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Comparison theorems for functional differential equations with deviating arguments

By Takaŝi KUSANO and Manabu NAITO

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Introduction.

We consider the functional differential equations with deviating arguments

(L_n^+, F, g) $$L_n x(t)+F(t, x(g(t)))=0,$$

(L_n^-, F, g) $$L_n x(t)-F(t, x(g(t)))=0,$$

where $n \geqq 2$ and L_n denotes the disconjugate differential operator

$$L_n=\frac{1}{p_n(t)}\frac{d}{dt}\frac{1}{p_{n-1}(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{p_1(t)}\frac{d}{dt}\frac{\cdot}{p_0(t)}. \tag{1}$$

We always assume that:

(L-1) $p_i, g:[a, \infty)\to R$ are continuous, $p_i(t)>0$, $0\leqq i\leqq n$, and $g(t)\to\infty$ as $t\to\infty$;

(L-2) $F:[a, \infty)\times R\to R$ is continuous, and $\operatorname{sgn} F(t, x)=\operatorname{sgn} x$ for each $t\in[a, \infty)$.

We introduce the notation:

$$\begin{aligned} &D^0(x\,;p_0)(t)=\frac{x(t)}{p_0(t)}, \\ &D^i(x\,;p_0, \cdots, p_i)(t)=\frac{1}{p_i(t)}\frac{d}{dt}D^{i-1}(x\,;p_0, \cdots, p_{i-1})(t), \quad 1\leqq i\leqq n. \end{aligned} \tag{2}$$

The operator L_n can then be rewritten as

$$L_n=D^n(\cdot\,;p_0, \cdots, p_n).$$

The domain $\mathcal{D}(L_n)$ of L_n is defined to be the set of all functions $x:[T_x, \infty)\to R$ such that $D^i(x\,;p_0, \cdots, p_i)$, $0\leqq i\leqq n$, exist and are continuous on $[T_x, \infty)$. By a proper solution of equation $(L_n^+, F, g)[(L_n^-, F, g)]$ is meant a function $x\in\mathcal{D}(L_n)$ which satisfies $(L_n^+, F, g)[(L_n^-, F, g)]$ for all sufficiently large t and $\sup\{|x(t)|: t\geqq T\}>0$ for every $T\geqq T_x$. We make the standing hypothesis that equations

$(L_n^{\pm}, F, g)$ do possess proper solutions. A proper solution of (L_n^+, F, g) or (L_n^-, F, g) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (L_n^+, F, g) is said to be oscillatory if all of its proper solutions are oscillatory.

We say that the operator L_n is in canonical form if

$$\int_a^\infty p_i(t)dt=\infty \qquad \text{for} \quad 1 \leqq i \leqq n-1. \tag{3}$$

It is known that any differential operator of the form (1) can always be represented in canonical form in an essentially unique way (see Trench [**30**]).

Let $i_k \in \{1, \cdots, n-1\}$, $1 \leqq k \leqq n-1$, and $t, s \in [a, \infty)$. We define

$$\begin{aligned} &I_0=1, \\ &I_k(t, s; p_{i_k}, \cdots, p_{i_1})=\int_s^t p_{i_k}(r) I_{k-1}(r, s; p_{i_{k-1}}, \cdots, p_{i_1})dr. \end{aligned} \tag{4}$$

In case (3) holds, the functions

$$p_0(t)I_k(t, a; p_1, \cdots, p_k), \quad 0 \leqq k \leqq n-1,$$

form a fundamental set of solutions of the differential equation $L_n x=0$.

DEFINITION 1. Let L_n be in canonical form. Equation (L_n^+, F, g) is said to have property (A) if

(i) for n even, equation (L_n^+, F, g) is oscillatory, and

(ii) for n odd, every nonoscillatory solution $x(t)$ of (L_n^+, F, g) is strongly decreasing in the sense that

$$\left|\frac{x(t)}{p_0(t)}\right| \downarrow 0 \quad \text{as} \quad t \uparrow \infty. \tag{5}$$

Equation (L_n^-, F, g) is said to have property (B) if

(i) for n odd, every nonoscillatory solution $x(t)$ of (L_n^-, F, g) is strongly increasing in the sense that

$$\left|\frac{x(t)}{p_0(t)I_{n-1}(t, a; p_1, \cdots, p_{n-1})}\right| \uparrow \infty \quad \text{as} \quad t \uparrow \infty, \tag{6}$$

and

(ii) for n even, every nonoscillatory solution is either strongly decreasing or strongly increasing.

We are interested in comparing the oscillatory and asymptotic properties of equations (L_n^+, F, g), (L_n^-, F, g) with those of the equations

$$(M_n^+, G, h) \qquad M_n y(t)+G(t, y(h(t)))=0,$$

(M_n^-, G, h) $$M_n y(t) - G(t, y(h(t))) = 0,$$

where

$$(7) \qquad M_n = \frac{1}{q_n(t)} \frac{d}{dt} \frac{1}{q_{n-1}(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{1}{q_1(t)} \frac{d}{dt} \frac{\cdot}{q_0(t)}$$

and the following conditions are always assumed to hold:

(M-1) $q_i, h : [a, \infty) \to R$ are continuous, $q_i(t) > 0$, $0 \leqq i \leqq n$, and $h(t) \to \infty$ as $t \to \infty$;

(M-2) $G : [a, \infty) \times R \to R$ is continuous, and $\operatorname{sgn} G(t, y) = \operatorname{sgn} y$ for each $t \in [a, \infty)$.

The prototype of results we wish to establish is the following theorem which is a consequence of Sturm's classical comparison theorem.

THEOREM 0. *Let $p_i, q_i, a, b : [a, \infty) \to (0, \infty)$ be continuous, $0 \leqq i \leqq 2$. Suppose that*

$$(8) \qquad p_i(t) \geqq q_i(t), \quad 0 \leqq i \leqq 2, \quad \text{and} \quad a(t) \geqq b(t) \quad \text{for} \quad t \in [a, \infty).$$

If the equation $M_2 y + b(t) y = 0$ is oscillatory, then so is the equation $L_2 x + a(t) x = 0$.

An n-th order nonlinear analogue of this theorem has been given by Čanturija [3], who has compared the ordinary differential equations $L_n x \pm F(t, x) = 0$ with $M_n y \pm G(t, y) = 0$. The first purpose of this paper is to extend Čanturija's results [3] to the functional differential equations $(L_n^\pm, F, g)$ and $(M_n^\pm, G, h)$ by means of a variation of his comparison principle. We shall prove a theorem (Theorem 1) to the effect that if equation (M_n^+, G, h) $[(M_n^-, G, h)]$ with M_n in canonical form has property (A) [(B)], then so does equation (L_n^+, F, g) $[(L_n^-, F, g)]$ which majorizes the former in a sense similar to (8). An attempt (Theorem 3) will also be made to compare equations whose differential operators are not in canonical form.

In a recent paper [**24**] Mahfoud has presented a useful comparison principle which enables us to deduce the oscillation of a delay differential equation of the form $x^{(n)}(t) + F(t, x(g(t))) = 0$ from that of an ordinary differential equation of the form $y^{(n)} + G(t, y) = 0$. Our second purpose is to generalize Mahfoud's result to differential equations involving general canonical disconjugate operators (see Theorems 4 and 5). Several examples illustrating the main theorems will also be provided.

For other related comparison results regarding the oscillatory and asymptotic behavior of differential equations with or without functional arguments the reader is referred to the papers [**1, 2, 4-10, 13, 14, 17, 19-23, 25-29**].

1. Preliminaries.

We begin by formulating several preparatory results which are basic to the discussions developed in later sections. See also [12].

First note that the following formulas hold for the functions $I_k(t, s; p_{i_k}, \cdots, p_{i_1})$, $1 \leqq k \leqq n-1$, defined by (4):

$$I_k(t, s; p_{i_k}, \cdots, p_{i_1}) = (-1)^k I_k(s, t; p_{i_1}, \cdots, p_{i_k}), \tag{9}$$

$$I_k(t, s; p_{i_k}, \cdots, p_{i_1}) = \int_s^t p_{i_1}(r) I_{k-1}(t, r; p_{i_k}, \cdots, p_{i_2}) dr. \tag{10}$$

LEMMA 1. *If $x \in \mathcal{D}(L_n)$, then for $t, s \in [T_x, \infty)$ and $0 \leqq i \leqq k \leqq n-1$*

$$\begin{aligned} &D^i(x; p_0, \cdots, p_i)(t) \\ &= \sum_{j=i}^{k} (-1)^{j-i} D^j(x; p_0, \cdots, p_j)(s) I_{j-i}(s, t; p_j, \cdots, p_{i+1}) \\ &\quad + (-1)^{k-i+1} \int_t^s I_{k-i}(r, t; p_k, \cdots, p_{i+1}) p_{k+1}(r) D^{k+1}(x; p_0, \cdots, p_{k+1})(r) dr. \end{aligned} \tag{11}$$

This is a generalization of Taylor's formula with remainder encountered in calculus. The proof is straightforward.

