J. Math. Soc. Japan
Vol. 33, No. 4, 1981

# Zero-dimensional automorphisms having a dense orbit

By Nobuo AOKI

(Received Nov. 29, 1979)
(Revised March 28, 1980)

## § 1. Introduction.

Let $X$ be a compact metric group and $\sigma$ be an automorphism of $X$. It is proved in [7] that if some point of the group $X$ has a dense orbit under $\sigma$ then $\sigma$ is ergodic with respect to the Haar measure of $X$. It is known that the converse holds.

When $X$ is abelian and connected, it is known ([10]) that every automorphism $\sigma$ is densely periodic if $X$ contains a subgroup $H$ such that $X/H$ is a torus and $\bigcap \sigma^n H=\{e\}$. When $X$ is totally disconnected, it is proposed in [3] that an automorphism $\sigma$ with a dense orbit is densely periodic. This is obtained by using the following proposition in [3]. If $X$ admits an automorphism $\sigma$ with a dense orbit, then $X$ contains a sequence $X=F_0\supset F_1\supset \cdots$ of $\sigma$-invariant subgroups such that $\bigcap F_n=\{e\}$, and for every $n\geqq 0$ the subgroup $F_{n+1}$ is normal in $F_n$ and $F_n/F_{n+1}$ splits into a direct product $F_n/F_{n+1}=\times_{-\infty}^{\infty}\sigma^j \dot{F}_n$ of simple subgroups $\sigma^j \dot{F}_n$. However, this proposition is not true when $X$ is abelian, which is reformed as Theorem 2 (ii), and there is an example of $(X, \sigma)$ with a dense orbit which is not densely periodic (Theorem 1).

Throughout this paper, the term "subgroup" will be applied only to closed subgroups. The restriction and the factor of $\sigma$ will be denoted by the same symbol if there is no possibility of confusion.

It is obvious that an automorphism $\sigma$ preserves the normalized Haar measure of $X$. Therefore we can consider ergodic theoretical properties of $(X, \sigma)$ and we shall use ergodic properties for the proof of our results. For results in the ergodic theory and the theory of groups used here, the reader may refer to [9], [5] and [6].

In this paper we shall prove the following two theorems.

THEOREM 1. *There exist a totally disconnected compact metric abelian group $Y$ and an automorphism $\sigma$ with a dense orbit of $Y$ such that $\sigma$ on $Y$ has no periodic points except the identity of $Y$.*

REMARK. It is mentioned in [4] that zero-dimensional ergodic automorphisms satisfy specification in many cases. And in [4] the following is conjectured:

every zero-dimensional ergodic automorphism satisfies specification. By definition we get easily that every automorphism with specification is densely periodic. From this fact and Theorem 1, we shall obtain that the conjecture is false.

THEOREM 2. *Let $X$ be a totally disconnected compact metric group and $\sigma$ be an automorphism of $X$. If $\sigma$ has a dense orbit, then $X$ contains a sequence $X=F_0\supset F_1\supset\cdots$ of $\sigma$-invariant subgroups such that $\bigcap F_n=\{e\}$ and for every $n\geqq 0$, $F_{n+1}$ is normal in $F_n$ and*

(i) *when $F_n/F_{n+1}$ is non-abelian $F_n/F_{n+1}$ splits into a direct product $F_n/F_{n+1}=\times_{-\infty}^{\infty}\sigma^j\dot{F}_n$ of simple subgroups*) $\sigma^j\dot{F}_n$,*

(ii) *when $F_n/F_{n+1}$ is abelian there is in $F_n$ a decreasing sequence $\{Y_{n,i}\}$ of $\sigma$-invariant subgroups such that $\bigcap_i Y_{n,i}=F_{n+1}$ and for every $i$, $F_n/Y_{n,i}$ splits into a direct product $F_n/Y_{n,i}=\times_{-\infty}^{\infty}\sigma^j\dot{H}_n$ of simple subgroups $\sigma^j\dot{H}_n$.*

