Differentiability of solutions of some unilateral problem of parabolic type

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Let us begin with the following simple example of a parabolic unilateral problem

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &\geq 0, \quad u \geq \Psi \\
(\frac{\partial u}{\partial t} - \Delta u)(u - \Psi) &= 0 \\
u &= 0 \\
u(x, 0) &= u_0(x) \geq \Psi(x)
\end{align*}
\]

Here \( \Omega \) is a domain in \( \mathbb{R}^N \) with sufficiently smooth boundary \( \Gamma \), and \( \Psi \) is a function such that \( \Psi \in W^{2,p}(\Omega) \) and \( \Psi|_{\Gamma} \leq 0 \). We wish to make \( p \) small; however, assume

\[ 1 < p < 2 < p^* = pN/(N-p). \]

In view of Sobolev's imbedding theorem it follows that

\[ W^{2,p}(\Omega) \subset H^1(\Omega) \subset L^{p'}(\Omega), \quad p' = p/(p-1). \]

Let \( L_\Psi \) be the realization of \(-\Delta\) in \( L^p(\Omega) \) under the Dirichlet boundary condition, and \( M_\Psi \) be the multivalued mapping defined by

\[ D(M_\Psi) = \{ u \in L^p(\Omega) : u \geq \Psi \ \text{a.e. in} \ \Omega \}, \]

\[ M_\Psi u = \{ g \in L^p(\Omega) : g \leq 0 \ \text{a.e. in} \ \Omega, \quad g(x) = 0 \ \text{if} \ u(x) > \Psi(x) \}. \]

Note that \( M_\Psi = \partial I_K \) where \( I_K \) is the indicatrix of the closed convex set \( K = D(M_\Psi) \). The problem (0.1)-(0.3) is formulated in \( L^p(\Omega) \) as

\[ du(t)/dt + (L_\Psi + M_\Psi)u(t) \ni 0 \]

\[ u(0) = u_0. \]

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It can be shown that $L_p + M_p$ is $m$-accretive, and hence we can apply a result of M.G. Crandall and T.M. Liggett \cite{6} to construct the solution $u(t)$ of (0.8) in some sense by an exponential formula. We are interested in the differentiability of this solution with respect to $t$ assuming only $\mathcal{F} \subseteq u_0 \in L^p(\Omega)$ or $u_0 \in D(L_p + M_p)$ for the initial value $u_0$. With the aid of a comparison theorem we can show $u(t) \in L^r(\Omega)$ for $t > 0$. Hence noting that $\mathcal{F} \subseteq H^1(\Omega)$ in view of (0.5) we may consider $u(t)$ as the solution of

$$
\frac{d}{dt}u(t) + \partial \phi(u(t)) \ni 0 \quad (0.10)
$$

in $(0, T]$, where $\phi: L^2(\Omega) \to [0, \infty]$ is the convex function

$$
\phi(u) = \begin{cases}
\frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx & \text{if } \mathcal{F} \subseteq u \in H^1_0(\Omega), \\
\infty & \text{otherwise}.
\end{cases}
$$

Thus we may apply a general result on the subdifferential of a convex function to establish the differentiability of $u(t)$ in $L^r(\Omega)$. With the aid of another application of a comparison theorem we can show that $\frac{d}{dt}u(t) \in L^r(\Omega)$ for any $r > 2$, if $t > 0$. We note $L_2 + M_2 \subsetneq \partial \phi$ in general under our hypothesis as the following counter example shows. Suppose $\mathcal{F} \in W^{2, p}(\Omega) \cap W^{1, p}_0(\Omega) = D(L_p)$ and $0 \leq -\Delta \mathcal{F} \in L^2(\Omega)$. Let $v$ be an arbitrary element of $D(\phi)$. Then $v - \mathcal{F} \in L^r(\Omega)$ by virtue of (0.5). Hence with the aid of an integration by part

$$
0 \leq (-\Delta \mathcal{F}, v - \mathcal{F}) = (\nabla \mathcal{F}, \nabla v - \nabla \mathcal{F}) \leq \phi(v) - \phi(\mathcal{F}),
$$

which implies $\mathcal{F} \in D(\partial \phi)$. However $\mathcal{F} \in D(L_2 + M_2) = D(L_2) \cap D(M_2)$ since $\Delta \mathcal{F} \in L^2(\Omega)$.

In this paper we consider the more general problem

$$
\begin{align*}
\partial u/\partial t + \mathcal{L} u \geq f, & \quad u \geq \mathcal{F} & \text{in } \Omega \times (0, T] \quad (0.11) \\
(\partial u/\partial t + \mathcal{L} u - f)(u - \mathcal{F}) = 0 & \quad & \text{on } \Gamma \times (0, T] \quad (0.12) \\
-\partial u/\partial n \in \beta(x, u) & \quad & \text{on } \Gamma \quad (0.13)
\end{align*}
$$

Here $\Omega$ is not assumed to be bounded. $\mathcal{L}$ is a not necessarily symmetric linear elliptic operator of second order, and $\partial/\partial n$ is the differentiation in the outward conormal direction with respect to $\mathcal{L}$. $\beta(x, \cdot)$ is a maximal monotone graph in $\mathbb{R}^2$ with $0 \in \beta(x, 0)$ for each fixed $x \in \Gamma$. $\mathcal{F}$ is a function such that

$$
\mathcal{F} \in W^{2, p}(\Omega), \quad \partial \mathcal{F}/\partial n + \beta^-(x, \mathcal{F}) \leq 0 \quad \text{on } \Gamma \quad (0.14)
$$

with $p$ satisfying (0.4). $\beta^-(x, r)$, which will be defined later, is roughly speaking $\min \beta(x, r)$. 