LEMMA 2. *Let* (3) *hold and suppose $x \in \mathcal{D}(L_n)$ satisfies*

$$x(t) L_n x(t) < 0 \quad [x(t) L_n x(t) > 0] \quad on \quad [t_0, \infty).$$

Then there exist a $t_1 \in [t_0, \infty)$ and an integer $l \in \{0, 1, \cdots, n\}$ such that $l \not\equiv n$ (mod 2) $[l \equiv n$ (mod 2)$]$ and

$$\begin{aligned} &x(t) D^i(x; p_0, \cdots, p_i)(t) > 0 \quad on \quad [t_1, \infty),\ 1 \leqq i \leqq l, \\ &(-1)^{i-l} x(t) D^i(x; p_0, \cdots, p_i)(t) > 0 \quad on \quad [t_1, \infty),\ l+1 \leqq i \leqq n. \end{aligned} \tag{12}$$

This lemma generalizes a well-known lemma of Kiguradze [11] and can be proved similarly.

In the next three lemmas, which extend Lemmas 2, 3, 4 of Čanturija [3], we let t_0 and T be such that $T \geqq t_0$ and $g(t) \geqq t_0$ for $t \geqq T$, and assume that $u: [t_0, \infty) \to (0, \infty)$, $w: [T, \infty) \to (0, \infty)$, $H: [T, \infty) \times [0, \infty) \to [0, \infty)$ and Φ, $\Psi: \Delta \to [0, \infty)$ are continuous, where $\Delta = \{(t, s): t \geqq s \geqq T\}$, and H is nondecreasing in the second variable.

LEMMA 3. *Suppose that the functions g, u, w, H, Φ, Ψ satisfy*

$$\int_T^\infty \Psi^*(t) H(t, u(g(t))) dt < \infty,$$

(13) $$u(t)\geqq w(t)+\int_T^t \Phi(t, s)\int_s^\infty \Psi(r, s)H(r, u(g(r)))drds \quad for \quad t\geqq T,$$

where $\Psi^*(t)=\max\{\Psi(t, s): s\in[T, t]\}$. *Then the integral equation*

(14) $$v(t)=w(t)+\int_T^t \Phi(t, s)\int_s^\infty \Psi(r, s)H(r, v(g(r)))drds$$

has a solution $v\in C([t_0, \infty), (0, \infty))$ *satisfying*

(15) $$w(t)\leqq v(t)\leqq u(t) \quad for \quad t\geqq T.$$

LEMMA 4. *If in Lemma* 3 *condition* (13) *is replaced by*

$$u(t)\geqq w(t)+\int_t^\infty \Psi(s, t)H(s, u(g(s)))ds \quad for \quad t\geqq T,$$

then the integral equation

$$v(t)=w(t)+\int_t^\infty \Psi(s, t)H(s, v(g(s)))ds$$

has a solution $v\in C([t_0, \infty), (0, \infty))$ *satisfying* (15).

LEMMA 5. *Suppose that the functions* g, u, w, H, Φ *satisfy*

$$u(t)\geqq w(t)+\int_T^t \Phi(t, s)H(s, u(g(s)))ds \quad for \quad t\geqq T.$$

Then the integral equation

$$v(t)=w(t)+\int_T^t \Phi(t, s)H(s, v(g(s)))ds$$

has a solution $v\in C([t_0, \infty), (0, \infty))$ *satisfying* (15).

We give an outline of the proof of Lemma 3. Let C be the vector space of all continuous functions $x:[t_0, \infty)\to R$ with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$. Denote by X the set of functions $v\in C$ satisfying the inequality $0\leqq v(t)\leqq u(t)$ on $[t_0, \infty)$ and let $\mathcal{S}: X\to C$ be the operator defined by

$$(\mathcal{S}v)(t)=w(t)+\int_T^t \Phi(t, s)\int_s^\infty \Psi(r, s)H(r, v(g(r)))drds, \quad t\geqq T,$$

$$(\mathcal{S}v)(t)=\frac{w(T)}{u(T)}u(t), \quad t_0\leqq t\leqq T.$$

It is easy to verify that $\mathcal{S}$ maps X into itself, $\mathcal{S}$ is continuous and $\overline{\mathcal{S}X}$ is compact. Since X is convex and closed, from the Schauder-Tychonoff fixed-point theorem it follows that the operator $\mathcal{S}$ has a fixed point v in X, which provides a solution of (14) satisfying (15). Lemmas 4 and 5 are proved similarly.

2. Equations with operators in canonical form.

In this section we compare equations (L_n^+, F, g) and (L_n^-, F, g) with equations (M_n^+, G, h) and (M_n^-, G, h), respectively, under the assumption that the differential operators L_n and M_n are in canonical form. The main result (Theorem 1) asserts that if equation (M_n^+, G, h) $[(M_n^-, G, h)]$ has property (A) [(B)], then so does equation (L_n^+, F, g) $[(L_n^-, F, g)]$ which majorizes the former in a certain sense.

THEOREM 1. *Suppose that the following conditions are satisfied:*

(16) $$g(t) \geqq h(t) \quad for \quad t \in [a, \infty);$$

(17) $$p_0(g(t)) \geqq q_0(h(t)) \quad for \quad t \in [a, \infty);$$

(18) $$p_i(t) \geqq q_i(t) \quad for \quad t \in [a, \infty), \quad 1 \leqq i \leqq n-1;$$

(19) $$\int_a^\infty q_i(t)dt = \infty, \quad 1 \leqq i \leqq n-1;$$

(20) $$p_n(t)F(t, x)\operatorname{sgn} x \geqq q_n(t)G(t, x)\operatorname{sgn} x \quad for \quad (t, x) \in [a, \infty) \times R;$$

(21) $$G(t, x) \text{ is nondecreasing in } x \text{ for each } t \in [a, \infty).$$

(i) *Equation (L_n^+, F, g) has property* (A) *if equation (M_n^+, G, h) has property* (A).

(ii) *Equation (L_n^-, F, g) has property* (B) *if equation (M_n^-, G, h) has property* (B).

This theorem is equivalent to the following.

THEOREM 1′. *Suppose that conditions* (16)-(21) *are satisfied.*

(i) *If equation (L_n^+, F, g) has a nonoscillatory solution $x(t)$ satisfying*

(22) $$\liminf_{t\to\infty} |D^0(x; p_0)(t)| > 0,$$

then equation (M_n^+, G, h) has a nonoscillatory solution $y(t)$ satisfying

(23) $$\liminf_{t\to\infty} |D^0(y; q_0)(t)| > 0.$$

(ii) *If equation (L_n^-, F, g) has a nonoscillatory solution $x(t)$ satisfying* (22) *and*

(24) $$\limsup_{t\to\infty} |D^{n-1}(x; p_0, \cdots, p_{n-1})(t)| < \infty,$$

then equation (M_n^-, G, h) *has a nonoscillatory solution* $y(t)$ *satisfying* (23) *and*

$$\limsup_{t\to\infty} |D^{n-1}(y\,;q_0,\cdots,q_{n-1})(t)|<\infty\,. \tag{25}$$

As a matter of fact we are able to prove a more general comparison theorem as stated below.

THEOREM 2. *Suppose that conditions* (16)-(21) *are satisfied.*

(i) *If there exists a nonoscillatory function* $x\in\mathcal{D}(L_n)$ *satisfying* $\liminf_{t\to\infty} |D^0(x\,;p_0)(t)|>0$ *and the inequality*

$$\{L_n x(t)+F(t,\ x(g(t)))\}\operatorname{sgn} x(t)\leqq 0 \tag{26}$$

for all sufficiently large t, *then equation* (M_n^+, G, h) *has a nonoscillatory solution* $y(t)$ *satisfying* $\liminf_{t\to\infty} |D^0(y\,;q_0)(t)|>0$.

