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Isometric immersions into spheres

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Recently ([1], [2]) new quantitative results concerning isometric immersions of complete Riemannian manifolds into euclidean space were obtained using a powerful theorem of Omori's [6]. Here we shall prove analogous results (theorem 1 and 2 below) concerning immersions into spheres. We begin with some auxiliary formulae for the sphere.

Let S_λ^{n+q} be the $(n+q)$-sphere of radius λ with the standard Riemannian metric induced by inclusion in R^{n+q+1}. For a point P_0 of S_λ^{n+q} say the north pole, and a nonnegative number h, let $C(P_0, h)$ be the $(n+q-1)$-hypersphere of S_λ^{n+q} with constant mean curvature h centered at P_0 and lying in the northern hemisphere. Note that $C(P_0, 0)$ is a great $(n+q-1)$-hypersphere in S_λ^{n+q}, the "equator". Let $D(P_0, h)$ be the closed geodesic ball around P_0 with $\partial D(P_0, h)=C(P_0, h)$. We take as origin of R^{n+q+1} the point P_0 and let φ be the position vector of a point in $D(P_0, h)$. If O is the center of the sphere S_λ^{n+q}, we set $e_0=\lambda^{-1}\overrightarrow{P_0O}$. If we denote by N the outer unit normal of S_λ^{n+q} in R^{n+q+1}, then by easy computations we obtain

$$d^2(P_0, C(P_0, h))=2\lambda^2[1-h\lambda(1+h^2\lambda^2)^{-1/2}], \tag{1}$$

where $d(P_0, C(P_0, h))$ is the distance in R^{n+q+1}, and

$$\lambda^{-1}\langle N, \varphi\rangle=\frac{1}{2}\lambda^{-2}\langle\varphi, \varphi\rangle\leqq\frac{1}{2}\lambda^{-2}d^2(P_0, C(P_0, h))=1-h\lambda(1+h^2\lambda^2)^{-1/2}, \tag{2}$$

where $\langle\,,\rangle$ is the standard inner product in R^{n+q+1}.
Also, for all unit vectors e which are tangent to S_λ^{n+q} at any point of $D(P_0, h)$ we have

$$|\langle e, e_0\rangle|\leqq(1+h^2\lambda^2)^{-1/2} \tag{3}$$

The proofs of the results in this paper will consist in simple applications of by Omori's theorem A in [6] which we now formulate.

THEOREM A. *Let M be a complete Riemannian manifold with sectional curvature bounded from below, consider a smooth function $f: M\to R$ with* $\sup f<\infty$; *then for any $\varepsilon>0$ there exists a point $p\in M$, which depends on ε, where* $\|\operatorname{grad} f\|<\varepsilon$ *and $\nabla^2 f(X, X)<\varepsilon$, for all unit vectors X of T_pM (by $\nabla^2 f$ we denote the*

Hessian form of f).

A useful modified form of Theorem A is the following [2]

THEOREM B. *Let M be a complete Riemannian manifold satisfying, for some constant a the condition $-\infty<-a^2\leqq Ric(X, X)$ for all unit vectors X; if the smooth function $f: M\to R$ is bounded from above, then for any $\varepsilon>0$, there exists a point on M where $\|\mathrm{grad} f\|<\varepsilon$ and $\Delta f<\varepsilon$ (By Δf we denote the Laplacian of f).*

Now, we come to the main results of this paper.

THEOREM 1. *Let M be a complete n-dimensional Riemannian manifold with scalar curvature R bounded from below; assume that there exists an isometric immersion φ of M into the euclidean sphere S_λ^{n+q} with $q\leqq n-1$, so that $\varphi(M)$ is included in $D(P_0, h)$ with $h>0$; then the sectional curvature K of M satisfies:*

$$\limsup_M K\geqq \lambda^{-2}+\frac{1}{2}h^2[1+h\lambda(1+h^2\lambda^2)^{-1/2}].$$