First we formulate the elliptic boundary value problem
\[ Lu = f \text{ in } \Omega, \quad -\partial u / \partial n \in \beta(x, u) \text{ on } \Gamma \]
(0.15)
in \( L^q(\Omega) \) as some variational problem. With the aid of a result of H. Brézis \[2\] the problem thus formulated is expressed as \( L_q u = f \) with some single-valued \( m \)-accretive operator \( L_q \) in \( L^q(\Omega) \). Since \( (1 + \lambda L_q)^{-1} \) is a contraction for \( \lambda > 0 \) also in \( L^q \) norm, \( 1 \leq q < \infty \), an \( m \)-accretive operator \( L_q \) in \( L^q(\Omega) \) is defined as the smallest closed extension of the operator with graph \( G(L_q) \cap (L^q(\Omega) \times L^q(\Omega)) \), where \( G(L_q) \) is the graph of \( L_q \). Thus for \( 1 \leq q < \infty \) the problem (0.15) is formulated in \( L^q(\Omega) \) as \( L_q u = f \). Following the idea of B. D. Calvert and C. P. Gupta [5] it is shown that \( D(L_q) \subset W^{1,q}(\Omega) \) for \( 1 < q \leq 2 \), which will be used frequently in the subsequent argument.

In addition to (0.14) we assume also
\[ \Psi \in W^{1,1}(\Omega), \quad \mathcal{L} \Psi \in L^1(\Omega). \]
Then it is shown that \( A_q = L_q + M_q \) is \( m \)-accretive in \( L^q(\Omega) \) for \( 1 \leq q \leq p \), where \( M_q \) is the mapping defined by (0.6) and (0.7). For \( p < q \leq 2 \) \( A_q \) is defined as the \( m \)-accretive extension of \( L_q + M_q \). If \( f \in W^{1,1}(0, T; L^q(\Omega)) \) and \( \Psi \leq u_0 \in L^q(\Omega) \), the problem (0.11)-(0.13) is expressed as
\[ du(t)/dt + A_q u(t) \ni f(t), \quad 0 < t \leq T, \]
\[ u(0) = u_0. \]
With the aid of Theorem 5.1 of M. G. Crandall and A. Pazy [7] it is possible to construct the solution of this problem by an exponential formula. Suppose further \( f \in W^{1,1}(0, T; L^q(\Omega) \cap L^r(\Omega)) \) for \( 1 \leq q \leq 2 \leq r \). Then by a comparison theorem it follows that \( u(t) \in L^q(\Omega) \) for \( t > 0 \). Instead of (0.10) we have
\[ du(t)/dt + A u(t) \ni f(t) \]
(0.16)
this time where \( A \) is the mapping defined by \( A u = (Lu + \partial \phi(u)) \cap L^q(\Omega) \), \( L \) is the linear isomorphism from \( H^1(\Omega) \) onto \( H^s(\Omega)^* \) associated with \( \mathcal{L} \) and \( \phi \) is some proper convex function on \( H^s(\Omega) \) associated with \( \beta \) and \( \Psi \). It will be shown that the solution of (0.16) constructed by the exponential formula is differentiable a.e. (Theorem 6.1). As in the problem (0.1)-(0.3) we can show that \( du(t)/dt \in L^q(\Omega) \) for \( t > 0 \) with the aid of a comparison theorem following F. J. Massey, III [11] and L. C. Evans [8], [9]. The main theorem of the present paper is Theorem 7.1. Related results are found in the above papers of Massey and Evans. In [11] the equation of the form
\[ \partial u/\partial t + \mathcal{L} u + \beta(u) \ni f \]
(0.17)
is studied, and in [8], [9] various types of problems including (0.17) are
investigated.

The result of this paper was announced in [13] and [14].

§ 1. Assumptions and notations.

All functions considered in this paper are real valued.

Let $\Omega$ be a not necessarily bounded domain in $\mathbb{R}^N$. We assume that the boundary $\Gamma$ of $\Omega$ is uniformly regular of class $C^2$ and locally regular of class $C^4$ in the sense of F. E. Browder [3]. $W^{m,p}(\Omega)$ denotes the usual Sobolev space and $H^m(\Omega) = W^{m,2}(\Omega)$. The norm of $W^{m,p}(\Omega)$ is denoted by $\| \cdot \|_{m,p}$ and that of $L^p(\Omega)$ is simply by $\| \cdot \|_p$ if there is no fear of confusion. $W^{1-1/p,p}(\Gamma)$ is the set of the boundary values of functions belonging to $W^{1,p}(\Omega)$. $W^{1-1/p,p}(\Gamma)$ is a Banach space with norm

$$\| h \|_{1-1/p,p} = \inf \{ \| u \|_{1,p} : u \in W^{1,p}(\Omega), u = h \text{ on } \Gamma \}.$$ 

We denote by $\rightarrow$ strong convergence and by $\rightharpoonup$ weak convergence. For a mapping $A$ multivalued in general $D(A)$, $R(A)$ and $G(A)$ stand for its domain, range and graph respectively.

Let

$$a(u, v) = \int_{\Omega} \left( \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_{i}} \frac{\partial v}{\partial x_{j}} + \sum_{i=1}^{N} b_{i} \frac{\partial u}{\partial x_{i}} v + cuv \right) dx$$

be a bilinear form defined in $H^1(\Omega) \times H^1(\Omega)$. The coefficients $a_{ij}, b_i$ are bounded and continuous in $\overline{\Omega}$ together with first derivatives and $c$ is bounded and measurable in $\Omega$. \{a_{ij}(x)\} is uniformly positive definite in $\Omega$, i.e. for some positive constant $\delta$

$$\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq \delta |\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^N.$$  

(1.2)

We assume that there exists a positive constant $\alpha$ such that

$$c \geq \alpha, \quad c - \sum_{i=1}^{N} \partial b_i / \partial x_i \geq \alpha \quad \text{a.e. in } \Omega.$$  

(1.3)

We denote by $\mathcal{L}$ the linear differential operator associated with the bilinear form (1.1):

$$\mathcal{L} = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} \left( a_{ij} \frac{\partial}{\partial x_i} \right) + \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i} + c.$$ 

The conormal derivative with respect to $\mathcal{L}$ is denoted by

$$\partial / \partial n = \sum_{i,j=1}^{N} a_{ij} \nu_j \partial / \partial x_j.$$
where $\nu=(\nu_1, \cdots, \nu_N)$ is the outward normal vector to $\Gamma$.