(ii) *If there exists a nonoscillatory function* $x\in\mathcal{D}(L_n)$ *satisfying*

$$\liminf_{t\to\infty} |D^0(x\,;p_0)(t)|>0,\quad \limsup_{t\to\infty} |D^{n-1}(x\,;p_0,\cdots,p_{n-1})(t)|<\infty, \tag{27}$$

and the inequality

$$\{L_n x(t)-F(t,\ x(g(t)))\}\operatorname{sgn} x(t)\geqq 0 \tag{28}$$

for all sufficiently large t, *then equation* (M_n^-, G, h) *has a nonoscillatory solution* $y(t)$ *satisfying*

$$\liminf_{t\to\infty} |D^0(y\,;q_0)(t)|>0,\quad \limsup_{t\to\infty} |D^{n-1}(y\,;q_0,\cdots,q_{n-1})(t)|<\infty. \tag{29}$$

PROOF OF THEOREM 2. (i) Let $x\in\mathcal{D}(L_n)$ be a function satisfying (26) and $\liminf_{t\to\infty} |D^0(x\,;p_0)(t)|>0$. Without loss of generality we may suppose $x(t)$ is eventually positive. According to Lemma 2 there exist a t_1 and an integer $l\in\{0, 1,\cdots, n-1\}$ such that $l\not\equiv n \pmod 2$ and inequalities (12) hold.

Let $l\in\{1,\cdots,n-1\}$. Then, applying Lemma 1 to $x(t)$ with $i=0$, $k=l-1$, $s=t_1$, $t\geqq s$, and using (9), we have

$$D^0(x\,;p_0)(t)=\sum_{j=0}^{l-1} D^j(x\,;p_0,\cdots,p_j)(t_1)I_j(t,\ t_1\,;p_1,\cdots,p_j)$$
$$+\int_{t_1}^{t} I_{l-1}(t,\ s\,;p_1,\cdots,p_{l-1})p_l(s)D^l(x\,;p_0,\cdots,p_l)(s)ds\,,$$

which, in view of (12), implies

$$D^0(x\,;p_0)(t)\geqq c+\int_{t_1}^{t} I_{l-1}(t,\ s\,;p_1,\cdots,p_{l-1})p_l(s)D^l(x\,;p_0,\cdots,p_l)(s)ds \tag{30}$$

for $t\in[t_1, \infty)$, where $c=D^0(x;p_0)(t_1)$. Again, from Lemma 1 ($i=l$, $k=n-1$, $s\geqq t\geqq t_1$) we obtain

$$\text{(31)}\qquad D^l(x;p_0,\cdots,p_l)(t) \geqq -\int_t^\infty I_{n-l-1}(r,t;p_{n-1},\cdots,p_{l+1})p_n(r)D^n(x;p_0,\cdots,p_n)(r)dr$$

for $t\in[t_1,\infty)$. Combining (30) with (31) and noting that $D^n(x;p_0,\cdots,p_n)(r)\leqq -F(r,x(g(r)))$, we get

$$\text{(32)}\qquad \begin{aligned} D^0(x;p_0)(t)\geqq c+\int_{t_1}^t I_{l-1}(t,s;p_1,\cdots,p_{l-1})p_l(s)\cdot \\ \cdot\int_s^\infty I_{n-l-1}(r,s;p_{n-1},\cdots,p_{l+1})p_n(r)F(r,p_0(g(r))D^0(x;p_0)(g(r)))drds \end{aligned}$$

for $t\in[t_1,\infty)$. On the other hand, since $l\geqq 1$, $D^0(x;p_0)(t)$ is increasing, and so $D^0(x;p_0)(g(t))\geqq D^0(x;p_0)(h(t))$ by (16). Taking this fact into account and using (17), (18), (20), (21) we obtain from (32) that

$$\begin{aligned} D^0(x;p_0)(t)\geqq c+\int_{t_1}^t I_{l-1}(t,s;q_1,\cdots,q_{l-1})q_l(s)\cdot \\ \cdot\int_s^\infty I_{n-l-1}(r,s;q_{n-1},\cdots,q_{l+1})q_n(r)G(r,q_0(h(r))D^0(x;p_0)(h(r)))drds \end{aligned}$$

for $t\in[t_1,\infty)$. Applying Lemma 3 with $u(t)=D^0(x;p_0)(t)$ we see that the integral equation

$$\begin{aligned} z(t)=c+\int_{t_1}^t I_{l-1}(t,s;q_1,\cdots,q_{l-1})q_l(s)\cdot \\ \cdot\int_s^\infty I_{n-l-1}(r,s;q_{n-1},\cdots,q_{l+1})q_n(r)G(r,q_0(h(r))z(h(r)))drds \end{aligned}$$

has a solution $z(t)$ satisfying

$$c\leqq z(t)\leqq D^0(x;p_0)(t) \quad \text{for} \quad t\in[t_1,\infty).$$

Put $y(t)=q_0(t)z(t)$. Then it is easy to verify that $y(t)$ is a solution of equation (M_n^+, G, h) such that $\liminf_{t\to\infty} D^0(y;q_0)(t)\geqq c>0$.

Next let $l=0$. Note that this is possible only when n is odd. From (12) with $l=0$ it follows that $D^0(x;p_0)(t)$ is decreasing on $[t_1,\infty)$, so that the limit $\lim_{t\to\infty} D^0(x;p_0)(t)=c_0$ exists. The hypothesis of the theorem asserts that $c_0>0$, and so there exists $t_2\geqq t_1$ such that

(33) $$c_0 \leqq D^0(x\,;\,p_0)(t) \leqq \frac{3}{2}c_0 \quad \text{for} \quad t \in [t_2, \infty).$$

From Lemma 1 ($i=0$, $k=n-1$) we have

$$D^0(x\,;\,p_0)(t) \geqq c_0$$
$$+\int_t^\infty I_{n-1}(s,\ t\,;\,p_{n-1}, \cdots, p_1)p_n(s)F(s,\ p_0(g(s))D^0(x\,;\,p_0)(g(s)))ds\,,$$

which, in view of (17), (18), (20), (21) and (33), implies

$$\frac{3}{2}c_0 \geqq c_0 + \int_t^\infty I_{n-1}(s,\ t\,;\,q_{n-1}, \cdots, q_1)q_n(s)G(s,\ c_0q_0(h(s)))ds$$

for $t \in [t_2, \infty)$. Consequently,

(34) $$c_0 \geqq \frac{c_0}{2} + \int_t^\infty I_{n-1}(s,\ t\,;\,q_{n-1}, \cdots, q_1)q_n(s)G(s,\ c_0q_0(h(s)))ds$$

for $t \in [t_2, \infty)$. Applying Lemma 4 to (34), we conclude that the integral equation

$$z(t) = \frac{c_0}{2} + \int_t^\infty I_{n-1}(s,\ t\,;\,q_{n-1}, \cdots, q_1)q_n(s)G(s,\ q_0(h(s))z(h(s)))ds$$

has a solution $z(t)$ satisfying

$$\frac{c_0}{2} \leqq z(t) \leqq c_0 \quad \text{for} \quad t \in [t_2, \infty).$$

If we put $y(t)=q_0(t)z(t)$, then $y(t)$ is clearly a nonoscillatory solution of equation (M_n^+, G, h) with the property that $\liminf_{t\to\infty} D^0(y, q_0)(t) \geqq c_0/2 > 0$. This completes the proof in the case $l=0$.

(ii) Let $x \in \mathcal{D}(L_n)$ be a function satisfying (27) and (28). We may suppose $x(t)$ is eventually positive. By Lemma 2 we can find a t_1 and an integer $l \in \{0, 1, \cdots, n\}$ such that $l \equiv n \pmod 2$ and (12) holds. If $l \in \{1, \cdots, n-2\}$ or if n is even and $l=0$, then exactly as in Case (i) it can be shown that equation (M_n^-, G, h) has a solution $y(t)$ such that $\liminf_{t\to\infty} D^0(y\,;\,q_0)(t) > 0$. Since $l \leqq n-2$, it is obvious that $\limsup_{t\to\infty} |D^{n-1}(y\,;\,q_0, \cdots, q_{n-1})(t)| < \infty$.

