J. Math. Soc. Japan
Vol. 34, No. 2, 1982

A note on E. Michael's example and rectangular products

By Ken-ichi TAMANO

(Received July 16, 1980)

1. Introduction.

Throughout this paper, all spaces considered are to be completely regular.

Pasynkov [6] introduced the notion of rectangular product as follows;

DEFINITION. A product space $X\times Y$ is *rectangular* if every cozero subset of $X\times Y$ is a σ-locally finite union of cozero-set rectangles (i.e. products $U\times V$ of cozero subsets of $X\times Y$).

Rectangularity guarantees the product theorem for covering dimension: i.e. if a product $X\times Y$ is rectangular, then $\dim X\times Y\leqq\dim X+\dim Y$. He has shown in many cases the product is rectangular and asked; Is every product $X\times Y$ rectangular?

This question has been answered negatively by examples of Wage [8] and Przymusiński [7] which do not satisfy the product theorem for covering dimension. As a simpler non-rectangular product, it was announced [6] that V. Zolotarev had proved that (Sorgenfrey line)^2 is not rectangular. As another famous non-normal example, we know the example of E. Michael [3]. In this note we establish the following theorem;

THEOREM. *Let X_A be a Hannerization of a metric space X with respect to a subset A of X. Then $A\times X_A$ is rectangular if and only if $A\times X_A$ is normal if and only if A is F_σ in X.*

As a corollary we obtain that (Michael's line)$\times$(Irrationals) is not rectangular.

It is known [6] normality induces rectangularity in products with a metric factor. At the end of this note we give an example of non-normal rectangular product with a metric factor and we will show that rectangularity cannot be preserved under perfect maps.

2. Rectangularity means normality for product $A\times X_A$.

DEFINITION 1. Let A be a subspace of a space X. The family of all sets of the form $U\cup K$, where U is an open subset of X and $K\subset A$, is a topology on X: The set X with this topology is called a *Hannerization of X with* *respect to* A and denoted by X_A. For elementary properties of X_A, see [1, Chapter 5].

DEFINITION 2. A space $(X, \mathcal{T})$ is *submetrizable* if there is a metrizable topology $\mathcal{M}$ on X having $\mathcal{M}\subset\mathcal{T}$. Subspace A of a space $(X, \mathcal{M})$ is denoted by $(A, \mathcal{M})$.

We will show that if X is a perfectly normal submetrizable space and $A\subset X$, then the rectangularity of the product $(A, \mathcal{M})\times X_A$ means normality.

THEOREM 1. *Let X be a metric space and $A\subset X$. If the product $A\times X_A$ is rectangular, then A is F_σ in X.*

Claim 1. Let $B\subset A$ and $B=\bigcup_{i=1}^{\infty} F_i$ where each F_i is a discrete closed subset of X. Then $\Delta_B=\{(x, x)\in A\times X_A: x\in B\}$ is a zero set of $A\times X_A$.

PROOF. Let $\mathcal{U}_i=\{(V_{1/i}(x)\cap A)\times\{x\}: x\in F_i\}$, where $V_\varepsilon(x)$ is a ε-neighborhood of x in a metric space X. Then $\mathcal{U}=\bigcup_{i=1}^{\infty}\mathcal{U}_i$ is a locally finite cozero-set collection in $A\times X_A$, and for each $x\in F_i$, $\{(x, x)\}$ is a zero set in $A\times X_A$ contained in $(V_{1/i}(x)\cap A)\times\{x\}$. Then by [4, Lemma 2.3] Δ_B is a zero set in $A\times X_A$.

Claim 2. Let B be a dense subset of A, and $U\times V$ is a cozero-set rectangle of $A\times X_A$. If $U\times V\cap\Delta_B=\emptyset$, then $U\times(U\cap V)$ is a cozero-set rectangle in $A\times(cl_X A)_A$.

