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Nonselfadjoint crossed products II

By Michael MCASEY, Paul S. MUHLY*)
and Kichi-Suke SAITO

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1. Introduction.

This paper is a continuation of [7]. We are interested in the invariant subspace structure and ideal structure of certain subalgebras of von Neumann algebras constructed as crossed products of finite von Neumann algebras by trace preserving automorphisms. These subalgebras are called nonselfadjoint crossed products and most properly should be regarded as operator theoretic versions of twisted polynomial rings. We seek conditions under which an analogue of the theorem of Beurling (as generalized by Lax and Halmos) is valid. The theorem of Beurling, Lax and Halmos (hereafter abbreviated the BLH theorem) is usually regarded as describing the invariant subspaces of a unilateral shift (of arbitrary multiplicity). However, from a ring theoretic point of view it may be thought of as describing certain modules over the algebra $H^{\infty}(\Delta)$ of bounded analytic functions on the unit disc (regarded as a subalgebra of L^{∞} of the circle), and in particular the BLH theorem implies that every weak*-closed ideal in $H^{\infty}(\Delta)$ is principal. Thus, from an operator theoretic point of view $H^{\infty}(\Delta)$ is a principal ideal domain, a generalization of the polynomial algebra in one variable. Since von Neumann algebra crossed products may be viewed as noncommutative generalizations of L^{∞} of the circle and since our nonselfadjoint crossed products are generalizations of $H^{\infty}(\Delta)$, our search for analogues of the BLH theorem is tantamount to looking for conditions under which our algebras are noncommutative principal ideal rings. We shall find necessary and sufficient conditions for the validity of the BLH theorem for a nonselfadjoint crossed product and we shall prove that within the context of subdiagonal algebras defined and first studied by Arveson in [1], the validity of the BLH theorem essentially characterizes nonselfadjoint crossed products. More precisely, we shall show that with a minor qualification if a subdiagonal algebra has the property that every ultraweakly closed two-sided ideal is principal, then the algebra is a nonselfadjoint crossed product and every ultraweakly closed left ideal is principal.

The setting here is the following. Let M be a von Neumann algebra with

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a faithful, normal, finite and normalized trace ϕ and let α be a *-automorphism of M such that $\phi\circ\alpha=\phi$. We regard M as acting on the noncommutative Lebesgue space $L^2(M, \phi)$ (cf. [14]), i.e. we identify it, when convenient, with the von Neumann algebra of left multiplications on $L^2(M, \phi)$. Then α uniquely extends to a unitary operator u on $L^2(M, \phi)$ such that $\alpha(x)=uxu^*$, $x\in M$. From the Hilbert space $L^2=l^2(\boldsymbol{Z})\otimes L^2(M, \phi)$ and consider the operators L_x, $x\in M$, and L_δ defined on L^2 by the formulae $L_x=I\otimes x$ and $L_\delta=S\otimes u$ where S is the usual bilateral shift on $l^2(\boldsymbol{Z})$. Then the von Neumann algebra crossed product determined by M and α is defined to be the von Neumann algebra $\mathfrak{L}$ on L^2 generated by $\{L_x : x\in M\}$ $(=L(M))$ and L_δ, while the subalgebra which we call a nonselfadjoint crossed product is the ultraweakly closed subalgebra $\mathfrak{L}_+$ generated by $L(M)$ and the positive powers of L_δ. Let H^2 be the subspace $l^2(\boldsymbol{Z}_+)\otimes L^2(M, \phi)$ of L^2 and let $\mathfrak{Z}(\mathfrak{L})$ be the center of $\mathfrak{L}$. We shall denote by $\mathrm{Lat}(\mathfrak{L}_+)$ the lattice of subspaces invariant under $\mathfrak{L}_+$ such that $\bigcap_{n\geq 0} L_\delta^n\mathfrak{M}=\{0\}$. If every subspace $\mathfrak{M}$ in $\mathrm{Lat}(\mathfrak{L}_+)$ is of the form $\mathfrak{M}=R_vH^2$, where R_v is a partial isometry in the commutant $\mathfrak{R}$ of $\mathfrak{L}$, we shall say that the BLH theorem is valid.

In [7], we showed that the following three conditions are equivalent: i) M is a factor; ii) a conditioned form of the BLH theorem is valid; and iii) $\mathfrak{L}_+$ is maximal among the ultraweakly closed subalgebras of $\mathfrak{L}$. Moreover, we showed that if M is a factor, then the BLH theorem is valid without qualification. However, as is easily seen, the converse is not necessarily true.

Our objective in this paper is to study necessary and sufficient conditions under which the BLH theorem is valid. First of all, we find that α fixes the center $\mathfrak{Z}(M)$ of M elementwise if and only if the BLH theorem is valid. If M is not a factor, then $\mathfrak{L}_+$ is not a maximal ultraweakly closed subalgebra of $\mathfrak{L}$ [7, Theorem 4.1]. However, when α leaves the center $\mathfrak{Z}(M)$ of M elementwise invariant, then we can describe the ultraweakly closed subalgebras of $\mathfrak{L}$ containing $\mathfrak{L}_+$. It turns out that every ultraweakly closed subalgebra $\mathfrak{B}$ of $\mathfrak{L}$ containing $\mathfrak{L}_+$ is of the form $\mathfrak{B}=(1-L_p)\mathfrak{L}\oplus L_p\mathfrak{L}_+$ where L_p is a projection in $\mathfrak{Z}(\mathfrak{L})\cap L(M)$. And the converse is also true. Thus the validity of the BLH theorem is tied to the form of the ultraweakly closed subalgebras of $\mathfrak{L}$ containing $\mathfrak{L}_+$.