Suppose $l=n$. An application of Lemma 1 ($i=0$, $k=n-1$, $t \geqq s=t_1$) then shows that

$$D^0(x\,;\,p_0)(t) = \sum_{j=0}^{n-1} D^j(x\,;\,p_0, \cdots p_j)(t_1)I_j(t,\ t_1\,;\,p_1, \cdots p_j)$$
$$+\int_{t_1}^t I_{n-1}(t,\ s\,;\,p_1, \cdots, p_{n-1})p_n(s)F(s,\ p_0(g(s))D^0(x\,;\,p_0)(g(s)))ds$$

for $t\in[t_1, \infty)$. Proceeding as above, we get

$$D^0(x; p_0)(t)\geqq \sum_{j=0}^{n-1} D^j(x; p_0, \cdots, p_j)(t_1)I_j(t, t_1; q_1, \cdots, q_j)$$
$$+\int_{t_1}^{t} I_{n-1}(t, s; q_1, \cdots, q_{n-1})q_n(s)G(s, q_0(h(s))D^0(x; p_0)(h(s)))ds$$

for $t\in[t_1, \infty)$. Hence from Lemma 5 it follows that there exists a solution $z(t)$ of the integral equation

$$(35)\qquad \begin{aligned} z(t)= &\sum_{j=0}^{n-1} D^j(x; p_0, \cdots, p_j)(t_1)I_j(t, t_1; q_1, \cdots, q_j) \\ &+\int_{t_1}^{t} I_{n-1}(t, s; q_1, \cdots, q_{n-1})q_n(s)G(s, q_0(h(s))z(h(s)))ds \end{aligned}$$

satisfying

$$D^0(x; p_0)(t_1)\leqq z(t)\leqq D^0(x; p_0)(t) \quad \text{for} \quad t\in[t_1, \infty).$$

Put $y(t)=q_0(t)z(t)$. Then from (35) we see that $y(t)$ is a solution of equation (M_n^-, G, h) satisfying $\liminf_{t\to\infty} D^0(y; q_0)(t)>0$. On the other hand, since $|D^{n-1}(x; p_0, \cdots, p_{n-1})(t)|$ is bounded, integrating the inequality $L_n x(t)\geqq F(t, x(g(t)))$, we have

$$\int^{\infty} p_n(t)F(t, x(g(t)))dt<\infty.$$

This implies

$$(36)\qquad \int^{\infty} q_n(t)G(t, y(h(t)))dt<\infty,$$

since

$$y(h(t))\leqq q_0(h(t))z(g(t))\leqq q_0(h(t))\frac{x(g(t))}{p_0(g(t))}\leqq x(g(t))$$

for $t\in[t_1, \infty)$. An integration of (M_n^-, G, h) yields

$$D^{n-1}(y; q_0, \cdots, q_{n-1})(t)-D^{n-1}(y; q_0, \cdots, q_{n-1})(t_1)=\int_{t_1}^{t} q_n(s)G(s, y(h(s)))ds,$$

which, with the aid of (36), implies that $|D^{n-1}(y; q_0, \cdots, q_{n-1})(t)|$ is bounded. Thus the solution $y(t)$ obeys condition (29). The proof of Theorem 2 is complete.

In the particular case where $p_i=q_i$, $0\leqq i\leqq n$, $g=h$ and $F=G$ we have the following

COROLLARY 1. *Suppose L_n is in canonical form.*

(i) *Equation (L_n^+, F, g) has a solution $x(t)$ such that* $\liminf_{t\to\infty}|D^0(x; p_0)(t)|$

>0 *if and only if there exists a function* $y(t)$ *satisfying the inequality*

$$\{L_n y(t)+F(t,\ y(g(t)))\}\operatorname{sgn} y(t)\leqq 0$$

and $\liminf_{t\to\infty}|D^0(y\,;\,p_0)(t)|>0.$

(ii) *Equation* (L_n^-, F, g) *has a solution* $x(t)$ *such that*

$$\liminf_{t\to\infty}|D^0(x\,;\,p_0)(t)|>0 \quad and \quad \limsup_{t\to\infty}|D^{n-1}(x\,;\,p_0, \cdots, p_{n-1})(t)|<\infty$$

if and only if there exists a function $y(t)$ *satisfying the inequality*

$$\{L_n y(t)-F(t,\ y(g(t)))\}\operatorname{sgn} y(t)\geqq 0\,,$$

$$\liminf_{t\to\infty}|D^0(y\,;\,p_0)(t)|>0 \quad and \quad \limsup_{t\to\infty}|D^{n-1}(y\,;\,p_0, \cdots, p_{n-1})(t)|<\infty.$$

EXAMPLE 1. Consider the even order equations

$$(t^{\alpha+m}x^{(m)}(t))^{(m)}+t^{\beta-m}F(x(g(t)))=0\,, \quad t\geqq 1, \tag{37}$$

$$(t^{\gamma+m}y^{(m)}(t))^{(m)}+t^{\gamma-m}G(y(t))=0\,, \quad t\geqq 1, \tag{38}$$

where α, β, γ are constants such that $\gamma\leqq -m+1$, $\alpha\leqq\gamma\leqq\beta$, and $F, G: R\to R$ are continuous functions such that $F(x)\operatorname{sgn} x\geqq G(x)\operatorname{sgn} x$, $\operatorname{sgn} G(x)=\operatorname{sgn} x$, $G(x)$ is nondecreasing, and

$$\lim_{|x|\to\infty}\frac{|G(x)|}{|x|}=\infty\,.$$

According to a result of Kreith, Kusano and Naito [**14**] equation (38) is oscillatory, and so from Theorem 1 it follows that equation (37) is oscillatory if $g(t)\geqq t$.

3. Equations with non-canonical operators.

We now turn to equations whose differential operators are not in canonical form. According to the general theory of Trench [**30**], every non-canonical L_n of the form (1) can be represented in canonical form in an essentially unique way. However, actual computation leading to canonical form is in general not easy, so it is desirable to obtain comparison principles for general equations without knowing the canonical representation of the operators involved.

The following example shows that we can not expect much for results in this direction.

EXAMPLE 2. Consider the equations

$$(t^3x'(t))'+t^3x^3(t)=0\,, \quad t\geqq 1, \tag{39}$$

(40) $$(t^3y'(t))'+t^3y^3(t^{1/3})=0, \quad t\geqq 1.$$

Putting $z(t)=t^2y(t)$, equation (40) is transformed into

(41) $$(t^{-1}z'(t))'+t^{-1}z^3(t^{1/3})=0, \quad t\geqq 1.$$

It is not hard to see that equation (41) is oscillatory (see, for example, [**15**]). Hence the delay equation (40) is oscillatory. However, the ordinary equation (39) is not oscillatory, since it has a nonoscillatory solution $x(t)=t^{-1}$. Thus, for equations with non-canonical differential operators, Theorem 1 is false even if $L_n=M_n$ and $F=G$.

To modify the definitions of properties (A) and (B) we need the concept of a principal system for the operator L_n. By a principal system for L_n we mean a set of n solutions $X_1(t), \cdots, X_n(t)$ of the equation $L_nx=0$ which are eventually positive and satisfy the relation

(42) $$\lim_{t\to\infty}\frac{X_i(t)}{X_j(t)}=0 \quad \text{for} \quad 1\leqq i<j\leqq n.$$

For example, if L_n is in canonical form, then the set of functions

$$\{p_0(t),\ p_0(t)I_1(t),\ \cdots,\ p_0(t)I_{n-1}(t)\},$$

where $I_i(t)=I_i\ (t,\ a\ ;\ p_1,\ \cdots,\ p_i)$, $1\leqq i\leqq n-1$, is a principal system for L_n. A principal system for non-canonical L_n can easily be obtained by direct integration of the equation $L_nx=0$. A basic property of principal systems is that if both $\{X_1(t), \cdots, X_n(t)\}$ and $\{\tilde{X}_1(t), \cdots, \tilde{X}_n(t)\}$ are principal systems for L_n, then for each i, $1\leqq i\leqq n$, $X_i(t)$ and $\tilde{X}_i(t)$ have the same order of growth (or decay) as $t\to\infty$, that is, the limits

(43) $$\lim_{t\to\infty}\frac{\tilde{X}_i(t)}{X_i(t)}>0, \quad 1\leqq i\leqq n,$$

exist and are finite.

DEFINITION 2. Let $\{X_1(t), \cdots, X_n(t)\}$ be a principal system for L_n. Equation (L_n^+, F, g) is said to have property (A) if

(i) for n even, it is oscillatory, and

(ii) for n odd, every nonoscillatory solution $x(t)$ of (L_n^+, F, g) satisfies

(44) $$\lim_{t\to\infty}\frac{x(t)}{X_1(t)}=0.$$

Equation (L_n^-, F, g) is said to have property (B) if

(i) for n odd, every nonoscillatory solution $x(t)$ of (L_n^-, F, g) satisfies

(45)
$$\lim_{t\to\infty}\frac{|x(t)|}{X_n(t)}=\infty,$$

and

(ii) for n even, every nonoscillatory solution satisfies either (44) or (45).