Let $j(x, r)$ be a function defined on $\Gamma \times R$ such that for each fixed $x \in \Gamma$ $j(x, r)$ is a proper convex lower semicontinuous function of $r$ such that

$$j(x, r) \geq j(x, 0) = 0.$$  \hspace{1cm} (1.4)

We denote by $\beta(x, \cdot) = \partial j(x, \cdot)$ the subdifferential of $j(x, r)$ with respect to $r$. As for the regularity with respect to $x$ we assume that for each $t \in R$ and $\lambda > 0$, $(1 + \lambda \beta(x, \cdot))^{-1}(t)$ is a measurable function of $x$ (cf. B.D. Calvert and C.P. Gupta \[5\]). Unless $j(x, r) = \infty$ for $r \neq 0$ (namely the boundary condition is of Dirichlet type), we assume that

$$\sum_{i=1}^{N} b_i \nu_i \geq 0 \quad \text{on} \quad \Gamma.$$  \hspace{1cm} (1.5)

Let $\Psi(x)$ be a function satisfying

$$\Psi \in W^{2, p}(\Omega),$$  \hspace{1cm} (1.6)

$$\mathcal{L} \Psi \in L^1(\Omega),$$  \hspace{1cm} (1.7)

$$\partial \Psi(x)/\partial n + \beta^{-}(x, \Psi(x)) \leq 0 \quad x \in \Gamma$$  \hspace{1cm} (1.8)

where $p$ is an exponent satisfying

$$1 < p < 2 < p^* = Np/(N-p)$$  \hspace{1cm} (1.9)

and

$$\beta^{-}(x, r) = \begin{cases} \min \{z : z \in \beta(x, r)\} & \text{if} \ r \in D(\beta(x, \cdot)), \\ \infty & \text{if} \ r \in D(\beta(x, \cdot)) \text{ and } r \geq \sup D(\beta(x, \cdot)), \\ -\infty & \text{if} \ r \in D(\beta(x, \cdot)) \text{ and } r \leq \inf D(\beta(x, \cdot)) \end{cases}$$

(cf. p. 55 of H. Brézis [2]).

§ 2. Preliminaries (1).

In this section we collect some preliminary results mainly due to H. Brézis [2] and B.D. Calvert and C.P. Gupta [5] concerning the boundary value problem \(\mathcal{L}u = f\) in $\Omega$, $-\partial u / \partial n \in \beta(u)$ on $\Gamma$. Here $\beta(u)$ stands for the (multivalued in general) function $x \mapsto \beta(x, u(x))$. In our case the proofs are simpler than those of the corresponding results of [5] since $\mathcal{L}$ is linear and we can use the Yosida approximation of $\beta(x, \cdot)$ according to Proposition 2.1 below.

Let $a(u, v)$ be a bilinear form (1.1) such that

$$a(u, u) \geq c_0 \|u\|_{1,2}^2, \quad u \in H^1(\Omega)$$  \hspace{1cm} (2.1)

for some $c_0 > 0$. It will be shown in Lemma 2.2 that such a constant $c_0$ exists
under our hypothesis. Let $\Phi$ be a proper convex, lower semicontinuous convex function defined in $L^2(\Gamma)$ such that $\Phi \not\equiv \infty$ on $H^{1/2}(\Gamma)$. Then it is known that for any $f \in L^2(\Omega)$ there exists a unique solution $u \in H^1(\Omega)$, $\Phi(u|_r) < \infty$, of the inequality

$$a(u, v-u)+\Phi(v|_r)-\Phi(u|_r) \geq (f, v-u), \quad v \in H^1(\Omega).$$

(2.2)

Furthermore the solution is characterized by

$$-\partial u/\partial n \in \partial F(u|_r)$$

(2.4)

where $F$ is the restriction of $\phi$ to $H^{1/2}(\Gamma)$ (cf. Theorem 1.7 of [2]).

For $\varepsilon > 0$ let

$$\Phi_{\varepsilon}(u) = \frac{1}{2\varepsilon} \int (u-J_{\varepsilon}u)^2 d\Gamma + \Phi(J_{\varepsilon}u)$$

be the Yosida approximation of $\Phi$ where $J_\varepsilon = (1+\varepsilon \partial \Phi)^{-1}$.

**PROPOSITION 2.1.** For $f \in L^2(\Omega)$ let $u_{\varepsilon} \in H^1(\Omega)$ be the solution of the inequality

$$a(u_{\varepsilon}, v-u_{\varepsilon}) + \Phi_{\varepsilon}(v|_r) - \Phi_{\varepsilon}(u_{\varepsilon}|_r) \geq (f, v-u_{\varepsilon}), \quad v \in H^1(\Omega).$$

Then

$$-\partial u_{\varepsilon}/\partial n = \Phi_{\varepsilon}'(u_{\varepsilon}|_r) \in L^2(\Gamma)$$

(2.5)

As $\varepsilon \to 0$ $u_{\varepsilon}$ converges to the solution $u$ of (2.2) in the strong topology of $H^1(\Omega)$ and

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int (u_{\varepsilon}-J_{\varepsilon}u_{\varepsilon})^2 d\Gamma = 0.$$  

(2.6)