In § 2, we define the nonselfadjoint crossed products. In § 3, we show the equivalence of the assertions that α fixes the center $\mathfrak{Z}(M)$ of M elementwise, the BLH theorem is valid, every ultraweakly closed subalgebra of $\mathfrak{L}$ containing $\mathfrak{L}_+$ is of the form $(1-L_p)\mathfrak{L}\oplus L_p\mathfrak{L}_+$ where L_p is a projection in $\mathfrak{Z}(\mathfrak{L})\cap L(M)$, etc. In § 4, we consider a generalization of the results of § 6 in [7]. That is, we prove that if a version of the BLH theorem is valid in a subdiagonal algebra, then the algebra must be a nonselfadjoint crossed product determined by a finite von Neumann algebra M, and a *-automorphism of M which fixes the center $\mathfrak{Z}(M)$ of M elementwise. Finally, in § 5 we rephrase our results in terms of ideals.

2. Nonselfadjoint crossed products.

Let M be a von Neumann algebra with a trace ϕ. In this paper, all traces without exception will be assumed to be faithful, normal, finite and normalized. We assume M to be in standard form and identify it with the von Neumann algebra of left multiplications on the noncommutative Lebesgue space $L^2(M, \phi)$ associated with M and ϕ (see [14]). Also, we fix once and for all a *-automorphism α of M which preserves ϕ; i.e., $\phi\circ\alpha=\phi$. Then we have the following proposition.

PROPOSITION 2.1. *Let $L_0^2=\{f: \boldsymbol{Z}\to M \mid f(n)=0$ for all but finitely many $n\}$. Then, with respect to pointwise addition and scalar multiplication and the operations defined by equations (1)-(3), L_0^2 is a Hilbert algebra with identity ψ defined by $\psi(0)=I_M$, and $\psi(n)=0$, $n\neq 0$.*

(1) $(f*g)(n)=\sum_{k\in\boldsymbol{Z}} f(k)\alpha^k(g(n-k))$,

(2) $(f^*)(n)=[\alpha^n(f(-n))]^*$,

(3) $(f, g)=\sum_{k\in\boldsymbol{Z}}(f(k), g(k))_{L^2(M,\phi)}$.

Observe, too, that the Hilbert space completion L^2 of L_0^2 is precisely $\{f: \boldsymbol{Z}\to L^2(M, \phi) \mid \sum_{n\in\boldsymbol{Z}} \|f(n)\|^2_{L^2(M,\phi)}<\infty\}$ and may be identified with $l^2(\boldsymbol{Z})\otimes L^2(M, \phi)$. For f in L_0^2, we define operators L_f and R_f on L^2 by the formulae $L_f g=f*g$ and $R_f g=g*f$, $g\in L^2$. As is customary, we set $\mathfrak{L}=\{L_f: f\in L_0^2\}''$ and $\mathfrak{R}=\{R_f: f\in L_0^2\}''$. Also we define L^∞ to be the achieved Hilbert algebra of all bounded elements in L^2. That is, L^∞ consists of those f in L^2 such that the map $g\to f*g$, $g\in L_0^2$, extends to a bounded operator on all of L^2. For such an f, we write L_f and R_f for the operators it determines. From Hilbert algebra theory (cf. [2, Chapter 1, §5]), we have $\mathfrak{L}=\{L_f: f\in L^\infty\}$ and $\mathfrak{R}=\{R_f: f\in L^\infty\}$. Since α preserves ϕ on M, α uniquely extends to a unitary operator u on $L^2(M, \phi)$. Consequently, the canonical antiunitary involution J on L^2, extending the *-operation on L_0^2, is given by the formula (2) in Proposition 2.1.

The original algebra M is identified with the subalgebra $\{x\psi: X\in M\}$ of L^∞, and we abbreviate $L_{x\psi}$ and $R_{x\psi}$ by L_x and R_x respectively. We put $L(M)=\{L_x: x\in M\}$ and $R(M)=\{R_x: x\in M\}$. More generally, if S is a subset of L^∞, we will write $L(S)$ (resp. $R(S)$) for $\{L_\sigma: \sigma\in S\}$ (resp. $\{R_\sigma: \sigma\in S\}$). The function δ defined by the formula $\delta(1)=I_M$ and $\delta(n)=0$, $n\neq 1$, plays a very important role. It is clear that $\mathfrak{L}=\{L(M), L_\delta\}''$ and $\mathfrak{R}=\{R(M), R_\delta\}''$.

Next we define $H^2=\{f\in L^2: f(n)=0, n<0\}$, we define H^∞ to be $L^\infty\cap H^2$, and we refer to it as the nonselfadjoint crossed product determined by M and α. Also $\mathfrak{L}_+=\{L_f: f\in H^\infty\}$ and $\mathfrak{R}_+=\{R_f: f\in H^\infty\}$. Then $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) is the ultraweakly closed subalgebra of $\mathfrak{L}$ (resp. $\mathfrak{R}$) generated by L_δ and $L(M)$ (resp.