PROOF. Take an open set U' in X such that $U=U'\cap A$. As U' is a cozero set in X, so in X_A, $U'\cap V\cap(cl_X A)_A$ is a cozero set in $(cl_X A)_A$. It suffices to show that $U'\cap V\cap cl_X A\subset A$ because then $U'\cap V\cap cl_X A=U'\cap V\cap A=(U'\cap A)\cap V=U\cap V$. Suppose there exists a point $x\in U'\cap V\cap cl_X A\setminus A$. Since B is dense in A, $cl_{X_A}A\setminus A=cl_{X_A}B\setminus B$. Hence there is a point $b\in U'\cap V\cap B$. Then $(b, b)\in U\times V\cap\Delta_B$. Contradiction.

For a collection of rectangles $\mathcal{U}=\{U_\alpha\times V_\alpha\}_{\alpha\in\Omega}$ in $X\times Y$, we put $\pi_X\mathcal{U}=\{U_\alpha\}_{\alpha\in\Omega}$.

Claim 3. Let X be a metric space and Y be any space. If $G=\cup(\bigcup_{i=1}^{\infty}\mathcal{U}_i)$ where each $\mathcal{U}_i$ is a locally finite collection of cozero-set rectangles in $X\times Y$, then G can be written as $G=\cup(\bigcup_{i=1}^{\infty}\mathcal{V}_i)$ such that each $\mathcal{V}_i$ is a locally finite collection of cozero-set rectangles in $X\times Y$ and furthermore $\pi_X\mathcal{V}_i$ is locally finite in X.

PROOF. Let $\{\mathcal{B}_i\}_{i=1}^{\infty}$ be a σ-locally finite base of X. Fix i and j. For each $U\in\mathcal{B}_i$, put

$$V_U=\cup\{V: \text{there exists } U'\times V\in\mathcal{U}_i \text{ such that } U\times V\subset U'\times V\}.$$

Because of the local finiteness of the right hand collection, V_U is a cozero set of Y. Put $\mathcal{V}_{ij}=\{U\times V_U: U\in\mathcal{B}_i\}$. Then $\{\mathcal{V}_{ij}\}_{ij}$ is the desired collection.

PROOF OF THEOREM 1. Take a dense subset B of A such that B can be written as $B=\bigcup_{i=1}^{\infty} F_i$ where each F_i is a discrete closed subset of X. By Claim 1, Δ_B is a zero set of $A\times X_A$. Let $G=A\times X_A\setminus\Delta_B$. G is a cozero set and by the rectangularity of the product and by Claim 3, G can be written as $G=\cup(\bigcup_{i=1}^{\infty}\mathcal{V}_i)$ where each $\mathcal{V}_i=\{U_\alpha\times V_\alpha : \alpha\in\Omega_i\}$ is a collection of cozero-set rectangles and $\{U_\alpha\}_{\alpha\in\Omega_i}$ is locally finite in A. In addition we can assume $\{U_\alpha\}_{\alpha\in\Omega_i}$ is locally finite in X.

Now, by Claim 2, for each i, $\mathcal{V}'_i=\{U_\alpha\times(U_\alpha\cap V_\alpha)\}_{\alpha\in\Omega_i}$ is a collection of cozero-set rectangles in $A\times(cl_X A)_A$, and $\cup(\bigcup_{i=1}^{\infty}\mathcal{V}'_i)$ covers $\Delta_A\cap G=\Delta_A\setminus\Delta_B$. Since $\{U_\alpha\}_{\alpha\in\Omega_i}$ is locally finite in X, $\{U_\alpha\cap V_\alpha\}_{\alpha\in\Omega_i}$ is locally finite in X_A. Hence $W_i=\bigcup_{\alpha\in\Omega_i}(U_\alpha\cap V_\alpha)$ is a cozero set of $(cl_X A)_A$ and $\bigcup_{i=1}^{\infty} W_i=A\setminus B$. Thus $A\setminus B$ is F_σ in $(cl_X A)_A$, so in X_A. Since B is F_σ in X, A is F_σ in X. This completes the proof.