## §2. Proof of Theorem 1.

Let $G$ be a countable discrete abelian group and $\gamma$ be an automorphism of $G$ such that

$$G=\bigoplus_{-\infty}^{\infty}\gamma^n\langle g\rangle$$

where $\langle g\rangle$ is a cyclic group of order $p$ ($p$ is a prime number). The group operation of $G$ will be written by additive form. The notation $\bigoplus_{-\infty}^{\infty}G_n$ used here means the restricted direct sum for an infinite family of subgroups $G_n$. Let $I$ denote the identity map. It will be easily obtained that

$$(\gamma^j-I)G\subsetneqq G$$

for all $j>0$. Let $P$ denote the restricted direct product

$$P=\mathop{\times}_{1}^{\infty}G_i$$

where $G_i=G$ for $i\geqq 1$. Define the maps $\tilde{\gamma}$ and $\tilde{\beta}$ by

$$\tilde{\gamma}(x)=(\gamma x_1,\cdots,\gamma x_n,0,\cdots)\qquad\text{and}$$

$$\tilde{\beta}(x)=(0,(\gamma-I)x_1,\cdots,(\gamma^n-I)x_n,0,\cdots)$$

for every $x=(x_1,\cdots,x_n,0,\cdots)\in P$. Then it follows that $\tilde{\gamma}$ is an automorphism of $P$ and $\tilde{\beta}$ is a 1-1 homomorphism from $P$ into itself. Obviously $\tilde{\gamma}$ has no periodic points except the identity, and

$$Q=\{x-\tilde{\beta}(x):\ x\in P\}$$

---

*) The term "simple subgroup" will be applied only to algebraic simple subgroups.

is a $\tilde{\gamma}$-invariant subgroup. Denote

$$P_n=\{(0, \cdots, 0, x_n, 0, \cdots): x_n \in G_n\}$$

for every $n \geqq 1$, then we get easily that for every $n \geqq 1$, $Q \cap P_n=\{0\}$ and

$$P_n \oplus Q \subsetneqq P_{n+1} \oplus Q .$$

Hence,

$$\bigcup_1^\infty (P_n \oplus Q)=P ,$$

from which we get

$$\bigcup_1^\infty [(P_n \oplus Q)/Q]=P/Q .$$

Since, for all $j \geqq 1$, $\tilde{\gamma}^j-I$ is 1-1 on $P$, we have that for every $j \geqq 1$

$$P_j \oplus Q=[(\tilde{\gamma}^j-I)P_{j+1}] \oplus Q .$$

Since $((P_j \oplus Q)/Q, \tilde{\gamma})$ is isomorphic to $(G, \gamma)$ for all $j$, $\tilde{\gamma}_{P/Q}$ has no periodic points except the identity. We claim that for all $j \geqq 1$

$$(*) \qquad (\tilde{\gamma}^j-I)(P/Q)=P/Q .$$

Indeed, we have

$$\begin{aligned} P_{jn} \oplus Q &= [(\tilde{\gamma}^{jn}-I)P_{jn+1}] \oplus Q \\ &= [(\tilde{\gamma}^{j(n-1)}+\tilde{\gamma}^{j(n-2)}+ \cdots +I)(\tilde{\gamma}^j-I)P_{jn+1}] \oplus Q \\ &\subset [(\tilde{\gamma}^j-I)P_{jn+1}] \oplus Q . \end{aligned}$$

Hence,

$$(P_{jn} \oplus Q)/Q \subset (\tilde{\gamma}^j-I)[(P_{jn+1} \oplus Q)/Q] .$$

Since $(P_n \oplus Q)/Q \nearrow P/Q$, we have (*).

Now let $(Y, \tilde{\sigma})$ be the dual of $(P/Q, \tilde{\gamma})$. Since $P/Q$ is a torsion group, $Y$ is totally disconnected. Using (*), we see that $\tilde{\sigma}$ has no periodic points except the identity of $Y$. Since $\tilde{\gamma}$ has no periodic points except the identity, $(Y, \tilde{\sigma})$ is ergodic; i.e. some point of $Y$ has a dense orbit under $\tilde{\sigma}$. Therefore $(Y, \tilde{\sigma})$ satisfies all the conditions of Theorem 1. The proof is completed.

## § 3. Proof of Theorem 2.

For the proof we need the following results proved in [3]. Let $X$ be as in Theorem 2 and $\sigma$ be an automorphism of $X$.

(A) Let $A$ be an open normal subgroup of $X$ such that $\bigcap \sigma^n A=\{e\}$. If $\sigma$ is ergodic and $X/A$ is simple, then $X$ splits into a direct product $X=\times_{-\infty}^{\infty} \sigma^j W$

of simple subgroups $\sigma^j W$ (Proposition 11.5).