R_δ and $R(M)$) (cf. [7, Theorem 2.2]).

Definition 2.2. Let $\mathfrak{M}$ be a closed subspace of L^2. We shall say that $\mathfrak{M}$ is: left invariant, if $\mathfrak{L}_+\mathfrak{M}\subseteq\mathfrak{M}$; left reducing, if $\mathfrak{L}\mathfrak{M}\subseteq\mathfrak{M}$; left-pure, if $\mathfrak{M}$ contains no nontrivial left reducing subspace; and left-full, if the smallest left reducing subspace containing $\mathfrak{M}$ is all of L^2. The right-hand versions of these concepts are defined similarly, and a closed subspace which is both left and right invariant will be called two-sided invariant.

In this paper all results will be formulated in terms of left invariant subspaces. We leave it to the reader to rephrase them to obtain "right-hand" statements.

3. Validity of the BLH theorem.

We shall say that the BLH theorem is valid if every left-pure, left invariant subspace $\mathfrak{M}$ of L^2 has the form $\mathfrak{M}=R_vH^2$ for some partial isometry v in L^∞. Our main objective in this section is to find necessary and sufficient conditions that the BLH theorem is valid. If M is a factor, then the BLH theorem is valid. However, the converse is not necessarily true.

Let $\mathfrak{Z}(M)$ be the center of M and $\mathfrak{Z}(\mathfrak{L})$ (resp. $\mathfrak{Z}(\mathfrak{R})$) the center of $\mathfrak{L}$ (resp. $\mathfrak{R}$). Since the commutant $\mathfrak{L}'$ of $\mathfrak{L}$ is $\mathfrak{R}$ and the commutant $\mathfrak{R}'$ of $\mathfrak{R}$ is $\mathfrak{L}$, $\mathfrak{Z}(\mathfrak{L})=\mathfrak{Z}(\mathfrak{R})$; put $C=\{z\in\mathfrak{Z}(M):\alpha(z)=z\}$. Then we have the following lemma. The proof is straightforward and so will be omitted.

Lemma 3.1. (1) *For every* $z\in C$, $L_z=R_z$.

(2) $\mathfrak{Z}(\mathfrak{L})\cap L(M)=\mathfrak{Z}(\mathfrak{R})\cap R(M)=L(C)$.

Our first result is

Theorem 3.2. *The following three statements are equivalent:*

(1) *α fixes the center $\mathfrak{Z}(M)$ of M elementwise;*

(2) *Every left-pure, left invariant subspace of L^2 has the form R_vH^2 for some partial isometry v in L^∞; and*

(3) *Every left invariant subspace of H^2 has the form R_vH^2 for some partial isometry v in L^∞.*

Proof. (1)⇒(2). Suppose that α fixes the center $\mathfrak{Z}(M)$ of M elementwise. Let $\mathfrak{M}$ be a left-pure, left invariant subspace in L^2, let P be the projection of L^2 onto $\mathfrak{M}\ominus L_\delta\mathfrak{M}$ and let P_0 be the projection of L^2 onto $H^2\ominus L_\delta H^2$. By [7, Theorem 3.2], P and P_0 lie in the commutant $L(M)'$ of $L(M)$. By the Comparability theorem (cf. [2, p. 218, Théoremè 1]), there exists a projection z in $\mathfrak{Z}(M)$ such that $L_zP\prec L_zP_0$ and $(1-L_z)P\succ(1-L_z)P_0$. From Lemma 3.1 and the hypothesis that (1) is satisfied, we have $L_z\in\mathfrak{Z}(\mathfrak{L})\cap L(M)$. Since $L_z\mathfrak{M}$ and L_zH^2 are left-pure, left invariant subspaces of L^2, by [7, Theorem 3.2], there is a partial isometry R_{v_1} in $\mathfrak{R}$ such that $L_z\mathfrak{M}=R_{v_1}L_zH^2$. If necessary, we may suppose that $R_{v_1}R_{v_1}^*$, $R_{v_1}^*R_{v_1}\leqq L_z$. Similarly, there is a partial isometry R_{v_2} in $\mathfrak{R}$ such that $(1-L_z)H^2=R_{v_2}(1-L_z)\mathfrak{M}$ and $R_{v_2}R_{v_2}^*$, $R_{v_2}^*R_{v_2}\leqq 1-L_z$. But then, since R_{v_2} and L_δ commute and H^2 is left-full, we find that

$$R_{v_2}(1-L_z)L^2\geqq R_{v_2}(\bigvee_{n\in Z}L_\delta^n(1-L_z)\mathfrak{M})=\bigvee_{n\in Z}L_\delta^n R_{v_2}(1-L_z)\mathfrak{M}$$

$$=\bigvee_{n\in Z}L_\delta^n(1-L_z)H^2=(1-L_z)L^2;$$

that is, $R_{v_2}R_{v_2}^*=1-L_z$. Since, however, $\mathfrak{R}$ is finite, $R_{v_2}^*R_{v_2}=1-L_z$ and we may consequently write $(1-L_z)\mathfrak{M}=R_{v_2}^*(1-L_z)H^2$. Therefore, putting $R_v=R_{v_1}L_z+R_{v_2}^*(1-L_z)$, R_v is a partial isometry in $\mathfrak{R}$ and $\mathfrak{M}=R_vH^2$. This completes the proof that (1) implies (2).