**Proof.** This proposition was proved by H. Brézis (Theorem 1.8 of [2]) under the assumption that $\Omega$ is bounded. In case $\Omega$ is unbounded the proof is essentially unchanged and hence we only sketch it. If we put

$$\tilde{a}(u, v) = \frac{1}{2}(a(u, v)+a(v, u)),$$

then $\tilde{a}(u, u) = a(u, u) \geq c_0 \| u \|_{1,2}$. Hence $a(u, u)^{1/2} = \tilde{a}(u, u)^{1/2}$ may be considered as a norm of $H^1(\Omega)$. As in Theorem 1.8 of [2], $a(u_{\varepsilon}, u_{\varepsilon})$ and $\varepsilon^{-1} \int (u_{\varepsilon}-J_{\varepsilon}u_{\varepsilon})^2 d\Gamma$ are bounded as $\varepsilon \to 0$. If $u_{\varepsilon} \to u^*$ in $H^1(\Omega)$, then $u_{\varepsilon}|_r \to u^*|_r$ in $H^{1/2}(\Gamma)$ and $J_{\varepsilon}u_{\varepsilon}|_r \to u^*|_r$ in $L^2(\Gamma)$. Letting $\varepsilon = \varepsilon_n \to 0$ in

$$a(u_{\varepsilon}, v) + \Phi_{\varepsilon}(v|_r) \geq (f, v-u_{\varepsilon}) + a(u_{\varepsilon}, u_{\varepsilon}) + \Phi(J_{\varepsilon}u_{\varepsilon}|_r)$$

we get

$$a(u^*, v) + \Phi(v|_r) \geq (f, v-u^*) + \lim \sup a(u_{\varepsilon_n}, u_{\varepsilon_n}) + \Phi(u^*|_r).$$

Hence $\Phi(u^*|_r) < \infty$ and
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\[ a(u^*, u^*) \leq \lim \inf a(u_{\epsilon_n}, u_{\epsilon_n}) \leq \lim \sup a(u_{\epsilon_n}, u_{\epsilon_n}) \]

\[ \leq a(u^*, v) + \Phi(v \mid I) - \Phi(u^* \mid I) - (f, v - u^*) \]

Letting \( v = u^* \) we get \( a(u^*, u^*) = \lim a(u_{\epsilon_n}, u_{\epsilon_n}) \), and hence \( u_{\epsilon_n} \to u^* \) in \( H^1(\Omega) \).

\[ (2.2) \]

is established in Theorem 1.8 of [2]. The proof of (2.6) is easy and is omitted.

Lemma 2.1. Suppose \( \chi \) is a uniformly Lipschitz continuous increasing function in \( R \) such that \( \chi(0) = 0 \). Then for any \( u \in H^1(\Omega) \)

\[ a(u, \chi(u)) \geq \alpha(u, \chi(u)). \]

(2.7)

Proof. Let \( \zeta \) be the indefinite integral of \( \chi \) such that \( \zeta(0) = 0 \). (2.7) is easily established by noting

\[ \partial u/\partial x_i \cdot \chi(u) = \partial \zeta(u)/\partial x_i, \quad u \zeta(u) \geq \zeta(u) \]

and using \([1.2], [1.3], (1.5)\).

Lemma 2.2. For any \( u \in H^1(\Omega) \)

\[ a(u, u) \geq \min \{ \delta, \alpha \} \| u \|_{1.2}^2. \]

(2.8)

Proof. (2.8) is clear from the proof of Lemma 2.1.

In what follows \( \Phi : L^2(\Gamma) \to [0, \infty] \) denotes the function

\[ \Phi(u) \begin{cases} & \int \Gamma \mid j(x, u(x))d\Gamma \quad \text{if } j(u) \in L^1(\Gamma) \\ & \infty \quad \text{otherwise} \end{cases} \]

(2.9)

where \( j(u) \) is the function \( j(x, u(x)) \). By the proof of Lemma 3.1 of [5], \( j(x, u(x)) \) is measurable for \( u \in L^2(\Gamma) \) and \( \Phi \) is proper convex, lower semicontinuous on \( L^2(\Gamma) \).

Definition 2.1. \( L_2 \) is the operator with domain and range contained in \( L^2(\Omega) \) such that \( L_2 u = f \) if \( f \in L^2(\Omega), u \in H^1(\Omega), \Phi(u \mid I) < \infty \) and (2.2) holds.

Note that \( L_2 \) is single valued since (2.3) holds if \( L_2 u = f \). It is known that the following proposition holds.

Proposition 2.2 \( L_2 \) is m-accretive and \( R(L_2) = L^2(\Omega) \).

For the Yosida approximation \( \Phi_\epsilon \) of \( \Phi \) we have

\[ \partial \Phi_\epsilon(u)(x) = \beta_\epsilon(x, u(x)) \]

(2.10)

where \( \beta_\epsilon(x, \cdot) \) is the Yosida approximation of \( \beta(x, \cdot) \). To simplify the notation we write \( \beta_\epsilon(u) \) to denote the function \( \beta_\epsilon(x, u(x)) \).

We denote by \( L_{2, \epsilon} \) the operator defined as \( L_2 \) with \( \Phi_\epsilon \) in place of \( \Phi \) in Definition 2.1. If \( L_{2, \epsilon} u_\epsilon = f \), then
\[-\partial u_{\epsilon}/\partial n = \beta_{\epsilon}(u_{\epsilon}) \in L^{\infty}(\Gamma) \]  
(2.11)

(Theorem 1.8 of [2]) and in view of Proposition 2.1 \( u_{\epsilon} \to u \) in \( H^{1}(\Omega) \) where \( u \) is the solution of \( L_{2}u=f \).

**Definition 2.2.** For \( 1 \leq q < \infty \) the operator \( L_{q} \) with domain and range contained in \( L^{q}(\Omega) \) is defined by

\[
G(L_{q}) = \text{the closure of } G(L_{2}) \cap (L^{q}(\Omega) \times L^{q}(\Omega)) \text{ in } L^{q}(\Omega) \times L^{q}(\Omega).
\]

**Lemma 2.3.** Let \( \chi \) be a uniformly Lipschitz continuous increasing function in \( R \) such that \( \chi(0)=0 \). Then for any \( u, v \in D(L_{2}) \)

\[
(L_{2}u-L_{2}v, \chi(u-v)) \geq a(u-v, \chi(u-v)).
\]  
(2.12)

**Proof.** (2.12) is easily established by approximating \( u, v \) by the solutions of \( L_{2,\epsilon}u_{\epsilon}=L_{2}u, \ L_{2,\epsilon}v_{\epsilon}=L_{2}v \), and noting (2.11).