(B) Let $H$ be a $\sigma$-invariant normal subgroup of $X$. Then there exists in $H$ a $\sigma$-invariant normal subgroup $W$ of $X$ such that $(H/W, \sigma)$ has zero entropy and $(W, \sigma)$ is a $K$-system (Proposition 11.1).

(C) Let $H$ be as in (B) and $\sigma_H$ be the restriction of $\sigma$ to $H$. If $h(\sigma_H)=0$, then $H$ contains a decreasing sequence $\{H_n\}$ of $\sigma$-invariant normal subgroups of $X$ such that for every $n$, $H_n$ is open in $H$ and $\bigcap H_n=\{e\}$ (Proposition 11.2).

(D) Let $Z$ be the center of $X$. Assume that $Z$ is finite and $X/Z$ splits into a direct product $X/Z=\times_{-\infty}^{\infty}\sigma^j\dot{W}$ of simple subgroups $\sigma^j\dot{W}$. If $\dot{W}$ is non-abelian, then $X$ contains a $\sigma$-invariant normal subgroup $H$ such that $X=Z\times H$ (Proposition 3.7).

(E) Let $H$ be a $\sigma$-invariant finite central normal subgroup of $X$ such that $X/H$ is abelian. If the factor automorphism $\sigma_{X/H}$ of $\sigma$ in $X/H$ is densely periodic and ergodic, then $X$ is abelian (Proposition 10.7).

(F) If $X$ splits into a direct product $X=X_{i\in I}L_i$ of simple non-abelian groups $L_i$, then this splitting is unique, and an arbitrary normal subgroup of $X$ is equal to the direct product of some set of groups $L_i$ (Proposition 3.4).

(G) If $H$ is a $\sigma$-invariant finite normal subgroup of $X$ and $\sigma$ is ergodic, then $H$ is central in $X$ (Proposition 3.5).

(H) If $W$ is a $\sigma$-invariant normal subgroup of $X$, then $h(\sigma)=h(\sigma_{X/W})+h(\sigma_W)$ (Proposition 7.5).

We shall now prove Theorem 2. Since $X$ is totally disconnected, $X$ contains a sequence $X\supset A_1\supset A_2\supset\cdots$ of open normal subgroups such that $\bigcap A_n=\{e\}$. Writing

$$H'_n=\bigcap_k \sigma^k A_n \qquad (n\geqq 1),$$

there is in $H'_n$ a normal subgroup $H_n$ of $X$ such that $(H'_n/H_n, \sigma)$ has zero entropy and $(H_n, \sigma)$ is a $K$-system (by (B)). It is easy to see that

$$(**) \qquad X=H_0\supset H_1\supset\cdots\supset\bigcap H_n=\{e\}.$$

For every $n\geqq 1$, put

$$\dot{X}=X/H'_n \quad\text{and}\quad \dot{A}=A_n/H'_n .$$

Then $\dot{A}$ is open in $\dot{X}$. Let $\rho$ be the finite partition of $\dot{X}$ consisting of the cosets of $\dot{A}$, then we get that $\bigvee\sigma^k\rho$ is the partition of $\dot{X}$ into single points. Hence, $h(\sigma_{\dot{X}})=h(\sigma, \rho)<\infty$ and by (H)

$$h(\sigma_{H_{n-1}/H_n})\leqq h(\sigma_{X/H_n})=h(\sigma_{X/H'_n})+h(\sigma_{H'_n/H_n})<\infty \qquad (n\geqq 1).$$

PROPOSITION 1. *$(X, \sigma)$ $(=(H_0, \sigma))$ is a $K$-system.*

PROOF. Assume that $(X, \sigma)$ is not a $K$-system. By (B) there is a $\sigma$-invariant normal subgroup $W$ of $X$ such that $(X/W, \sigma)$ has zero entropy and $(W, \sigma)$

is a $K$-system. Using (C), we have an open normal proper subgroup $\dot{K}$ of $X/W$. Hence $(X/W)/\dot{K}$ is finite. But, since $((X/W)/\dot{K}, \sigma)$ is a factor of $(X, \sigma)$, it is ergodic and so $X/W=\dot{K}$. This is a contradiction.