(2)⇒(3). Since H^2 contains no nonzero left reducing subspace, it is clear that (2) implies (3).

(3)⇒(1). Suppose that α does not fix the center $\mathfrak{Z}(M)$ of M elementwise. Then there is a nonzero projection e in $\mathfrak{Z}(M)$ such that $\alpha(e)e=0$. Put $\mathfrak{M}=\{f\in H^2: ef(0)=f(0)\}$. As in the proof that (2) implies (1) of [7, Theorem 4.1], it is clear that $\mathfrak{M}$ is a left-pure, left-full and left invariant subspace of H^2 and $L_eL_\delta^*\mathfrak{M}\subseteqq\mathfrak{M}$. Now we suppose that $\mathfrak{M}=R_vH^2$ where R_v is a partial isometry in $\mathfrak{R}$. Since $\mathfrak{M}$ is left-full, R_v is a unitary operator (cf. [7, Theorem 3.3]). Thus we have

$$L_eL_\delta^*H^2=L_eL_\delta^*R_v^*\mathfrak{M}=R_v^*L_eL_\delta^*\mathfrak{M}\subseteqq R_v^*\mathfrak{M}=H^2.$$

Consequently $L_eL_\delta^*\in\mathfrak{L}_+$. This is a contradiction and so completes the proof that (3) implies (1).

Next, in connection with validity of the BLH theorem, we shall study the form of ultraweakly closed subalgebras of $\mathfrak{L}$ containing $\mathfrak{L}_+$ and the form of left-pure, two-sided invariant subspaces of L^2. In this section the closure of a subset S of L^2 in the L^2-norm will be denoted by $[S]_2$.

THEOREM 3.3. *The following statements are equivalent to* (2) (*and hence to* (1) *and* (3)) *in Theorem* 3.2:

(4) *If* $\mathfrak{B}$ *is an ultraweakly closed subalgebra of* $\mathfrak{L}$ *which contains* $\mathfrak{L}_+$, *then there is a projection* e *in* C *such that* $\mathfrak{B}=(1-L_e)\mathfrak{L}\oplus L_e\mathfrak{L}_+$; *and*

(5) *If* $\mathfrak{M}$ *is a two-sided invariant subspace of* H^2, *then* $\mathfrak{M}$ *may be expressed as* R_vH^2 *where* v *is a partial isometry in* L^∞ *such that* $v^*v=vv^*\in C$.

PROOF. (3)⇒(5). Let $\mathfrak{M}$ be a two-sided invariant subspace of H^2. By Theorem 3.2, $\mathfrak{M}=R_vH^2$ where v is a partial isometry in L^∞. If $e=v^*v$, then we have $R_e\in\mathfrak{Z}(\mathfrak{R})\cap R(M)=L(\mathfrak{Z}(M))=L(C)$ as in the proof of [7, Proposition 4.5]. Since $R_e=L_e$, by Lemma 3.1, $\mathfrak{M}$ may be expressed as R_vH^2 where v is a partial isometry in L^∞ such that $v^*v=vv^*\in C$ and so this completes the proof that (3) implies (5).

(5)$\Rightarrow$(4). If $\mathfrak{B}$ is a proper ultraweakly closed subalgebra of $\mathfrak{L}$ containing $\mathfrak{L}_+$, then $[\mathfrak{B}\phi]_2$ is a two-sided invariant subspace of L^2. Then $K=L^2\ominus[\mathfrak{B}\phi]_2\neq\{0\}$ by [7, Corollary 1.5] and JK is a two-sided invariant subspace in H^2. By hypothesis, $JK=R_vH^2$ for a partial isometry v in L^∞ such that $v^*v=vv^*\in C$. Put $e=v^*v$ and note that $L_e=R_e$ by Lemma 3.1. Then we have

$$(1-L_e)K=(1-L_e)JJK=(1-L_e)JR_vH^2=L_{1-e}L_{v*}JH^2=\{0\}$$

and so

$$(1-L_e)L^2=(1-L_e)([\mathfrak{B}\phi]_2\oplus K)=(1-L_e)[\mathfrak{B}\phi]_2\,.$$

By [13, Theorem 1], $(1-L_e)\mathfrak{L}=(1-L_e)\mathfrak{B}$. On the other hand,

$$\begin{aligned}L_e[\mathfrak{B}\phi]_2&=L_eL^2\ominus K=L_eL^2\ominus JJK=L_{v*}L^2\ominus L_{v*}JH^2\\&=L_{v*}(L^2\ominus JH^2)=L_{v*}H_0^2=L_{v*}L_\delta H^2.\end{aligned}$$

Since $L_{v*}L_\delta(L_{v*}L_\delta)^*=(L_{v*}L_\delta)^*L_{v*}L_\delta=L_e$, we have

$$\begin{aligned}L_e[\mathfrak{B}\phi]_2&=(L_{v*}L_\delta)^*L_{v*}L_\delta[R(\mathfrak{B}\phi)H^2]_2=(L_{v*}L_\delta)^*[L_{v*}L_\delta R(\mathfrak{B}\phi)H^2]_2\\&=(L_{v*}L_\delta)^*[R(\mathfrak{B}\phi)L_{v*}L_\delta H^2]_2=(L_{v*}L_\delta)^*[R(\mathfrak{B}\phi)L_e[\mathfrak{B}\phi]_2]_2\\&=(L_{v*}L_\delta)^*[\mathfrak{B}\phi]_2=L_eH^2.\end{aligned}$$

Therefore $L_e\mathfrak{B}=L_e\mathfrak{L}_+$ and so $\mathfrak{B}=(1-L_e)\mathfrak{L}\oplus L_e\mathfrak{L}_+$. This completes the proof that (5) implies (4).