The main result of this section is the following

THEOREM 3. *Suppose that $F(t, x)\operatorname{sgn} x\geqq G(t, x)\operatorname{sgn} x$ and $G(t, x)$ is non-decreasing in x.*

(i) *If equation (L_n^+, G, g) has property* (A), *then so does equation (L_n^+, F, g).*

(ii) *If equation (L_n^-, G, g) has property* (B), *then so does equation (L_n^-, F, g).*

This theorem is restated as follows.

THEOREM 3′. *Suppose that $F(t, x)\operatorname{sgn} x\geqq G(t, x)\operatorname{sgn} x$ and $G(t, x)$ is non-decreasing in x. Let $\{X_1(t), \cdots, X_n(t)\}$ be a principal system for L_n.*

(i) *If equation (L_n^+, F, g) has a nonoscillatory solution $x(t)$ satisfying*

(46)
$$\liminf_{t\to\infty}\frac{|x(t)|}{X_1(t)}>0,$$

then equation (L_n^+, G, g) has a nonoscillatory solution $y(t)$ satisfying

(47)
$$\liminf_{t\to\infty}\frac{|y(t)|}{X_1(t)}>0.$$

(ii) *If equation (L_n^-, F, g) has a nonoscillatory solution $x(t)$ satisfying*

(48)
$$\liminf_{t\to\infty}\frac{|x(t)|}{X_1(t)}>0 \quad \textit{and} \quad \limsup_{t\to\infty}\frac{|x(t)|}{X_n(t)}<\infty,$$

then equation (L_n^-, G, g) has a nonoscillatory solution $y(t)$ satisfying

(49)
$$\liminf_{t\to\infty}\frac{|y(t)|}{X_1(t)}>0 \quad \textit{and} \quad \limsup_{t\to\infty}\frac{|y(t)|}{X_n(t)}<\infty.$$

We shall prove statement (ii) of Theorem 3′. Suppose L_n is not in canonical form. Let

$$L_n=\frac{1}{\tilde{p}_n(t)}\frac{d}{dt}\frac{1}{\tilde{p}_{n-1}(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{\tilde{p}_1(t)}\frac{d}{dt}\frac{\cdot}{\tilde{p}_0(t)}$$

be the canonical representation of L_n, so that

$$\int_a^\infty \tilde{p}_i(t)dt=\infty \quad \text{for} \quad 1\leqq i\leqq n-1.$$

Applying Theorem 1′ to equations (L_n^-, F, g) and (L_n^-, G, g) with L_n thus transformed, we see that if equation (L_n^-, F, g) has a nonoscillatory solution $x(t)$ satisfying

(50) $\liminf_{t\to\infty}|D^0(x;\tilde{p}_0)(t)|>0$ and $\limsup_{t\to\infty}|D^{n-1}(x;\tilde{p}_0,\cdots,\tilde{p}_{n-1})(t)|<\infty$,

then equation (L_n^-, G, g) has a nonoscillatory solution $y(t)$ satisfying (50) with x replaced by y. We note that

$$\{\tilde{p}_0(t), \tilde{p}_0(t)\tilde{I}_1(t), \cdots, \tilde{p}_0(t)\tilde{I}_{n-1}(t)\},$$

where $\tilde{I}_i(t)=I_i(t, a;\tilde{p}_1,\cdots,\tilde{p}_i)$, $1\leqq i\leqq n-1$, forms a principal system for L_n, and that (50) is equivalent to

$$\liminf_{t\to\infty}\frac{|x(t)|}{\tilde{p}_0(t)}>0 \quad\text{and}\quad \limsup_{t\to\infty}\frac{|x(t)|}{\tilde{p}_0(t)\tilde{I}_{n-1}(t)}<\infty. \tag{51}$$

In view of (43) $\tilde{p}_0(t)$ and $X_1(t)$ has the same order of growth (or decay) as $t\to\infty$, and the same is true of $\tilde{p}_0(t)\tilde{I}_{n-1}(t)$ and $X_n(t)$. Therefore (50) [resp. (50) with x replaced by y] is equivalent to (48) [resp. (49)].

Statement (i) of Theorem 3′ can be proved similarly.

EXAMPLE 3. Consider the fourth order elliptic equation

$$\Delta^2 u+c(|\xi|)u=0 \tag{52}$$

in an exterior domain E of Euclidean N-space R^N of points $\xi=(\xi_1,\cdots,\xi_N)$, where $\Delta=\sum_{i=1}^N\partial^2/\partial\xi_i^2$, $|\xi|$ denotes the Euclidean length of ξ, and $c(|\xi|)$ is continuous and positive in E. Equation (52) is called oscillatory if every nontrivial solution $u\in C^4(E)$ of (52) has arbitrarily large zeros in E, that is, the set $\{\xi\in E: u(\xi)=0\}$ is unbounded.

Recently Kusano and Yoshida [18] have shown that equation (52) is oscillatory in E if and only if the ordinary differential equation

$$\frac{d}{dt}t^{N-1}\frac{d}{dt}t^{1-N}\frac{d}{dt}t^{N-1}\frac{d}{dt}w+t^{N-1}c(t)w=0, \quad t\geqq 1,$$

is oscillatory. From this fact and Theorem 3 we see that if equation (52) is oscillatory in E, then so is the equation

$$\Delta^2 v+C(|\xi|)v=0 \quad\text{with}\quad C(t)\geqq c(t).$$

4. More on comparison theory.

In Section 2 we have established comparison theorems to the effect that if a differential equation with deviating argument $h(t)$ has property (A) or (B), then so does another related equation with larger deviating argument $g(t)$.

We are interested in comparison results in the opposite direction, that is, we wish to derive property (A) or (B) of an equation with deviating argument

$h(t)$ from the corresponding property of another equation with larger deviating argument $g(t)$. Efforts in this direction have been undertaken by several authors; see, for example, Erbe [**5, 6**] and Mahfoud [**24**] in which delay equations are compared with ordinary equations (without delay). The main purpose of this section is to extend Mahfoud's theory [**24**] to much more general situations.

THEOREM 4. *Let L_n be in canonical form. Suppose that $F(t, x)$ is nondecreasing in x and that $g(t)$ and $h(t)$ are subject to the conditions*

$$g,\ h\in C^1,\quad g'(t)>0,\quad h'(t)>0,\quad h(t)\leqq g(t),\quad \lim_{t\to\infty} h(t)=\infty\,. \tag{53}$$

Suppose that the differential equation

$$\langle L_n^+,\ F,\ g,\ h\rangle \qquad L_n z(t)+\frac{g'(t)p_n(h^{-1}(g(t)))}{h'(h^{-1}(g(t)))p_n(t)}F(h^{-1}(g(t)),\ z(g(t)))=0$$

has property (A). *Then equation $(L_n^+,\ F,\ h)$ has property* (A).

PROOF. Let $x(t)$ be a nonoscillatory solution of equation $(L_n^+,\ F,\ h)$ such that $\liminf_{t\to\infty}|D^0(x\,;\,p_0)(t)|>0$. We may suppose that $x(t)$ is eventually positive. Let t_1 and $l\in\{0,\ 1,\ \cdots,\ n-1\}$ be the numbers associated with $x(t)$ (see Lemma 2).