By the same technique as above, we can prove the next theorem.

THEOREM 1′. *Let $(X, \mathcal{T})$ be a submetrizable space with a metrizable topology $\mathcal{M}$, and $A\subset X$. If the product $(A, \mathcal{M})\times X_A$ is rectangular, then A is F_σ in $(X,\mathcal{T})$.*

THEOREM 2. *Let X be a metric space, $((X, \mathcal{T})$ be a perfectly normal submetrizable space with a metrizable topology $\mathcal{M})$, and $A\subset X$. Then the following conditions are equivalent.*

(a) *$A\times X_A$ $((A, \mathcal{M})\times X_A)$ is rectangular.*
(b) *A is F_σ in X.*
(c) *X_A is metrizable (perfectly normal).*
(d) *$A\times X_A$ $((A, \mathcal{M})\times X_A)$ is normal.*

PROOF. From the above theorem, we have (a)⇒(b). For (b)⇒(c), see [**1**, 5.5.2]. (c)⇒(d); It is well known that every product of a metric space and a perfectly normal space is perfectly normal. (d)⇒(a); It is known [**6**] that normality induces rectangularity in products with a metric factor.

3. Examples.

EXAMPLE 1 (Michael's line). Let R be a real line with a usual topology, and P be all irrational numbers. Then R_P is called as Michael's line. From our previous theorem, $P\times R_P$ is not rectangular. Replacing P by another subset of R, we can make Michael's line to be Lindelöf [**3**]. Thus even a product of a separable metric space and a Lindelöf space need not to be rectangular.

REMARK. Wage's example [**8**] is also a non-rectangular product of a separable metric space and a Lindelöf space. Terasawa [**5**] has shown a product theorem of covering dimension for products with a factor of Michael's line type.

We can see that his theorem does not follow from Pasynkov's theorem for rectangular products.

The following example is suggested by T. Hoshina.

EXAMPLE 2 (Non-normal rectangular product). For every space Y, there is an extremally disconnected space $E(Y)$ called an absolute of Y, and a perfect irreducible map $E: E(Y)\rightarrow Y$. Let X be a metrizable space and Y be a space such that the product $X\times Y$ is not normal. We consider the perfect map $1_X\times E: X\times E(Y)\rightarrow X\times Y$. Ohta [2] proved that every product of a metric space and an extremally disconnected space is rectangular. Since perfect maps preserves normality, $X\times E(Y)$ is a non-normal rectangular product. Let X and Y be spaces of example 1, then we can find a non-normal rectangular product of a separable metric space and a Lindelöf space. Further in this case we can see that rectangularity cannot be preserved under perfect maps.

References

[1] R. Engelking, General Topology, Polish Scientific Publishers, Warsaw, 1977.
[2] H. Ohta, Some new characterization of metrizable spaces, preprint.
[3] E. Michael, The product of a normal space and a metric space need not to be normal, Bull. Amer. Math. Soc., **69** (1963), 375-376.
[4] K. Morita and T. Hoshina, *P*-embedding and product spaces, Fund. Math., **93** (1976), 71-80.
[5] J. Terasawa, On the zero-dimensionality of some non-normal product spaces, Sci. Rep. Tokyo Kyoiku Daigaku, Sect A, **11** (1971), 95-102.
[6] B.A. Pasynkov, On the dimension of rectangular products, Soviet Math. Dokl., **16** (1975), 344-347.
[7] T.C. Przymusiński, On the dimension of product spaces and an example of M. Wage, Proc. Amer. Math. Soc., **76** (1979), 315-321.
[8] M. L. Wage, The dimension of product spaces, Proc. Nat. Acad. Sci. U.S.A., **75** (1978), 4671-4672.

Ken-ichi TAMANO
Department of Mathematics
University of Tsukuba
Ibaraki 305
Japan