(4)$\Rightarrow$(1). Suppose that α does not fix the center $\mathfrak{Z}(M)$ of M elementwise. As in the proof that (3) implies (1) in Theorem 3.2, we construct the left-pure, left-full and left invariant space $\mathfrak{M}$ of H^2. Let $\mathfrak{B}$ be the ultraweakly closed subalgebra of $\mathfrak{L}$ generated by $L_eL_\delta^*$ and $\mathfrak{L}_+$. Then it is clear that $\mathfrak{L}_+\subsetneqq\mathfrak{B}\subsetneqq\mathfrak{L}$. By hypothesis, there is a projection p in C such that $\mathfrak{B}=(1-L_p)\mathfrak{L}\oplus L_p\mathfrak{L}_+$. Since $L_p\mathfrak{B}=L_p\mathfrak{L}_+$, we have $L_pL_eL_\delta^*\in L_p\mathfrak{L}_+\subset\mathfrak{L}_+$ and so $L_pL_eL_\delta^*=0$. Thus $(1-L_p)\mathfrak{L}$ is the ultraweakly closed subalgebra generated by $L_eL_\delta^*$ and $(1-L_p)\mathfrak{L}_+$. Since $(1-L_p)L_\delta^2\in(1-L_p)\mathfrak{B}$ and $((1-L_p)L_eL_\delta^*)^2=0$, this is a contradiction and completes the proof that (4) implies (1).

There is a useful variation of condition (5) in Theorem 3.3; it looks mildly stronger, but in fact it is equivalent. We present it in

Proposition 3.4. *The following statement is equivalent to each of the statements* (1) *through* (5) *appearing in Theorems* 3.2 *and* 3.3:

(6) *if $\mathfrak{M}$ is a left-pure two-sided invariant subspace of L^2, then $\mathfrak{M}$ may be expressed as R_vH^2 where v is a partial isometry in L^∞ such that $v^*v=vv^*\in C$.*

Proof. Since H^2 contains no nonzero left reducing subspace, it is clear that (6) implies (5). To prove the reverse implication, it suffices to prove that (1) implies (6) by Theorem 3.3. Let $\mathfrak{M}$ be a left-pure, two-sided invariant subspace of L^2. By Theorem 3.2, $\mathfrak{M}=R_vH^2$ for some partial isometry v in L^∞. If $e=v^*v$, then $R_e=R_vR_{v^*}$, and

$$R_eL^2=R_vL^2=R_v(\bigvee_{n\leqq 0}L_\delta^nH^2)=\bigvee_{n\leqq 0}L_\delta^nR_vH^2=\bigvee_{n\leqq 0}L_\delta^n\mathfrak{M}$$

is invariant under $\mathfrak{R}_+$. By [7, Corollary 4.3], e lies in the center of L^∞. Since L^∞ is finite, $vv^*=e$. Putting $\mathfrak{B}=\{x\in\mathfrak{L}: x\mathfrak{M}\subseteqq\mathfrak{M}\}$, it is clear that $\mathfrak{B}$ is a proper ultraweakly closed subalgebra of $\mathfrak{L}$ containing $\mathfrak{L}_+$. By Theorem 3.3, there is a projection p in $C=\mathfrak{Z}(M)$ such that $\mathfrak{B}=(1-L_p)\mathfrak{L}\oplus L_p\mathfrak{L}_+$. Since $(1-L_p)\mathfrak{M}$ is left reducing and $(1-L_p)\mathfrak{M}\subseteqq\mathfrak{M}$, $(1-L_p)\mathfrak{M}=\{0\}$ and so we have

$$R_{1-p}R_eL^2=L_{1-p}R_eL^2=\bigvee_{n\leqq 0}L_\delta^nL_{1-p}\mathfrak{M}=\{0\}.$$

Thus $(1-p)e=0$. Since $R_e\in\mathfrak{Z}(\mathfrak{L})$, $R_e=L_e$ and so $L_e\in\mathfrak{B}$, because $L_e\mathfrak{M}\subseteqq\mathfrak{M}$. Therefore $L_e=L_eL_p\in L_p\mathfrak{L}_+\subset\mathfrak{L}_+$. This implies that $L_e\in\mathfrak{Z}(\mathfrak{L})\cap L(M)=L(\mathfrak{Z}(M))=L(C)$ and completes the proof.

In Theorem 3.2, 3.3 and Proposition 3.4, we need not use reduction theory to the factor case. Therefore we have the following corollary.