For any fixed $n \geqq 0$ we write

$$F=H_n \quad \text{and} \quad H=H_{n+1}.$$

Notice that $F \supset H$, $(F, \sigma)$ and $(H, \sigma)$ are $K$-systems and

$$(***) \qquad h(\sigma_{F/H})<\infty .$$

In the following Propositions 2, 3 and 4, we shall show that $F$ contains a finite sequence

$$F=F_{-1} \supset F_0 \supset \cdots \supset F_k=H$$

of $\sigma$-invariant subgroups such that for every $i \geqq 0$, $F_i$ is normal in $F_{i-1}$ and when $F_{i-1}/F_i$ is non-abelian, $F_{i-1}/F_i$ satisfies (i) of Theorem 2 and when $F_{i-1}/F_i$ is abelian, $F_{i-1}$ satisfies (ii) of Theorem 2.

We can find an open normal proper subgroup $B$ such that $F \supset B \supset H$ and $F/B$ is simple. Put

$$B_0=\bigcap \sigma^k B ,$$

by (A) we have

$$F/B_0=\times_{-\infty}^{\infty} \sigma^j(F'/B_0)$$

where $F'/B_0$ is a simple group. By $(B)$, $B_0$ contains a $\sigma$-invariant normal subgroup $F_0$ such that

$$F \supset B_0 \supset F_0 \supset H ,$$

$(B_0/F_0, \sigma)$ has zero entropy and $(F_0, \sigma)$ is a $K$-system.

PROPOSITION 2. *If $F/F_0$ is non-abelian, then $F/F_0$ splits into a direct product $F/F_0=\times_{-\infty}^{\infty} \sigma^j \dot{F}$ of simple subgroups $\sigma^j \dot{F}$.*

PROOF. It is enough to see that $F_0=B_0$. Since $h(\sigma_{B_0/F_0})=0$, $B_0$ contains a decreasing sequence $\{H_k\}$ of $\sigma$-invariant subgroups such that $\bigcap H_k=F_0$ and for every $k$ the subgroup $H_k$ is normal in $F$ and $H_k$ is open in $B_0$ (by (C)). For every $k$, $B_0/H_k$ is a finite normal subgroup of $F/H_k$, so that it is central in $F/H_k$ (by (G)).

Assume that $(F/H_k)(B_0/H_k)$ is abelian. Since $F/B_0=\times_{-\infty}^{\infty} \sigma^j(F'/B_0)$, $\sigma_{F/B_0}$ is densely periodic and ergodic. For every $k$, $(F/H_k)/(B_0/H_k) \cong F/B_0$ and $B_0/H_k$ is central in $F/H_k$, hence $F/H_k$ is abelian by $(E)$. Since $H_k \searrow F_0$, $F/F_0$ is also abelian, which is a contradiction. Therefore $(F/H_k)/(B_0/H_k)$ is non-abelian and by (F) the group $B_0/H_k$ is the center of $F/H_k$ for all $k$. Hence by $(D)$, $F/H_k$ contains a $\sigma$-invariant subgroup $\dot{C}_k$ such that $F/H_k=\dot{C}_k \times (B_0/H_k)$. Thus $(B_0/H_k, \sigma)$ is a factor of the $K$-system $(F/H_k, \sigma)$, but $B_0/H_k$ is finite. Hence we have $B_0=H_k$ and so $B_0=F_0$.

Proposition 3. *If $F/F_0$ is abelian, then $h(\sigma_{F/F_0})=\log p$ where $p$ is a prime number, and $F$ contains a decreasing sequence $\{Y_{0,k}\}$ of $\sigma$-invariant normal subgroups such that $\bigcap_k Y_{0,k}=F_0$ and for every $k$, $F/Y_{0,k}$ splits into a direct product $F/Y_{0,k}=\times_{-\infty}^{\infty}\sigma^j\dot{F}$ of cyclic groups $\sigma^j\dot{F}$ of order $p$.*

Proof. Let $(G, \gamma)$ be the dual of $(F/F_0, \sigma)$. Obviously $G$ is a torsion group. Since $(F/F_0, \sigma)$ is a $K$-system, $\gamma$ has no periodic points except the identity of $G$. As before the group operation of $G$ is written by additive form.