PROOF. If $\inf K=-\infty$, then $\inf R>-\infty$ easily implies $\sup K=\infty$ and the theorem follows. We may therefore assume $\inf K>-\infty$. We take as origin the point P_0 and we consider the function $f=\langle\varphi, \varphi\rangle/2$ on M. Identifying φ with a tangent vector to R^{n+q+1}, we compute easily

$$\nabla^2 f(X, X)=\langle X, X\rangle+\langle L(X, X), \varphi\rangle, \tag{4}$$

where L stands for the second fundamental form of M in R^{n+q+1}. The function f is bounded and thus by Theorem A for any natural number m there exists a point $P_m\in M$ so that

$$\nabla^2 f(X, X)<\frac{1}{m},$$

for all unit vectors X tangent to M at P_m. Now, we have

$$L(X, Y)=L_1(X, Y)-\frac{1}{\lambda}\langle X, Y\rangle N, \tag{5}$$

where L_1 is the second fundamental form of M in S_λ^{n+q}. Thus, for a nonzero vector $X\in T_{P_m}M$ we must have

$$1+\langle L_1(X, X), \varphi\rangle\|X\|^{-2}-\frac{1}{\lambda}\langle N, \varphi\rangle<\frac{1}{m}$$

or, using (2)

$$h\lambda(1+h^2\lambda^2)^{-1/2}-\frac{1}{m}<-\langle L_1(X, X), \varphi\rangle\|X\|^{-2}.$$

Thus for all nonzero vectors X of $T_{P_m}M$ we have

$$\|\varphi\|^{-1}\left[h\lambda(1+h^2\lambda^2)^{-1/2}-\frac{1}{m}\right]<\|L_1(X, X)\|\|X\|^{-2}. \tag{6}$$

From (6) and $h>0$ we conclude that, for a nonzero X at $P_m \in M$ and m sufficiently large, we have $L_1(X, X)\neq 0$ and therefore we can use as in [3] the following well-known algebraic lemma ([4], p. 28): let $L_1: R^n\times R^n\to R^q$ be a symmetric bilinear mapping satisfying $L_1(X, X)\neq 0$ for $X\neq 0$; if $q\leqq n-1$, there exist linearly independent X, Y so that $L_1(X, Y)=0$ and $L_1(X, X)=L_1(Y, Y)$. Applying (6) for two such vectors X, Y in $T_{P_m}M$ we get:

$$\|\varphi\|^{-2}\Big[h\lambda(1+h^2\lambda^2)^{-1/2}-\frac{1}{m}\Big]^2<\|L_1(X, X)\|\cdot\|L_1(Y, Y)\|\|X\|^{-2}\|Y\|^{-2}$$

$$\leqq(\langle L_1(X, X), L_1(Y, Y)\rangle-\|L_1(X, Y)\|^2)(\|X\|^2\|Y\|^2-\langle X, Y\rangle^2)^{-1}$$

or

$$\|\varphi\|^{-2}\Big[h\lambda(1+h^2\lambda^2)^{-1/2}-\frac{1}{m}\Big]^2<K_M(X\wedge Y)-\lambda^{-2} \tag{7}$$

since by the Gauss equation we have

$$K_M(X\wedge Y)=\lambda^{-2}+(\langle L_1(X, X), L_1(Y, Y)\rangle-\|L_1(X, Y)\|^2)(\|X\|^2\|Y\|^2-\langle X, Y\rangle^2)^{-1}$$

where $X\wedge Y$ is the plane spanned by X and Y.

Now, $\|\varphi\|^2\leqq 2\lambda^2[1-h\lambda(1+h^2\lambda^2)^{-1/2}]$ and using (7) we get

$$\frac{1}{2}\lambda^{-2}[1-h\lambda(1+h^2\lambda^2)^{-1/2}]^{-1}\Big[h\lambda(1+h^2\lambda^2)^{-1/2}-\frac{1}{m}\Big]^2<K_m(X\wedge Y)-\lambda^{-2}.$$

Then, letting m go to infinity, we deduce

$$\limsup K\geqq\lambda^{-2}+\frac{1}{2}h^2(1+h^2\lambda^2)^{-1}[1-h\lambda(1+h^2\lambda^2)^{-1/2}]^{-1}$$

or

$$\limsup K\geqq\lambda^{-2}+\frac{1}{2}h^2[1+\lambda h(1+h^2\lambda^2)^{-1/2}].$$

The following corollary is an easy consequence of Theorem 1.