COROLLARY 3.5 ([7]). *The following statements are equivalent:*

(1) *M is a factor:*

(2) *$C=\{CI\}$ and each left-pure, left invariant subspace of L^2 has the form R_vH^2 for some partial isometry v in L^∞;*

(3) *$C=\{CI\}$ and each left invariant subspace of H^2 has the form R_vH^2 for some partial isometry v in L^∞;*

(4) *$\mathfrak{L}_+$ is a maximal ultraweakly closed subalgebra of $\mathfrak{L}$;*

(5) *if $\mathfrak{M}$ is a two-sided invariant subspace of H^2, then $\mathfrak{M}$ may be expressed as R_vH^2 where v is a unitary operator in L^∞; and*

(6) *if $\mathfrak{M}$ is a two-sided invariant subspace of L^2 which is not left reducing, then there exists a unitary operator v in L^∞ such that $\mathfrak{M}=R_vH^2$.*

PROOF. It is evident that each of the conditions (1), (4), (5) and (6) implies that $C=\{CI\}$. Indeed, if $C\neq\{CI\}$, then C contains a projection p different from 0 and 1. The subspace $\mathfrak{M}=R_pH^2$ violates (5) and (6), while the algebra $\mathfrak{B}=(1-L_p)\mathfrak{L}\oplus L_p\mathfrak{L}_+$ violates (4). Thus, from Theorems 3.2, 3.3 and Proposition 3.4, we may conclude that assertions (1) through (5) are equivalent and that they are implied by (6). It suffices to prove that (4) implies (6). Let $\mathfrak{M}$ be a two-sided invariant subspace of L^2 which is not left reducing. To prove that (4) implies (6), it is sufficient to prove that $\mathfrak{M}$ is left-pure. Let P be the projection of L^2 onto $\bigcap_{n\geqq 0}L_\delta^n\mathfrak{M}$. Since $\bigcap_{n\geqq 0}L_\delta^n\mathfrak{M}$ is left reducing, P lies in $\mathfrak{L}'=\mathfrak{R}$. In addition, since $\bigcap_{n\geqq 0}L_\delta^n\mathfrak{M}$ is right invariant, P commutes with $R(M)$ and $R_\delta PR_\delta^*\leqq P$. Since $\mathfrak{R}$ is finite, $R_\delta PR_\delta^*=P$ and so $P\in\mathfrak{Z}(\mathfrak{L})$. But also $P\mathfrak{M}\subset PL^2=\bigcap_{n\geqq 0}L_\delta^n\mathfrak{M}\subset\mathfrak{M}$. By hypothesis (4), $P\in\mathfrak{L}_+$ and so $P\in\mathfrak{L}_+\cap\mathfrak{L}_+^*=L(M)$. Therefore $P\in\mathfrak{Z}(\mathfrak{L})\cap L(M)=L(C)=\{CI\}$. Since $\mathfrak{M}$ is not left reducing, $P=0$ and so $\mathfrak{M}$ is left-pure. This completes the proof.

4. Which subdiagonal algebras are crossed products?

We fix once and for all a finite von Neumann algebra $\mathfrak{B}$ with trace ϕ and a subalgebra $\mathfrak{A}$ of $\mathfrak{B}$ which is a finite, maximal, subdiagonal algebra in $\mathfrak{B}$ with respect to ϕ and expectation Φ mapping $\mathfrak{B}$ onto $D=\mathfrak{A}\cap\mathfrak{A}^*$, that is, $\mathfrak{A}$ is an ultraweakly closed subalgebra of $\mathfrak{B}$ containing the identity operator 1 which satisfies the following conditions: 1) $\mathfrak{A}+\mathfrak{A}^*$ is ultraweakly dense in $\mathfrak{B}$; 2) Φ is multiplicative on $\mathfrak{A}$; 3) $\mathfrak{A}$ is maximal among those subalgebras of satisfying 1) and 2); and 4) $\phi\circ\Phi=\phi$. We shall denote the noncommutative Lebesgue space associated with $\mathfrak{B}$ and ϕ by $L^2(\mathfrak{B}, \phi)$ and write $L^2=L^2(\mathfrak{B}, \phi)$. As before, the closure S of L^2 in the L^2-norm is denoted by $[S]_2$. We put $H^2=[\mathfrak{A}]_2$ and $H_0^2=[\mathfrak{A}_0]_2$ where $\mathfrak{A}_0=\{x\in\mathfrak{A}: \Phi(x)=0\}$.

If x is in $\mathfrak{B}$, we shall write L_x (resp. R_x) for the operator defined by the equations $L_xf=xf$ (resp. $R_xf=fx$), $f\in L^2$, and we let $\mathfrak{L}=\{L_x: x\in\mathfrak{B}\}$ (resp. $\mathfrak{R}=\{R_x: x\in\mathfrak{B}\}$). One may regard $\mathfrak{B}$ as a finite, achieved Hilbert algebra whose completion is L^2, and $\mathfrak{L}$ and $\mathfrak{R}$ are the left and right von Neumann algebras of $\mathfrak{B}$. Also we put $\mathfrak{L}_+=\{L_x: x\in\mathfrak{A}\}$ (resp. $\mathfrak{R}_+=\{R_x: x\in\mathfrak{A}\}$). Finally, the canonical conjugate-linear, isometric involution on L^2 which extends the map $x\to x^*$ on $\mathfrak{B}$ will be denoted by J. As in Definition 2.2, we define the concept of invariant subspaces of L^2.