Since $F/B_0=\times_{-\infty}^{\infty}\sigma^j(F'/B_0)$ where $F'/B_0$ is simple, and since $F/B_0$ is abelian, $F'/B_0$ is a cyclic group of order prime $p$. Hence $h(\sigma_{F/B_0})=\log p$ and by (H)

$$h(\sigma_{F/F_0})=h(\sigma_{F/B_0})+h(\sigma_{B_0/F_0})=\log p\,.$$

We get that $G$ is a $p$-group. For, since $G$ is a torsion group, $G$ splits into a direct sum

$$G=\bigoplus_{a\geq 1}G^{(a)}$$

of $\gamma$-invariant prime groups $G^{(a)}$ (p. 137 of [5]). Hence we have

$$F/F_0=\times_{a\geq 1}\dot{F}^{(a)}$$

where $\dot{F}^{(a)}$ $(a\geqq 1)$ is a $\sigma$-invariant subgroup with character group $G^{(a)}$. Let $G_{B_0}$ be the annihilator of $B_0/F_0$, then $G_{B_0}$ is the character group of $F/B_0$. Since $F/B_0=\times_{-\infty}^{\infty}\sigma^j(F'/B_0)$ and $F'/B_0$ is a cyclic group of order $p$, $G_{B_0}$ is annihilated by multiplication by $p$, and hence $G_{B_0}\subset G^{(a)}$ for some $a$. So we have $h(\sigma_{\dot{F}^{(a)}})\geqq\log p$. Since $h(\sigma_{F/F_0})=\log p$, it follows that $G=G^{(a)}$; i.e. $G$ is a $p$-group.

Let us put

$$G_1=\{g\in G:\ pg=0\}\,.$$

Then we have $G=G_1$. Indeed, let $\dot{W}_1$ be the annihilator of $G_1$ in $F/F_0$. Assume $\dot{W}_1\neq\{e\}$. By (A), $h(\sigma_{(F/F_0)/\dot{W}_1})\geqq\log p$ and so $h(\sigma_{\dot{W}_1})=0$ (by (H)). But, since $\gamma_{G/G_1}$ has no periodic points except the identity, $(\dot{W}_1, \sigma)$ is ergodic and so is a $K$-system (Proposition 1). This is a contradiction and hence $\dot{W}_1=\{e\}$. So we get $G=G_1$.

Therefore we can consider $G$ to be a $\boldsymbol{Z}/p\boldsymbol{Z}[x, x^{-1}]$-module. Since $\boldsymbol{Z}/p\boldsymbol{Z}[x, x^{-1}]$ is a principal ideal domain, by (p. 85, Theorem 2 in Chapter 7 of [2]) there is an increasing sequence $\{U_j\}$ of free $\boldsymbol{Z}/p\boldsymbol{Z}[x, x^{-1}]$-modules of finite type such that $\bigcup U_j=G$ and for every $j\geqq 1$, $U_j$ is of the form

$$\begin{aligned}U_j&=\boldsymbol{Z}/p\boldsymbol{Z}[\gamma, \gamma^{-1}]g_1\oplus\cdots\oplus\boldsymbol{Z}/p\boldsymbol{Z}[\gamma, \gamma^{-1}]g_{k_j}\\&=\Big(\bigoplus_{-\infty}^{\infty}\gamma^n\langle g_1\rangle\Big)\oplus\cdots\oplus\Big(\bigoplus_{-\infty}^{\infty}\gamma^n\langle g_{k_j}\rangle\Big)\end{aligned}$$

for some $g_1, \cdots, g_{k_j}\in U_j$. However, since $h(\sigma_{F/F_0})=\log p$, we have that $k_j=$

1; i.e.

$$U_j = \boldsymbol{Z}/p\boldsymbol{Z}[\gamma, \gamma^{-1}]g_1 = \bigoplus_{-\infty}^{\infty} \gamma^n \langle g_1 \rangle .$$

If $Y_{0,j}/F_0$ is the annihilator of $U_j$ in $F/F_0$, then $Y_{0,j} \searrow F_0$ and for every $j \geqq 1$, $F/Y_{0,j}$ has the required splitting.