If $l\in\{1,\ \cdots,\ n-1\}$, then proceeding as in the proof of the first part of Theorem 2, we obtain the inequality

$$\begin{aligned} &D^0(x\,;\,p_0)(t)\geqq c+\int_{t_1}^{t} I_{l-1}(t,\ s\,;\,p_1,\ \cdots,\ p_{l-1})p_l(s)\cdot \\ &\cdot\int_s^\infty I_{n-l-1}(r,\ s\,;\,p_{n-1},\ \cdots,\ p_{l+1})p_n(r)F(r,\ p_0(h(r))D^0(x\,;\,p_0)(h(r)))drds \end{aligned} \tag{54}$$

for $t\in[t_1,\ \infty)$, where $c>0$ is a constant. By the change of variables $r=h^{-1}(g(\rho))$ we find

$$\begin{aligned} &\int_s^\infty I_{n-l-1}(r,\ s\,;\,p_{n-1},\ \cdots,\ p_{l+1})p_n(r)F(r,\ p_0(h(r))D^0(x\,;\,p_0)(h(r)))dr \\ &=\int_{g^{-1}(h(s))}^\infty I_{n-l-1}(h^{-1}(g(\rho)),\ s\,;\,p_{n-1},\ \cdots,\ p_{l+1})\cdot \\ &\qquad\cdot\frac{g'(\rho)p_n(h^{-1}(g(\rho)))}{h'(h^{-1}(g(\rho)))}F(h^{-1}(g(\rho)),\ p_0(g(\rho))D^0(x\,;\,p_0)(g(\rho)))d\rho \\ &\geqq\int_s^\infty I_{n-l-1}(\rho,\ s\,;\,p_{n-1},\ \cdots,\ p_{l+1})\cdot \\ &\qquad\cdot\frac{g'(\rho)p_n(h^{-1}(g(\rho)))}{h'(h^{-1}(g(\rho)))}F(h^{-1}(g(\rho)),\ p_0(g(\rho))D^0(x\,;\,p_0)(g(\rho)))d\rho\,, \end{aligned}$$

where we have used the fact that $g^{-1}(h(s))\leqq s\leqq h^{-1}(g(s))$ which follows from

(53). Combining the above inequality with (54), we have for $t\in[t_1, \infty)$

$$D^0(x; p_0)(t)\geqq c+\int_{t_1}^{t} I_{l-1}(t, s; p_1, \cdots, p_{l-1})p_l(s)\int_{s}^{\infty} I_{n-l-1}(r, s; p_{n-1}, \cdots, p_{l+1})\cdot \tag{55}$$
$$\cdot\frac{g'(r)p_n(h^{-1}(g(r)))}{h'(h^{-1}(g(r)))}F(h^{-1}(g(r)), p_0(g(r))D^0(x; p_0)(g(r)))drds.$$

By lemma 3 applied to (55) there exists a positive solution $y(t)$ of the integral equation

$$y(t)=c+\int_{t_1}^{t} I_{l-1}(t, s; p_1, \cdots, p_{l-1})p_l(s)\int_{s}^{\infty} I_{n-l-1}(r, s; p_{n-1}, \cdots, p_{l+1})\cdot$$
$$\cdot\frac{g'(r)p_n(h^{-1}(g(r)))}{h'(h^{-1}(g(r)))}F(h^{-1}(g(r)), p_0(g(r))y(g(r)))drds.$$

Then the function $z(t)=p_0(t)y(t)$ is a solution of equation $\langle L_n^+, F, g, h\rangle$ satisfying $\liminf_{t\to\infty} D^0(z; p_0)(t)>0$. This contradicts the hypothesis.

If $l=0$ (which is possible only when n is odd), then arguing as in the proof of the second part of Theorem 2, we see that there exist constants $c_0>0$ and $t_2>t_1$ such that

$$c_0\geqq\frac{c_0}{2}+\int_{t}^{\infty} I_{n-1}(s, t; p_{n-1}, \cdots, p_1)p_n(s)F(s, c_0p_0(h(s)))ds \tag{56}$$

for $t\in[t_2, \infty)$. Noting that

$$\int_{t}^{\infty} I_{n-1}(s, t; p_{n-1}, \cdots, p_1)p_n(s)F(s, c_0p_0(h(s)))ds$$
$$=\int_{g^{-1}(h(t))}^{\infty} I_{n-1}(h^{-1}(g(\sigma)), t; p_{n-1}, \cdots, p_1)\cdot$$
$$\cdot\frac{g'(\sigma)p_n(h^{-1}(g(\sigma)))}{h'(h^{-1}(g(\sigma)))}F(h^{-1}(g(\sigma)), c_0p_0(g(\sigma)))d\sigma$$
$$\geqq\int_{t}^{\infty} I_{n-1}(\sigma, t; p_{n-1}, \cdots, p_1)\frac{g'(\sigma)p_n(h^{-1}(g(\sigma)))}{h'(h^{-1}(g(\sigma)))}F(h^{-1}(g(\sigma)), c_0p_0(g(\sigma)))d\sigma,$$

from (56) we have

$$c_0\geqq\frac{c_0}{2}+\int_{t}^{\infty} I_{n-1}(\sigma, t; p_{n-1}, \cdots, p_1)\frac{g'(\sigma)p_n(h^{-1}(g(\sigma)))}{h'(h^{-1}(g(\sigma)))}\cdot \tag{57}$$
$$\cdot F(h^{-1}(g(\sigma)), c_0p_0(g(\sigma)))d\sigma$$

for $t\in[t_2, \infty)$. Lemma 4 applied to (57) guarantees the existence of a positive solution $y(t)$ of the integral equation

$$y(t)=\frac{c_0}{2}+\int_t^{\infty} I_{n-1}(\sigma, t ; p_{n-1}, \cdots, p_1)\frac{g'(\sigma)p_n(h^{-1}(g(\sigma)))}{h'(h^{-1}(g(\sigma)))}\cdot$$

$$\cdot F(h^{-1}(g(\sigma)), p_0(g(\sigma))y(g(\sigma)))d\sigma .$$

The function $z(t)=p_0(t)y(t)$ then gives a solution of equation $\langle L_n^+, F, g, h\rangle$ satisfying $\liminf_{t\to\infty} D^0(z ; p_0)(t)>0$, which again contradicts the hypothesis. This completes the proof.

The following example shows that Theorem 4 fails to hold if L_n is not in canonical form.

EXAMPLE 4. The equation

$$(t^3z'(t))'+3t^3z^3(t^{1/3})=0, \quad t\geqq 1,$$

is oscillatory (see Example 2). However, the equation

$$(t^3x'(t))'+t^{1/3}x^3(t^{1/9})=0, \quad t\geqq 1,$$

has a nonoscillatory solution $x(t)$ such that $\lim_{t\to\infty} x(t)=\text{const}\neq 0$. This follows from Theorem 1 of Kusano and Naito [**16**].

Of particular interest is the case where $g(t)=t$, that is, the comparison equation $\langle L_n^+, F, g, h\rangle$ is an ordinary differential equation

$$\langle L_n^+, F, h\rangle \qquad L_nz+\frac{p_n(h^{-1}(t))}{h'(h^{-1}(t))p_n(t)}F(h^{-1}(t), z)=0 .$$

A specialization of Theorem 4 to this case yields the following corollary which is a generalization of a result of Mahfoud [**24**, Theorem 1].

COROLLARY 2. *Let L_n be in canonical form. Suppose that $F(t, x)$ is non-decreasing in x and that $h(t)$ satisfies*

$$h\in C^1, \quad h'(t)>0, \quad h(t)\leqq t, \quad \lim_{t\to\infty} h(t)=\infty . \tag{58}$$

If the ordinary differential equation $\langle L_n^+, F, h\rangle$ has property (A), *then so does the delay equation (L_n^+, F, h).*

EXAMPLE 5. Consider the linear delay equation

$$D^n(x ; 1, p, \cdots, p, 1)(t)+q(t)x(h(t))=0, \tag{59}$$

where $n\geqq 2$, $p, q : [a, \infty)\to(0, \infty)$ are continuous, and $h : [a, \infty)\to R$ satisfies condition (58). We wish to compare (59) with the ordinary differential equation

$$D^n(z ; 1, p, \cdots, p, 1)(t)+\frac{q(h^{-1}(t))}{h'(h^{-1}(t))}z(t)=0 . \tag{60}$$

Suppose that $\int_a^\infty p(t)dt=\infty$ and put $P(t)=\int_a^t p(s)ds$. According to the linear oscillation theory developed in [**17**] and [**29**] equation (60) has property (A) if either (i)

$$\int^\infty [P(s)]^{n-2}\frac{q(h^{-1}(s))}{h'(h^{-1}(s))}ds=\infty, \tag{61}$$

or (ii) (61) fails and

$$\liminf_{t\to\infty} P(t)\int_t^\infty [P(s)-P(t)]^{n-2}\frac{q(h^{-1}(s))}{h'(h^{-1}(s))}ds>\frac{(n-2)!}{4}. \tag{62}$$

If we let $\tau=h^{-1}(s)$, then (61) and (62) reduce respectively to

$$\int^\infty [P(h(s))]^{n-2}q(s)ds=\infty \tag{63}$$

and

$$\liminf_{t\to\infty} P(h(t))\int_t^\infty [P(h(s))-P(h(t))]^{n-2}q(s)ds>\frac{(n-2)!}{4}. \tag{64}$$

Applying now Corollary 2 to equations (59) and (60), we conclude that the delay equation (59) has property (A) if either (63) or (64) holds.