In [7], we proved that, if every nonzero two-sided invariant subspace $\mathfrak{M}$ of H^2 has the form $\mathfrak{M}=R_vH^2$ for some unitary operator v in $\mathfrak{B}$, then there is a *-automorphism α of D preserving ϕ such that $\mathfrak{B}$ is isomorphic to the crossed product L^∞ determined by D and α in such a way that $\mathfrak{A}$ becomes identified with the corresponding space H^∞. Further, D is a factor. In this section we shall consider a generalization of this result. Put $C=\mathfrak{Z}(\mathfrak{B})\cap D$, so that $C\subset\mathfrak{Z}(D)$.

Definition 4.1. Let $\mathfrak{A}$ be a finite, maximal, subdiagonal algebra with respect to Φ and ϕ. Then $\mathfrak{A}$ is called pure if there is no nonzero projection p in C such that $\mathfrak{A}p=\mathfrak{B}p$.

Remark 4.2. If $\mathfrak{A}$ is a finite, maximal, subdiagonal algebra, then $\mathfrak{A}$ is not necessarily pure. For example, let $\mathfrak{B}=L^\infty(\Delta)\oplus L^\infty(\Delta)$, where Δ is the unit circle. Put $\mathfrak{A}=H^\infty(\Delta)\oplus L^\infty(\Delta)$. Then $\mathfrak{A}$ is a finite, maximal, subdiagonal algebra with respect to expectation Φ where $\Phi(f\oplus g)=\left(\int f\,dm\right)\oplus g$, f, $g\in L^\infty(\Delta)$. However, it is clear that $\mathfrak{A}$ is not pure.

Theorem 4.3. *Suppose that every two-sided invariant subspace $\mathfrak{M}$ of H^2 has the form $\mathfrak{M}=R_vH^2$ for some partial isometry v in $\mathfrak{B}$ such that $v^*v=vv^*\in C$.*

*If $\mathfrak{U}$ is pure, then there is a *-automorphism α of D preserving ϕ such that $\mathfrak{B}$ is isomorphic to the crossed product L^∞ determined by D and α in such a way that $\mathfrak{U}$ becomes identified with the corresponding space H^∞.*

This and Theorem 3.3 immediately yield

COROLLARY 4.4. (1) $C=\{z\in\mathfrak{Z}(D): \alpha(z)=z\}$.

(2) *α fixes the center $\mathfrak{Z}(D)$ of D elementwise.*

We break the proof of Theorem 4.3 up into a series of lemmas. In the remainder of this section, we suppose that every two-sided invariant subspace $\mathfrak{M}$ of H^2 has the form $\mathfrak{M}=R_vH^2$ for some partial isometry v in $\mathfrak{B}$ such that $v^*v=vv^*\in C$ and $\mathfrak{U}$ is pure.

LEMMA 4.5. *$H_0^2=R_vH^2$ for some unitary operator v in $\mathfrak{B}$.*

PROOF. Since H_0^2 is a nonzero two-sided invariant subspace of H^2, by hypothesis, there is a partial isometry v in $\mathfrak{B}$ such that $H_0^2=R_vH^2$ and $v^*v=vv^*\in C$. Put $v^*v=1-p$. Then $\phi(pxpy)=0$ for every $y\in H_0^2$ and $x\in\mathfrak{B}$. By [**1**, Corollary 2.2.4], we have $\mathfrak{B}p=\mathfrak{U}p$. Since $\mathfrak{U}$ is pure, $p=0$ proving that v is unitary. This completes the proof.

The proof of Lemma 6.3 in [7] works here too, and yields

LEMMA 4.6. *If v is a unitary in $\mathfrak{B}$ such that $R_vH^2=H_0^2$, then for all $n\in Z$, $R_v^nH^2=L_v^nH^2$.*

LEMMA 4.7. *If $\mathfrak{T}$ is a proper ultraweakly closed subalgebra of $\mathfrak{B}$ containing $\mathfrak{U}$, then there is a projection p in C such that $\mathfrak{T}=(1-p)\mathfrak{B}\oplus p\mathfrak{U}$.*

PROOF. By [7, Corollary 1.5], $K=L^2\ominus[\mathfrak{T}]_2\neq\{0\}$. Then JK is a two-sided invariant subspace of H^2. By hypothesis, then $JK=R_wH^2$ for some partial isometry w in $\mathfrak{B}$ such that $w^*w=ww^*\in C$. As in the proof that (5) implies (4) in Theorem 3.3, we have $\mathfrak{T}=(1-p)\mathfrak{B}\oplus p\mathfrak{U}$ where $p=w^*w$. This completes the proof.

LEMMA 4.8. *Let v be a unitary in $\mathfrak{B}$ such that $H_0^2=R_vH^2$. Then $vDv^*=D$.*

PROOF. By Lemma 4.6, $H_0^2=L_vH^2$ and since $L(D)\subset\mathfrak{L}_+$, we find that $L_{v^*}L(D)L_vH^2\subset H^2$. Thus $L_{v^*}L(D)L_v\subset\mathfrak{L}_+\cap\mathfrak{L}_+^*=L(D)$. On the other hand, $vDv^*H_0^2=L_vL(D)L_{v^*}H_0^2=L_vL(D)H^2\subset L_vH^2=H_0^2$. This implies that $vDv^*\subset\mathfrak{B}\cap H^2=\mathfrak{U}$ and so $vDv^*\subset D$. Consequently $vDv^*=D$ as was to be proved.