PROPOSITION 4. *There exist a positive integer $n_0$ and a finite sequence*

$$F = F_{-1} \supset F_0 \supset \cdots \supset F_{n_0} = H$$

*of $\sigma$-invariant subgroups such that for every $i \geqq 0$, $F_i$ is normal in $F_{i-1}$ and*

(i)′ *when $F_{i-1}/F_i$ is non-abelian, $F_{i-1}/F_i$ has a direct product splitting satisfying Proposition* 2,

(ii)′ *when $F_{i-1}/F_i$ is abelian, $F_{i-1}$ contains a decreasing sequence $\{Y_{i-1,n}\}$ satisfying all the conditions of Proposition* 3.

PROOF. Since $F \supset F_0 \supset H$ and $(F_0, \sigma)$ is a $K$-system, we can apply the above argument to the pair $(F_0, H)$. Then it follows that $F_0$ contains a $\sigma$-invariant normal (in $F_0$) subgroup $F_1$ such that $F_0 \supset F_1 \supset H$ and $(F_1, \sigma)$ is a $K$-system. And we have that when $F_0/F_1$ is non-abelian, by Proposition 2, $F_0/F_1 = \times_{-\infty}^{\infty} \sigma^j \dot{F}_0$ where $\sigma^j \dot{F}_0$ $(-\infty < j < \infty)$ is a simple subgroup, and when $F_0/F_1$ is abelian, by Proposition 3 there is in $F_0$ a decreasing sequence $\{Y_{1,k}\}$ of $\sigma$-invariant subgroups such that $\bigcap_k Y_{1,k} = F_1$ and for every $k$, $F_0/Y_{1,k} = \times_{-\infty}^{\infty} \sigma^j \dot{F}_0$ where $\sigma^j \dot{F}_0$ $(-\infty < j < \infty)$ is a cyclic group of order $p$.

Since $(F_1, \sigma)$ is a $K$-system, we can repeate the same argument. Consequently we get that $F$ contains a sequence $F = F_{-1} \supset F_0 \supset \cdots \supset H$ of $\sigma$-invariant subgroups such that for every $j \geqq 0$, $F_j$ is normal in $F_{j-1}$, $(F_j, \sigma)$ is a $K$-system and $F_{j-1}/F_j$ satisfies either (i)′ or (ii)′.

It only remains to show the existence of an integer $n_0$ such that $F_{n_0} = H$. By (***), $h(\sigma_{F/H}) < \infty$. On the other hand, we have that $h(\sigma_{F_{j-1}/F_j}) \geqq \log 2$ if $F_j \subsetneqq F_{j-1}$ and by (H), $\sum_j h(\sigma_{F_{j-1}/F_j}) \leqq h(\sigma_{F/H}) < \infty$, from which our requirement is obtained.

Theorem 2 follows easily from (**) and Propositions 1, 2, 3 and 4.

ACKNOWLEDGEMENTS. The author would like to thank Professor H. Totoki and Professor I. Kubo for many helpful conversations.

## References

[1] N. Aoki, A simple proof of the Bernoullicity of ergodic automorphisms on compact abelian groups, Israel J. Math., **38** (1981), 189-198.

[2] N. Bourbaki, E'lements de Mathematique Algèbre, Hermann, Paris, 1964.

[3] S.A. Juzvinskii, Metric properties of endomorphisms of compact groups, Amer. Math. Soc. Transl., **66** (1968), 63-98.

[4] D. Lind, Ergodic group automorphisms and specification, Lecture notes in Math.,

**729** (edited by M. Denker and K. Jacobs), 93-104, 1979.
[5] A.G. Kurosch, The Theory of Gruops I and II, Chelsea, New York, 1960.
[6] L. Pontrjagin, Topological Groups, Princeton Univ. Press, Princeton, 1946.
[7] M. Rajagopalan and B.M. Screiber, Ergodic automorphisms and affine transformations, Proc. Japan Acad., **46** (1970), 633-636.
[8] V.A. Rohlin, Metric properties of endomorphisms of compact commutative groups, Amer. Math. Soc. Transl., **64** (1967), 244-272.
[9] H. Totoki, Ergodic Theory, Lecture notes series no. 14, Aarhus Univ., 1969.
[10] R. Laxton and W. Parry, On the periodic points of certain automorphisms and a system of polynomial identities, J. Algebra, **6** (1967), 388-393.

Nobuo Aoki
Department of Mathematics
Tokyo Metropolitan University
Tokyo 158
Japan