**Lemma 2.4.** Suppose \( 1 \leq q < \infty, \lambda > 0, \ f, g \in L^{q}(\Omega) \cap L^{q}(\Omega), \)

\[
u + \lambda L_{2}v = g.
\]  
Then \( u, v \in L^{q}(\Omega) \) and

\[
(1+\lambda\alpha)\|u-v\|_{q} \leq \|f-g\|_{q}.
\]  
(2.14)

**Proof.** First consider the case \( 1<q<2 \). Let

\[
\chi_{n}(t) = \begin{cases} t^{q-2}t & \text{if } |t| \geq 1/n, \\ n^{2-q}t & \text{if } |t| < 1/n. \end{cases}
\]

In view of Lemmas 2.1 and 2.3

\[
\alpha(u-v, \chi_{n}(u-v)) \leq (L_{2}u-L_{2}v, \chi_{n}(u-v)).
\]  
(2.15)

It follows from (2.13) and (2.15) that

\[
(1+\lambda\alpha)(u-v, \chi_{n}(u-v)) \leq (f-g, \chi_{n}(u-v)).
\]  
(2.16)

Applying Hölder's inequality to the right side of (2.16) and noting for \( q' = q/(q-1) \)

\[
\int_{\Omega} |\chi_{n}(u-v)|^{q} \, dx
\]

\[
\leq \int_{|u-v| \geq 1/n} |u-v|^{q} \, dx + n^{2-q} \int_{|u-v| < 1/n} (u-v)^{2} \, dx
\]

\[
= \int_{\Omega} (u-v)\chi_{n}(u-v) \, dx,
\]

we get

\[
(1+\lambda\alpha)\left\{ \int_{|u-v| \geq 1/n} |u-v|^{q} \, dx \right\}^{1/q}
\]  
(2.17)
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\[ \leq (1 + \lambda \alpha) \left\{ \int_{\Omega} (u - v) X_{n} (u - v) dx \right\}^{1/q} \leq \| f - g \|_{q}. \]

Letting \( n \to \infty \) in (2.17) we see that \( u - v \in L^{q}(\Omega) \) and (2.14) holds. Applying (2.14) for \( v = g = 0 \) we get \( u \in L^{q}(\Omega) \). Other cases are handled analogously.

In what follows we write for \( q > 1 \)

\[ F_{q}(r) = |r|^{q-2}r \]

and

\[ \text{sign}^{0}r = \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ -1 & \text{if } r < 0. \end{cases} \]

**Proposition 2.3.** For \( 1 \leq q < \infty \) \( L_{q} \) is \( m \)-accretive and \( R(L_{q}) = L^{q}(\Omega) \). For \( u, v \in D(L_{q}) \)

\[ \alpha \| u - v \|_{q} \leq (L_{q}u - L_{q}v, F_{q}(u - v)) \quad \text{if} \quad q > 1, \]

\[ \alpha \| u - v \|_{1} \leq (L_{1}u - L_{1}v, \text{sign}^{0}(u - v)) \quad \text{if} \quad q = 1, \]

\[ \alpha \| u \|_{q} \leq \| L_{q}u \|_{q} \quad \text{for} \quad q \geq 1. \]

**Proof.** The first part of the proposition is an easy consequence of Lemma 2.4. Letting \( n \to \infty \) in (2.15) we get (2.20) and (2.21) for \( u, v \in D(L_{2}) \cap L^{q}(\Omega), L_{q}u, L_{q}v \in L^{q}(\Omega) \). For general \( u, v \in D(L_{q}) \) these two inequalities are established by approximating \( u, v \) according to the definition of \( L_{q} \). Letting \( v = 0 \) in (2.20), (2.21) we obtain (2.22).

**Lemma 2.5.** If \( 1 < q < 2 \), then \( D(L_{q}) \subset W^{1,q}(\Omega) \) and there exists a constant \( c_{q} \) such that for any \( u, v \in D(L_{q}) \)

\[ (L_{q}u - L_{q}v, F_{q}(u - v)) \geq c_{q} \| u - v \|_{q} \].

For the proof of this lemma we refer to Proposition 3.2 of [5].

§ 3. Preliminaries (2).

Let \( \Psi \) and \( p \) be such that (1.6), (1.7), (1.8) and (1.9) hold. In view of Sobolev's imbedding theorem

\[ W^{2,p}(\Omega) \subset H^{1} (\Omega) \subset L^{p'} (\Omega), \quad p' = p/(p-1). \]

Let \( P \) be the operator defined by

\[ (Pu)(x) = \max \{ u(x), \Psi(x) \}. \]

Then

\[ u - Pu = - (\Psi - u)^{+}, \]

(3.2)
\[ F_q(u-Pu) = -((\Psi - u)^+)^{q - 1} \quad (3.3) \]

(recall (2.18) for the definition of \( F_q \)).

**Lemma 3.1.** If \( \phi(r) \) is a uniformly Lipschitz continuous function which vanishes for \( r < 0 \), then for any \( u \in D(L_2) \)

\[ (L_2u, \, \phi(\Psi - u)) \leq (L\Psi, \, \phi(\Psi - u)) \quad (3.4) \]

**Proof.** First note that by (3.1) \( \Psi - u \) belongs to \( L^q(\Omega) \cap L^p(\Omega) \), and so does \( \phi(\Psi - u) \) if \( u \in D(L_2) \), and hence both sides of (3.4) are meaningful. Let \( u_\epsilon \) be such that \( L_{2, \epsilon}u_\epsilon = L_2u \). Then by Proposition 2.1

\[
-\partial u_\epsilon/\partial n = \beta_\epsilon(u_\epsilon) \in L^2(\Gamma), \\
u_\epsilon \rightharpoonup u \quad \text{in} \quad H^1(\Omega), \\
\epsilon \int_{\Gamma} \beta_\epsilon(u_\epsilon)^2 d\Gamma \rightarrow 0. 
\]