COROLLARY 1. *If M is a complete n-dimensional submanifold of S_λ^{n+q} where $q\leqq n-1$, with $-\infty<-a^2\leqq$ sectional curvature $\leqq\lambda^{-2}$, then M has accumulation points in every great $(n+q-1)$-hypersphere of S_λ^{n+q}. If, in addition, M is compact, then it has points in common with every great $(n+q-1)$-hypersphere of S_λ^{n+q}.*

THEOREM 2. *Let M be a complete n-dimensional Riemannian manifold with Ricci curvature bounded from below; assume that there exists an isometric immersion φ of M into the euclidean sphere S_λ^{n+q}, so that $\varphi(M)$ is included in $D(P_0, h)$ $(h\geqq 0)$; if the mean curvature vector H_1 of the immersion φ satisfies $|H_1|\leqq l$, then $l\geqq h$.*

PROOF. We consider S_λ^{n+q} as included in R^{n+q+1}. If H, H_1 are respectively the mean curvature vectors of M in R^{n+q+1} and in S_λ^{n+q}, then by formula (5) above we get easily

$$H=H_1-\frac{1}{\lambda}N.$$

Consider again the bounded function $f=\langle\varphi, \varphi\rangle/2$ on M. Taking the trace of (4) we have

$$\Delta f=n(1+\langle H, \varphi\rangle)$$

or

$$\Delta f=n\left(1+\langle H_1, \varphi\rangle-\frac{1}{\lambda}\langle N, \varphi\rangle\right).$$

Now, using inequality (3) and the assumption, we get

$$|\langle H_1, \varphi\rangle|\leqq l\lambda(1+h^2\lambda^2)^{-1/2} \quad \text{and thus} \quad \langle H_1, \varphi\rangle\geqq -l\lambda(1+h^2\lambda^2)^{-1/2}.$$

Finally, by using the last inequality and the inequality (2) we deduce

$$\Delta f\geqq n(h-l)\lambda(1+h^2\lambda^2)^{-1/2}.$$

If, we had $h>l$, then $h-l=\varepsilon>0$ and so

$$\Delta f\geqq n\varepsilon\lambda(1+h^2\lambda^2)^{-1/2}=\text{const.}>0,$$

which contradicts Theorem B. So $l\geqq h$ and the proof is complete.

Note that if $\varphi: M\to S_\lambda^{n+q}$ with $\varphi(M)\subset D(P_0, h)$ is a minimal isometric immersion, we may take $l=0$ and thus $\Delta f\geqq nh\lambda(1+h^2\lambda^2)^{-1/2}\geqq 0$. Now, using the maximum principle, we obtain the following corollaries.

COROLLARY 2. *A compact connected minimal submanifold M of S_1^n intersects every great $(n-1)$-sphere of S_1^n. Moreover, if M is contained in a closed hemisphere of S_1^n then M must be contained in the boundary of this hemisphere.*

COROLLARY 3. *A complete connected non-compact minimal submanifold of S_1^n with Ricci curvature bounded below, has accumulation points on every great $(n-1)$-sphere of S_1^n. Moreover, if M is contained in a closed hemisphere and has at least one point on the boundary of this hemisphere, then M must be contained in this boundary.*

REMARK. Theorem 2 and Corollary 2 generalize the results in [7] concerning hypersurfaces to submanifolds. Corollaries 2 and 3 give partial answers to a question posed by Nakagawa and Shiohama [**5**; p. 415], manely whether a complete minimal submanifold of a euclidean sphere is contained in an open or closed hemisphere.

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Added in proof (May, 1981). Theorem 1 has been generalized recently by L. Jorge and D. Koutroufiotis, "An estimate for the curvature of bounded submanifolds", to appear in the Amer. J. Math. Theorem 2 has been generalized recently by L. Jorge and F. Xavier, "An inequality between the exterior diameter and the mean curvature of bounded immersions", to appear.