Next, we compare equation (L_n^-, F, h) with

$$\langle L_n^-, F, g, h\rangle \qquad L_n z(t)-\frac{g'(t)p_n(h^{-1}(g(t)))}{h'(h^{-1}(g(t)))p_n(t)}F(h^{-1}(g(t)), z(g(t)))=0.$$

THEOREM 5. *Let L_n be in canonical form. Suppose that $F(t, x)$ is nondecreasing in x and that $g(t)$ and $h(t)$ satisfy* (53). *If equation $\langle L_n^-, F, g, h\rangle$ has property* (B), *then equation (L_n^-, F, h) has property* (B).

PROOF. Suppose that equation (L_n^-, F, h) does not possess property (B). Let $x(t)$ be a nonoscillatory solution of (L_n^-, F, h). Then, we have

$$\liminf_{t\to\infty}|D^0(x;p_0)(t)|>0 \quad \text{and} \quad \limsup_{t\to\infty}|D^{n-1}(x;p_0,\cdots,p_{n-1})(t)|<\infty. \tag{65}$$

If $l<n$, where l is the integer associated with $x(t)$ by Lemma 2, then it can be shown as in the proof of Theorem 4 that equation $\langle L_n^-, F, g, h\rangle$ has a nonoscillatory solution $z(t)$ satisfying

$$\liminf_{t\to\infty}|D^0(z;p_0)(t)|>0 \quad \text{and} \quad \limsup_{t\to\infty}|D^{n-1}(z;p_0,\cdots,p_{n-1})(t)|<\infty,$$

which contradicts the assumption that $\langle L_n^-, F, g, h\rangle$ has property (B).

If $l=n$, then since $D^{n-1}(x;p_0,\cdots,p_{n-1})(t)$ is monotone,

(66) $$\lim_{t\to\infty}|D^{n-1}(x\,;\,p_0,\cdots,p_{n-1})(t)|=\text{const}\neq 0\,.$$

According to an analogue of Theorem 1 of Kitamura and Kusano [12], (66) implies that

(67) $$\int^{\infty}p_n(t)|F(t,\,cp_0(h(t))I_{n-1}(h(t)))|\,dt<\infty \quad \text{for some } c\neq 0,$$

where $I_{n-1}(t)=I_{n-1}(t,\,a\,;\,p_1,\cdots,p_{n-1})$. If we put $t=h^{-1}(g(s))$, then (67) is transformed into

(68) $$\int^{\infty}p_n(h^{-1}(g(s)))|F(h^{-1}(g(s)),\,cp_0(g(s))I_{n-1}(g(s)))|\frac{g'(s)}{h'(h^{-1}(g(s)))}ds<\infty\,.$$

Again an analogue of Theorem 1 of [12] shows that (68) is sufficient for equation $\langle L_n^-, F, g, h\rangle$ to have a nonoscillatory solution $z(t)$ satisfying

$$\lim_{t\to\infty}|D^{n-1}(z\,;\,p_0,\cdots,p_{n-1})(t)|=\text{const}\neq 0\,.$$

This is a contradiction, and the proof is complete.

COROLLARY 3. *Let L_n, F and h be as in Corollary 2. If the ordinary differential equation*

$\langle L_n^-, F, h\rangle$ $$L_n z-\frac{p_n(h^{-1}(t))}{h'(h^{-1}(t))p_n(t)}F(h^{-1}(t),\,z)=0$$

has property (B), *then so does the delay equation* (L_n^-, F, h).

THEOREM 6. *Let L_n be in canonical form. Suppose that $F(t, x)$ is nondecreasing in x and that $g(t)$ and $h(t)$ satisfy condition* (53). *Put $\tau(t)=h(g^{-1}(t))$ and define*

(69) $$\mathcal{L}_n=\frac{1}{p_n(t)}\frac{d}{dt}\frac{1}{p_{n-1}(\tau(t))\tau'(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{p_1(\tau(t))\tau'(t)}\frac{d}{dt}\frac{\cdot}{p_0(\tau(t))}\,.$$

If the equation

(70) $$\mathcal{L}_n y(t)+F(t,\,y(g(t)))=0$$

has property (A), *then so does equation* (L_n^+, F, h).

PROOF. Let $x(t)$ be a solution of equation (L_n^+, F, h) satisfying

$$\liminf_{t\to\infty}|D^0(x\,;\,p_0)(t)|>0\,.$$

We may suppose that $x(t)$ is eventually positive. Let l be the integer associated with $x(t)$ by Lemma 2. Integrating (L_n^+, F, h) and noting that $\tau(t)\leqq t$, we have

(71) $$D^{n-1}(x\,;\,p_0,\cdots,p_{n-1})(\tau(t))\geqq\int_t^\infty p_n(s)F(s,x(h(s)))ds$$

for $t\geqq T$, provided T is sufficiently large. Multiplying both sides of (71) by $p_{n-1}(\tau(t))\tau'(t)$ and integrating the resulting inequality, we get

$$D^{n-2}(x\,;\,p_0,\cdots,p_{n-2})(\tau(t))\geqq\int_t^\infty p_{n-1}(\tau(s_{n-1}))\tau'(s_{n-1})\cdot\int_{s_{n-1}}^\infty p_n(s)F(s,x(h(s)))ds\,ds_{n-1},\quad t\geqq T.$$

Repeating this procedure, we arrive at

(72) $$D^l(x\,;\,p_0,\cdots,p_l)(\tau(t))\geqq c_l+\int_t^\infty p_{l+1}(\tau(s_{l+1}))\tau'(s_{l+1})\cdot\int_{s_{l+1}}^\infty\cdots\int_{s_{n-1}}^\infty p_n(s)F(s,x(h(s)))ds\,ds_{n-1}\cdots ds_{l+1},\quad t\geqq T,$$

where $c_l=\lim_{t\to\infty}D^l(x\,;\,p_0,\cdots,p_l)(t)\geqq 0$. Note that $c_0>0$ by hypothesis.

Suppose $l\geqq 1$. We multiply (72) by $p_l(\tau(t))\tau'(t)$ and integrate over $[T, t]$, obtaining

$$D^{l-1}(x\,;\,p_0,\cdots,p_{l-1})(\tau(t))\geqq\int_T^t p_l(\tau(s_l))\tau'(s_l)\int_{s_l}^\infty p_{l+1}(\tau(s_{l+1}))\tau'(s_{l+1})\cdot\int_{s_{l+1}}^\infty\cdots\int_{s_{n-1}}^\infty p_n(s)F(s,x(h(s)))ds\,ds_{n-1}\cdots ds_l$$

for $t\geqq T$. Continuing in this manner, we have for $t\geqq T$

(73) $$D^0(x,p_0)(\tau(t))\geqq c+\int_T^t p_1(\tau(s_1))\tau'(s_1)\int_T^{s_1}\cdots\int_T^{s_{l-1}}p_l(\tau(s_l))\tau'(s_l)\cdot\int_{s_l}^\infty p_{l+1}(\tau(s_{l+1}))\tau'(s_{l+1})\int_{s_{l+1}}^\infty\cdots\int_{s_{n-1}}^\infty p_n(s)F(s,x(h(s)))ds\,ds_{n-1}\cdots ds_1,$$

where $c=D^0(x\,;\,p_0)(\tau(T))>0$. Denote the right hand side of (73) by $z(t)$ and define $\zeta(t)=p_0(\tau(t))z(t)$. Repeated differentiation of $\zeta(t)$ shows that

(74) $$\mathcal{L}_n\zeta(t)+F(t,x(h(t)))=0,\quad t\geqq T.$$

Since $x(\tau(t))\geqq p_0(\tau(t))z(t)$ by (73) and since $\tau(g(t))=h(t)$, we see that

$$x(h(t))=x(\tau(g(t)))\geqq p_0(\tau(g(t)))z(g(t))=\zeta(g(t)).$$

Combining this fact with (74), we have

(75) $$\mathcal{L}_n\zeta(t)+F(t,\zeta(g(t)))\leqq 0,\quad t\geqq T,$$

and $\liminf_{t\to\infty} D^0(\zeta; p_0\circ\tau)(t)>0$. It follows from Corollary 1 that equation (70) has a positive solution $y(t)$ such that $\liminf_{t\to\infty} D^0(y; p_0\circ\tau)(t)>0$. This contradicts the hypothesis that (70) has property (A).