In view of (3.1) \( \Psi - u_\epsilon \in H^1(\Omega) \), \( \phi(\Psi - u_\epsilon) \in H^1(\Omega) \). By Sobolev's imbedding theorem

\[
\partial \Psi/\partial n \in W^{1-1/p, p}(\Gamma) \subseteq L^{p(N-1)/(N-p)}(\Gamma), \\
\phi(\Psi - u_\epsilon)|_{\Gamma} \in W^{1/2, 2}(\Gamma) \subseteq L^{2(N-1)/(N-2)}(\Gamma). 
\]

Since

\[
\frac{N-p}{p(N-1)} + \frac{N-2}{2(N-1)} < 1, \quad \frac{1}{p} + \frac{1}{2} > 1
\]

there exist exponents \( q, r \) such that

\[
\frac{N-p}{p(N-1)} \leq \frac{1}{q} \leq \frac{1}{p}, \quad \frac{N-2}{2(N-1)} \leq \frac{1}{r} \leq \frac{1}{2}, \quad \frac{1}{q} + \frac{1}{r} = 1. \quad (3.10)
\]

In view of (3.8), (3.9), (3.10)

\[ \partial \Psi/\partial n \in L^q(\Gamma), \phi(\Psi - u_\epsilon)|_{\Gamma} \in L^r(\Gamma) \]

which implies

\[ \partial \Psi/\partial n \cdot \phi(\Psi - u_\epsilon)|_{\Gamma} \in L^1(\Gamma). \]

Therefore

\[
(L(\Psi - u_\epsilon), \phi(\Psi - u_\epsilon)) = -\int_{\Gamma} \left( \frac{\partial \phi}{\partial n} - \frac{\partial u_\epsilon}{\partial n} \right) \phi(\Psi - u_\epsilon) d\Gamma \\
+ a(\Psi - u_\epsilon, \phi(\Psi - u_\epsilon)).
\]

Hence

\[
(L_{2, \epsilon}u_\epsilon, \phi(\Psi - u_\epsilon)) = (L_\epsilon u_\epsilon, \phi(\Psi - u_\epsilon)) \\
= (L\Psi, \phi(\Psi - u_\epsilon)) - (L(\Psi - u_\epsilon), \phi(\Psi - u_\epsilon)). \quad (3.12)
\]
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\[= (L \Psi, \phi(\Psi-u)) + \int_{\Gamma} \left( \frac{\partial \Psi}{\partial n} - \frac{\partial u_{\epsilon}}{\partial n} \right) \phi(\Psi-u_{\epsilon}) d\Gamma \]

\[-a(\Psi-u_{\epsilon}, \phi(\Psi-u_{\epsilon})). \]

By (3.1) and (3.6) \(\Psi-u_{\epsilon} \rightarrow \Psi-u\) in \(L^{2}(\Omega) \cap L^{p'}(\Omega)\) as \(\epsilon \rightarrow 0\), and hence \(\phi(\Psi-u_{\epsilon}) \rightarrow \phi(\Psi-u)\) in \(L^{2}(\Omega) \cap L^{p'}(\Omega)\). Thus

\[
\lim_{\epsilon \rightarrow 0} (L_{2,\epsilon}u_{\epsilon}, \phi(\Psi-u_{\epsilon})) = (L_{2}u, \phi(\Psi-u)), \tag{3.13}
\]

\[
\lim_{\epsilon \rightarrow 0} (L \Psi, \phi(\Psi-u_{\epsilon})) = (L \Psi, \phi(\Psi-u)). \tag{3.14}
\]

As for the boundary integral in (3.12)

\[
\int_{\Gamma} \left( \frac{\partial \Psi}{\partial n} - \frac{\partial u_{\epsilon}}{\partial n} \right) \phi(\Psi-(1+\epsilon \beta)^{-1}u_{\epsilon}) d\Gamma
\]

\[
= \int_{\Gamma} \left( \frac{\partial \Psi}{\partial n} - \frac{\partial u_{\epsilon}}{\partial n} \right) \phi(\Psi-(1+\epsilon \beta)^{-1}u_{\epsilon}) d\Gamma \tag{3.15}
\]

\[
+ \int_{\Gamma} \left( \frac{\partial \Psi}{\partial n} - \frac{\partial u_{\epsilon}}{\partial n} \right) \phi(\Psi-u_{\epsilon}) - \phi(\Psi-(1+\epsilon \beta)^{-1}u_{\epsilon})) d\Gamma.
\]

If \(\phi(\Psi(x)-(1+\epsilon \beta)^{-1}u_{\epsilon}(x)) \neq 0, x \in \Gamma\), then \(\Psi(x)>(1+\epsilon \beta(x, \cdot))^{-1}u_{\epsilon}(x)\), which implies \(\beta^{-}(x, \Psi(x)) \geq \beta_{\epsilon}(x, u_{\epsilon}(x))\). Hence in view of (1.8) and (3.5) \(\partial \Psi(x)/\partial n \leq \partial u_{\epsilon}(x)/\partial n\). Consequently

\[
\int_{\Gamma} \left( \frac{\partial \Psi}{\partial n} - \frac{\partial u_{\epsilon}}{\partial n} \right) \phi(\Psi-(1+\epsilon \beta)^{-1}u_{\epsilon}) d\Gamma \leq 0. \tag{3.16}
\]

For some constant \(C\)

\[
|\phi(\Psi-u_{\epsilon})-\phi(\Psi-(1+\epsilon \beta)^{-1}u_{\epsilon})| \leq C |u_{\epsilon}-(1+\epsilon \beta)^{-1}u_{\epsilon}| = C \epsilon |\beta_{\epsilon}(u_{\epsilon})|,
\]

and hence by (3.7)

\[
\left| \int_{\Gamma} \left( \frac{\partial u_{\epsilon}}{\partial n} \phi(\Psi-u_{\epsilon})-\phi(\Psi-(1+\epsilon \beta)^{-1}u_{\epsilon}) \right) d\Gamma \right| \leq C \epsilon \int_{\Gamma} \beta_{\epsilon}(u_{\epsilon})^{2} d\Gamma \rightarrow 0 \tag{3.17}
\]

as \(\epsilon \rightarrow 0\). If \(\rho \geq 2N/(N+1)\), then

\(\partial \Psi/\partial n \in W^{1-1/p, p}(\Gamma) \subset L^{p}(\Gamma)\).

