descent for $n\geqq 0$, so that parts (a) and (b) follow from Lemmas (4.5) and (5.2)(b)(c). Also part (c) follows easily from Lemma (5.2)(d) and the formulas $R(\hat{T}^\infty)=R(T^\infty)$ and $R(\hat{V}^\infty)\subseteq R(V^\infty)$.

For the proof of (d), we consider the maps induced by T and V on $X/cl[N(T^\infty)]$. The map induced by T is bounded below, by Theorem (3.4)(d), so the map induced by V has finite ascent, by Lemma (5.1)(a), and is upper semi-Fredholm, by [**2**, Corollary (1.3.7)(a), p. 9]. Let the ascent of the map induced by V be p. Then $(N(V^\infty)+cl[N(T^\infty)])/cl[N(T^\infty)]$ is a subspace of the finite-dimensional space $V^{-p}(cl[N(T^\infty)])/cl[N(T^\infty)]$. Hence $cl[N(T^\infty)]$ has finite codimension in $N(V^\infty)+cl[N(T^\infty)]$, so that $N(V^\infty)+cl[N(T^\infty)]$ is closed, and therefore equals $cl[N(V^\infty)]+cl[N(T^\infty)]$. This completes the proof.

We can replace the inequalities in Theorem (5.7)(a)(b) with equalities and also improve (5.7)(c)(d) if we assume that both T and V have eventual topological uniform descent. We do this in the following theorem.

THEOREM (5.8). *Suppose that T and V are commuting bounded operators with eventual topological uniform descent. If $V-T$ is compact, then:*

(a) $\dim(R(V^n)/R(V^{n+1}))=\dim(R(T^m)/R(T^{m+1}))$ *for all sufficiently large m and n.*

(b) $\dim(N(V^{n+1})/N(V^n))=\dim(N(T^{m+1})/N(T^m))$ *for all sufficiently large m and n.*

(c) $[R(T^\infty)+R(V^\infty)]/[R(T^\infty)\cap R(V^\infty)]$ *is finite-dimensional.*

(d) $(cl[N(T^\infty)]+cl[N(V^\infty)])/(cl[N(T^\infty)]\cap cl[N(V^\infty)])$ *is finite-dimensional.*

PROOF. Parts (a) and (b) are immediate consequences of Theorem (5.7)(a) and (5.7)(b), respectively. From Theorem (5.7)(c) it follows that both $(R(T^\infty)+R(V^\infty))/R(V^\infty)$ and $(R(V^\infty)+R(T^\infty))/R(T^\infty)$ are finite-dimensional. Since $(R(V^\infty)+R(T^\infty))/R(T^\infty)$ is linearly isomorphic to $R(V^\infty)/(R(T^\infty)\cap R(V^\infty))$, part (c) now follows directly. The proof of (d) follows similarly from Theorem (5.7)(d).

In order to have Theorem (5.8) apply to all commuting compact perturbations of an operator T with eventual topological uniform descent, we need to consider those T for which all commuting compact perturbations have eventual topological uniform descent. That some restriction on T is necessary is clear from considering the special cases in which T is the zero operator or is a projection of finite rank. We see in the next theorem that when T has almost uniform descent, then V also has almost uniform descent and closed range, and, a fortiori, has eventual topological uniform descent, by Theorem (3.8).

THEOREM (5.9). *Suppose that T is a bounded operator with almost uniform descent and closed range, and that V is a bounded operator which commutes with T. If $V-T$ is compact, then V has almost uniform descent and closed range and therefore satisfies the conclusions of Theorem* (5.8).

PROOF. We let

$$R=R(T^{\infty})\cap R(V^{\infty}) \quad \text{and} \quad N=N(V^{\infty})+cl(N(T^{\infty}))\,.$$

The proof of the theorem will follow readily once we prove the formula

$$\dim(R+N)/R<\infty\,. \tag{5.10}$$

Since T has almost uniform descent and closed range, it follows from Lemmas (3.11) and (3.6)(a) that $(R(T^{\infty})+cl[N(T^{\infty})])/R(T^{\infty})$ is finite-dimensional. Since $R(T^{\infty})/R$ and $N/cl(N(T^{\infty}))$ are both finite-dimensional, by Theorem (5.7), $(R(T^{\infty})+N)/R(T^{\infty})\cong N/(N\cap R(T^{\infty}))$ and $N/(N\cap R)$ are also finite-dimensional. This proves formula (5.10).

Since $R\subseteq R(V^{\infty})$ and $N(V^{\infty})\subseteq N$, it follows from formula (5.10) that $(R(V^{\infty})+N(V^{\infty}))/R(V^{\infty})$ is finite-dimensional. It now follows from Lemma (3.11) that V has almost uniform descent.

We now prove that V has closed range. There is a nonnegative integer d for which T has eventual topological uniform descent for $n\geqq d$. So, by applying Lemma (5.2)(a) to the restriction of T to $R(T^{d})$, we can conclude that $R(VT^{d})$ is closed in the operator range topology on $R(T^{d})$. Therefore $T^{-d}(R(VT^{d}))=R(V)+N(T^{d})$ is closed in X. But $R\subseteq R(V)$ and $N(T^{d})\subseteq N$; so it follows from formula (5.10) that $R(V)$ has finite co-dimension in the closed subspace $R(V)+N(T^{d})$. Hence $R(V)$ must itself be closed, by [**2**, Corollary (3.2.5), p. 37]. This completes the proof.

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Sandy GRABINER
Department of Mathematics
Pomona College
Claremont, California 91711
U.S.A.