Next suppose $l=0$. From (72) we find

$$D^0(x; p_0)(\tau(t))\geqq c_0+\int_t^\infty p_1(\tau(s_1))\tau'(s_1)\int_{s_1}^\infty p_2(\tau(s_2))\tau'(s_2)\cdot \cdot\int_{s_2}^\infty\cdots\int_{s_{n-1}}^\infty p_n(s)F(s, x(h(s)))ds ds_{n-1}\cdots ds_1 \tag{76}$$

for $t\geqq T$. Denote the right hand side of (76) by $z(t)$ and put $\zeta(t)=p_0(\tau(t))z(t)$. Then, exactly as above, $\zeta(t)$ satisfies (75) and $\liminf_{t\to\infty} D^0(\zeta; p_0\circ\tau)(t)>0$. This implies the existence of a solution $y(t)$ of equation (70) with the property $\liminf_{t\to\infty} D^0(y; p_0\circ\tau)(t)>0$, again contradicting the hypothesis. Thus the proof is complete.

EXAMPLE 6. We show that Theorem 6 is not true for equations with non-canonical operators. Let $L_2=\frac{d}{dt}t^3\frac{d}{dt}$, $F(t, x)=t^{1/3}x^3$, $g(t)=t^{1/3}$ and $h(t)=t^{1/9}$ for $t\geqq 1$. Then, equations (L_n^+, F, h) and (70) are

$$(t^3x'(t))'+t^{1/3}x^3(t^{1/9})=0 \tag{77}$$

and

$$(t^{5/3}y'(t))'+\frac{1}{3}t^{1/3}y^3(t^{1/3})=0, \tag{78}$$

respectively. Equation (78) is oscillatory, since, by the change of variables $z(t)=t^{2/3}y(t)$, it is reduced to

$$(t^{1/3}z'(t))'+\frac{1}{3}t^{-1}z^3(t^{1/3})=0,$$

which is oscillatory. However, equation (77) has nonoscillatory solutions (see Example 4).

THEOREM 7. *Under the same assumptions as in Theorem* 6 *equation* (L_n^-, F, h) *has property* (B) *if the equation*

$$\mathcal{L}_n y(t)-F(t, y(g(t)))=0 \tag{79}$$

has property (B).

PROOF. Suppose that equation (L_n^-, F, h) has a nonoscillatory solution $x(t)$ satisfying (65). Let l be the integer associated with $x(t)$ by Lemma 2. If $l<n$, then the same argument as in the proof of Theorem 6 leads us to a contradic-

tion. If $l=n$, then there exists a constant $c\neq 0$ such that (67) holds. On the other hand, from a variant of Theorem 1 of [12] we see that equation (79) has a nonoscillatory solution $y(t)$ such that

$$\lim_{t\to\infty}|D^{n-1}(y\,;\,p_0\circ\tau,\,(p_1\circ\tau)\tau',\,\cdots,\,(p_{n-1}\circ\tau)\tau')(t)|=\text{const}\neq 0 \tag{80}$$

if and only if

$$\int^{\infty}p_n(t)|F(t,\,cp_0(\tau(g(t)))I_{n-1}(g(t),\,a\,;(p_1\circ\tau)\tau',\,\cdots,\,(p_{n-1}\circ\tau)\tau')|\,dt<\infty \tag{81}$$

for some $c\neq 0$. Since $\tau(g(t))=h(t)$ and

$$I_{n-1}(t,\,a\,;(p_1\circ\tau)\tau',\,\cdots,\,(p_{n-1}\circ\tau)\tau')=I_{n-1}(\tau(t),\,\tau(a)\,;\,p_1,\,\cdots,\,p_{n-1}),$$

(81) coincides with (67). Therefore, if $l=n$, then equation (79) has a nonoscillatory solution $y(t)$ satisfying (80). This again is a contradiction.

REMARK 1. It is easy to see that in case $g(h(t))=h(g(t))$ Theorem 6 and Theorem 7 are equivalent to Theorem 4 and Theorem 5, respectively.

In view of recent results of Brands [1] and Foster and Grimmer [7] we have the following conjecture.

CONJECTURE. *Let L_n be in canonical form and let $F(t, x)$ be nondecreasing in x. Let $g_1, g_2:[a, \infty)\to R$ be continuous functions such that $\lim_{t\to\infty} g_i(t)=\infty$, $i=1, 2$, and $|g_1(t)-g_2(t)|$ is bounded.*

(i) *Equation (L_n^+, F, g_1) has property* (A) *if and only if equation (L_n^+, F, g_2) has property* (A).

(ii) *Equation (L_n^-, F, g_1) has property* (B) *if and only if equation (L_n^-, F, g_2) has property* (B).

Below we give a partial answer to this conjecture.

LEMMA 6. *Let L_n be in canonical form. Suppose that the functions $p_i(t)$, $0\leqq i\leqq n-1$, are nonincreasing for $t\in[a, \infty)$. Let $g:[a, \infty)\to R$ be a C^1 function satisfying $g'(t)>0$ and $\lim_{t\to\infty} g(t)=\infty$. Then, for any constant $M\geqq 0$ equation (L_n^+, F, g) $[(L_n^-, F, g)]$ has property* (A) [(B)] *if and only if equation $(L_n^+, F, g-M)$ $[(L_n^-, F, g-M)]$ has property* (A) [(B)].

PROOF. Since $g(t)\geqq g(t)-M$, the "if" part of the lemma follows from Theorem 1. So, suppose that equation (L_n^+, F, g) $[(L_n^-, F, g)]$ has property (A) [(B)]. Put $h(t)=g(t)-M$ and $\tau(t)=h(g^{-1}(t))$. Then, clearly, $\tau(t)=t-M$, $\tau'(t)=1$, and $p_i(\tau(t))\geqq p_i(t)$ for $0\leqq i\leqq n-1$. Therefore, by Theorem 1, equation $(\mathcal{L}_n^+, F, g)$ $[(\mathcal{L}_n^-, F, g)]$, where $\mathcal{L}_n$ is defined by (69), has property (A) [(B)]. Applying now Theorem 6 [Theorem 7], we conclude that equation $(L_n^+, F, h)=(L_n^+, F, g-M)$ $[(L_n^-, F, h)=(L_n^-, F, g-M)]$ has property (A) [(B)], proving the "only if" part of the lemma.

THEOREM 8. *Let L_n and g be as in Lemma 6. Let $g_1, g_2:[a, \infty)\rightarrow R$ be continuous functions such that $\lim_{t\rightarrow\infty} g_i(t)=\infty$, $i=1, 2$, and $|g_1(t)-g(t)|$ and $|g_2(t)-g(t)|$ are bounded. Then, equation (L_n^+, F, g_1) $[(L_n^-, F, g_1)]$ has property* (A) [(B)] *if and only if equation (L_n^+, F, g_2) $[(L_n^-, F, g_2)]$ has property* (A) [(B)].

PROOF. There exists a constant $M>0$ such that $|g_1(t)-g(t)|\leqq M$, that is,

$$g(t)-M\leqq g_1(t)\leqq g(t)+M \quad \text{for} \quad t\in[a, \infty).$$

Theorem 1 implies that if equation (L_n^+, F, g_1) $[(L_n^-, F, g_1)]$ has property (A) [(B)], then so does equation $(L_n^+, F, g+M)$ $[(L_n^-, F, g+M)]$. Hence equation (L_n^+, F, g) $[(L_n^-, F, g)]$ has property (A) [(B)] by Lemma 6. Conversely, if equation (L_n^+, F, g) $[(L_n^-, F, g)]$ has property (A) [(B)], then, by Lemma 6 equation $(L_n^+, F, g-M)$ $[(L_n^-, F, g-M)]$ has the same property. From Theorem 1 it follows that equation (L_n^+, F, g_1) $[(L_n^-, F, g_1)]$ has property (A) [(B)]. Likewise, equation (L_n^+, F, g_2) $[(L_n^-, F, g_2)]$ has property (A) [(B)] if and only if (L_n^+, F, g) $[(L_n^-, F, g)]$ has property (A) [(B)]. This completes the proof.

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Takaŝi KUSANO
Department of Mathematics
Faculty of Science
Hiroshima University
Hiroshima 730
Japan

Manabu NAITO
Department of Mathematics
Faculty of Science
Hiroshima University
Hiroshima 730
Japan