Consequently as \(\epsilon \rightarrow 0\)

\[
\left| \int_{\Gamma} \left( \frac{\partial \Psi}{\partial n} \phi(\Psi-u_{\epsilon})-\phi(\Psi-(1+\epsilon \beta)^{-1}u_{\epsilon}) \right) d\Gamma \right|
\]
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\[ \leq C \varepsilon \int_{\Gamma} \left| \frac{\partial \Psi}{\partial n} \right| d\Gamma \]

\[ \leq C \varepsilon \int_{\Gamma} \left| \frac{\partial \Psi}{\partial n} \right|^2 d\Gamma \rightarrow 0. \]  

If \( p < 2N/(N+1) \), we put \( \theta = N+2-2N/p \). Then \( 0 < \theta < 1 \) and

\[ \frac{N-p}{p(N-1)} + \frac{\theta}{2} + \frac{(N-2)(1-\theta)}{2(N-1)} = 1. \]  

Noting \( |\beta_{\varepsilon}(u_{\varepsilon})| \leq \varepsilon^{-1}|u_{\varepsilon}| \),

\[ \left| \int_{\Gamma} \frac{\partial \Psi}{\partial n} (\phi(\Psi-u_{\varepsilon}) - \phi(\Psi-(1+\varepsilon \beta)^{-1}u_{\varepsilon})) d\Gamma \right| \]

\[ \leq C \varepsilon \int_{\Gamma} \left| \frac{\partial \Psi}{\partial n} \right| |\beta_{\varepsilon}(u_{\varepsilon})| d\Gamma \]

\[ \leq C \epsilon^\theta \int_{\Gamma} \left| \frac{\partial \Psi}{\partial n} \right| |\beta_{\varepsilon}(u_{\varepsilon})|^\theta |u_{\varepsilon}|^{1-\theta} d\Gamma. \]

By (3.15), (3.16), (3.17) we obtain

\[ \lim_{\varepsilon \rightarrow 0} \sup \int_{\Gamma} \left( \frac{\partial \Psi}{\partial n} - \frac{\partial u_{\varepsilon}}{\partial n} \right) \phi(\Psi-u_{\varepsilon}) d\Gamma \leq 0. \]  

(3.21)

By Lemma 2.1

\[ a(\Psi-u_{\varepsilon}, \phi(\Psi-u_{\varepsilon})) \geq 0. \]  

(3.22)

(3.4) follows from (3.12), (3.13), (3.14), (3.21) and (3.22).

**Lemma 3.2.** For \( u \in D(L_q), 1 < q \leq p, \)

\[ (L_q u, F_q(u-Pu)) \geq (L \Psi, F_q(u-Pu)). \]  

(3.23)

**Proof.** Let \( \phi_{n} \) be the function defined by

\[ \phi_{n}(r) = \begin{cases} r^{2-1} & \text{if } r \geq 1/n, \\ n^{2-2r} & \text{if } 0 < r < 1/n, \\ 0 & \text{if } r \leq 0, \end{cases} \]  

(3.24)

and \( u_{m} \in D(L_q) \cap L^q(\Omega) \) be such that \( L_2 u_{m} \in L^q(\Omega), u_{m} \rightharpoonup u, L_2 u_{m} \rightharpoonup L_2 u \) in \( L^q(\Omega) \).

By Lemma 3.1
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\[(L_2u_m, \phi_n(\Psi-u_m)) \leq (L\Psi, \phi_n(\Psi-u_m)).\]  \hspace{1cm} (3.25)

Letting \(n \to \infty\) in (3.25) we get

\[(L_2u_m, F_q(u_m-Pu_m)) \geq (L\Psi, F_q(u_m-Pu_m)).\]  \hspace{1cm} (3.26)

By Lemma 1.1 of [5] there exists a constant \(K\) such that

\[\|F_q u - F_q v\|_{q'} \leq K \|u - v\|_{q}^{q-1}, \quad q' = q/(q-1)\]  \hspace{1cm} (3.27)

for \(u, v \in L^q(\Omega)\). Hence letting \(m \to \infty\) in (3.26) we get the desired result.

Let \(\text{sign}^+\) be the function defined by

\[\text{sign}^+_r = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r \leq 0.\end{cases}\]

**Lemma 3.3.** For \(u \in D(L_1)\)

\[(L_1u, \text{sign}^+_\Psi (\Psi-u)) \leq (L\Psi, \text{sign}^+_\Psi (\Psi-u)).\]  \hspace{1cm} (3.28)

**Proof.** Let \(\phi_n\) be the function such that

\[\phi_n(r) = \begin{cases} 1 & \text{if } r \geq 1/n, \\ nr & \text{if } 0 < r < 1/n, \\ 0 & \text{if } r \leq 0,\end{cases}\]

and \(u_m \in D(L_2) \cap L^1(\Omega)\) be such that \(L_2u_m \in L^1(\Omega), u_m \to u, L_2u_m \to L_1u\) in \(L^1(\Omega)\), \(u_m(x) \to u(x)\) a.e. in \(\Omega\). By Lemma 3.1

\[(L_2u_m, \phi_n(\Psi-u_m)) \leq (L\Psi, \phi_n(\Psi-u_m)).\]  \hspace{1cm} (3.29)

Now,

\[(L_2u_m, \phi_n(\Psi-u_m)) - (L_1u, \phi_n(\Psi-u)) \]

\[= (L_2u_m - L_1u, \phi_n(\Psi-u_m)) \]

\[+ (L_1u, \phi_n(\Psi-u_m) - \phi_n(\Psi-u)).\]  \hspace{1cm} (3.30)

It is obvious that the first term on the right of (3.30) tends to 0 as \(m \to \infty\). The integrand of the second term is bounded by \(2|L_1u|\) in absolute value and converges to 0 a.e. as \(m \to \infty\). Hence as \(m \to \infty\)

\[(L_2u_m, \phi_n(\Psi-u_m)) \to (L_1u, \phi_n(\Psi-u)).\]  \hspace{1cm} (3.31)

Similarly we see that the right side of (3.29) tends to \((L\Psi, \phi_n(\Psi-u))\) as \(m \to \infty\). Hence

\[(L_1u, \phi_n(\Psi-u)) \leq (L\Psi, \phi_n(\Psi-u)).\]

Finally letting \(n \to \infty\) we get (3.28).
§ 4. Elliptic unilateral problem in $L^q(\Omega), 1 \leq q \leq 2$.