We may now define a *-automorphism α on D by the formula $\alpha(d)=vdv^*$, $d\in D$. Note that α preserves ϕ.

PROOF OF THEOREM 4.3. Fix once and for all a unitary operator v in $\mathfrak{B}$ such that $R_vH^2=H_0^2=L_vH^2$. By the proof of [7, Theorem 6.1], it suffices to prove that $\mathfrak{M}=\bigcap_{n\geq 0}L_v^nH^2=\{0\}$. Since $\mathfrak{M}$ is contained in H^2, $\mathfrak{M}$ does not reduce $\mathfrak{L}_+$ unless $\mathfrak{M}=\{0\}$. But if $\mathfrak{M}\neq\{0\}$, then $\mathfrak{T}=\{x\in\mathfrak{B}: L_x\mathfrak{M}\subset\mathfrak{M}\}\subsetneqq\mathfrak{B}$. By Lemma 4.7, there is a projection p in C such that $\mathfrak{T}=(1-p)\mathfrak{B}\oplus p\mathfrak{U}$. Since, however, $L_{v^*}\mathfrak{M}=\mathfrak{M}$, $v^*\in\mathfrak{T}$ and so $pv^*\in p\mathfrak{U}\subset\mathfrak{U}$. On the other hand, since $H_0^2=L_vH^2$, $pv\in\mathfrak{U}_0$. This implies that $pv=0$. Since v is unitary, $p=0$ and so $\mathfrak{T}=\mathfrak{B}$. This is a contradiction and completes the proof.

Finally we suppose that every nonzero two-sided invariant subspace $\mathfrak{M}$ of H^2 has the form $\mathfrak{M}=R_v H^2$ for some unitary v in $\mathfrak{B}$. Then it is clear that $C=\{CI\}$ and so $\mathfrak{U}$ is always pure. By Theorem 4.3, we obtain Theorem 6.1 of [**7**] as a corollary.

5. Ideals.

Our results have all been phrased in terms of the invariant subspace structure in L^2. However, thanks to the results of the third author in [**13**], we may extend them to cover subspaces in L^p, $1\leqq p\leqq\infty$. Of particular interest here is the case when $p=\infty$. For in this case invariant subspaces correspond to ideals-left, right, or two-sided according to circumstance. We continue with the notation of the previous section but supplement it as follows. The noncommutative Lebesgue space associated with $\mathfrak{B}$ and ϕ will be denoted by L^p ([**14**]), H^p denotes the closure of $\mathfrak{U}$ in L^p, $1\leqq p<\infty$. We identify $\mathfrak{B}$ with L^∞ and $\mathfrak{U}$ with H^∞. By a subspace of L^p we shall mean a closed subspace when $1\leqq p<\infty$ and an ultraweakly closed subspace when $p=\infty$.

The notions of invariant subspace, pure, full, etc. are defined as before. The following is proved as Theorem 1 of [**13**]. We state it here for reference in a slightly different form.

PROPOSITION 5.1. *Suppose $1\leqq p<s\leqq\infty$. The map which carries a left- (resp. right-) invariant subspace $\mathfrak{M}$ in L^p to $\mathfrak{M}\cap L^s$ sets up a one-to-one correspondence between the left- (resp. right-) invariant subspaces of L^p and those of L^s. Its inverse carries $\mathfrak{M}\subseteqq L^s$ to $[\mathfrak{M}]_p\subseteqq L^p$. For an invariant subspace $\mathfrak{M}\subseteqq L^p$, we have $\mathfrak{M}=[\mathfrak{M}\cap L^s]_p$ while for $\mathfrak{M}\subseteqq L^s$, $\mathfrak{M}=[\mathfrak{M}]_p\cap L^s$.*

Using this proposition, we see immediately that in our earlier results we may replace L^2 by L^p and H^2 by H^p, $1\leqq p\leqq\infty$, and not affect their validity. We formally spell out what happens when $p=\infty$.

THEOREM 5.2. *The conditions* (1) *through* (6) *of* 3.2, 3.3 *and* 3.4 *are each equivalent to:*

(7) *Every ultraweakly closed left- (or right-) ideal in H^∞ is principal and is generated by a partial isometry.*

With the notation as in the Theorem 4.3, we conclude with

THEOREM 5.3. *Suppose that every ultraweakly closed two-sided ideal in $\mathfrak{U}$ is principal and is generated by a partial isometry v such that $vv^*=v^*v\in C$, and suppose that $\mathfrak{U}$ is pure. Then there is a *-automorphism α of D preserving ϕ such that $\mathfrak{B}$ is isomorphic to the crossed product L^∞ determined by D and α in such a way that $\mathfrak{U}$ becomes identified with the corresponding space H^∞.*

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Michael McAsey
Department of Mathematics
Bradley University
Peoria, Illinois 61625
U. S. A.

Paul S. Muhly
Department of Mathematics
University of Iowa
Iowa City, Iowa 52242
U. S. A.

Kichi-Suke Saito
Department of Mathematics
Faculty of Science
Niigata University
Ikarashi, Niigata 950-21
Japan