Let $M_q$ be the multivalued mapping defined by

\[ D(M_q) = \{ u \in L^q(\Omega) : u \geq \Psi \text{ a.e. in } \Omega \}, \]

\[ M_q u = \{ g \in L^q(\Omega) : g \leq 0 \text{ a.e., } g(x) = 0 \text{ if } u(x) > \Psi(x) \}. \]

$D(M_q)$ is not empty for $1 \leq q \leq p^*$ since $\Psi \in L^q(\Omega)$ for these values of $q$. For $\lambda > 0$ and $u \in L^q(\Omega), 1 \leq q \leq p^*$

\[ Pu = (1 + \lambda M_q)^{-1} u. \quad (4.1) \]

**Definition 4.1.** For $1 \leq q \leq p^*$ the operator $A_q$ is defined as follows:

(i) $A_q = L_q + M_q$ for $1 \leq q \leq p$,

(ii) for $p < q \leq p^*$

\[ G(A_q) = \text{the closure of } G(A_p) \cap (L^q(\Omega) \times L^q(\Omega)) \text{ in } L^q(\Omega) \times L^q(\Omega). \]

For $\lambda > 0$ denote by $M_{q, \lambda}$ the Yosida approximation of $M_q$. By (4.1)

\[ M_{q, \lambda} u = (u - Pu) / \lambda. \quad (4.2) \]

**Proposition 4.1.** For $1 < q \leq p$ $A_q$ is $m$-accretive and $R(A_q) = L^q(\Omega)$.

**Proof.** It is easy to show that $A_q$ is accretive. For $f \in L^q(\Omega), \lambda > 0$, $u_\lambda$ be the solution of

\[ L_q u_\lambda + M_{q, \lambda} u_\lambda = f. \quad (4.3) \]

$u_\lambda$ is the fixed point of the mapping

\[ u \mapsto (1 + \lambda L_q)^{-1} (f + Pu) \quad (4.4) \]

which is a strict contraction from $L^q(\Omega)$ to itself in view of Lemma 2.4 and Proposition 2.3. Forming the scalar product of (4.3) and $F_q(u_\lambda - Pu_\lambda)$, and noting (4.2) we get

\[ (L_q u_\lambda, F_q(u_\lambda - Pu_\lambda)) + \| u_\lambda - Pu_\lambda \|^{q / 2} = (f, F_q(u_\lambda - Pu_\lambda)). \quad (4.5) \]

By Lemma 3.2

\[ (L_q u_\lambda, F_q(u_\lambda - Pu_\lambda)) \geq (L^q(\Omega), F_q(u_\lambda - Pu_\lambda)) \geq ((L^q(\Omega))^+, F_q(u_\lambda - Pu_\lambda)) \geq -\| (L^q(\Omega))^+ \| \| u_\lambda - Pu_\lambda \|^{q - 1}. \quad (4.6) \]

From (4.5) and (4.6) it follows that

\[ \| u_\lambda - Pu_\lambda \|^{q / 2} \leq (\| f \| + \| (L^q(\Omega))^+ \|) \| u_\lambda - Pu_\lambda \|^{q - 1}. \]
which implies
\[\|M_{q,\lambda}u_{\lambda}\|_{q} \leq \|f\|_{q} + \|(\mathcal{L}\Psi)^{+}\|_{q}\] (4.7)
\[\|L_{q}u_{\lambda}\|_{q} \leq 2\|f\|_{q} + \|(\mathcal{L}\Psi)^{+}\|_{q}\] (4.8)

Write (4.3) with \(\lambda, \mu > 0\), take the difference, multiply by \(F_{q}(u_{\lambda}-u_{\mu})\) and integrate over \(\Omega\). This yields
\[(L_{q}u_{\lambda}-L_{q}u_{\mu}, F_{q}(u_{\lambda}-u_{\mu})) + (M_{q,\lambda}u_{\lambda}-M_{q,\mu}u_{\mu}, F_{q}(u_{\lambda}-u_{\mu})) = 0.\] (4.9)

By Lemma 2.5 (4.9), and the accretiveness of \(M_{q}\)
\[c_{q}\|u_{\lambda}-u_{\mu}\|_{q} \leq \|M_{q,\lambda}u_{\lambda}-M_{q,\mu}u_{\mu}\|_{q}\] (4.10)
\[\leq K\|M_{q,\lambda}u_{\lambda}-M_{q,\mu}u_{\mu}\|_{q}^{q-1}.\] (4.11)

In view of (4.7) the right side of (4.11) goes to 0 as \(\lambda, \mu \to 0\). Hence there exists an element \(u\) of \(W^{1,q}(\Omega)\) such that
\[u_{\lambda} \to u \quad \text{in} \quad W^{1,q}(\Omega).\] (4.12)

From (4.8) and the demiclosedness of \(L_{q}\) it follows that \(L_{q}u_{\lambda} \to L_{q}u\) in \(L^{q}(\Omega)\), and also \(M_{q,\lambda}u_{\lambda} \to f\) in \(L^{q}(\Omega)\). By (4.7), (4.12) \(Pu_{\lambda} \to u\) in \(L^{q}(\Omega)\). By (4.1) \(M_{q,\lambda}u_{\lambda} \to M_{q}Pu_{\lambda}\). Hence \(f-L_{q}u \in M_{q}Pu_{\lambda}\). Since \(f \in L^{q}(\Omega)\) is arbitrary, \(A_{q}\) is surjective. From \((\mathcal{L}\Psi)^{+}\) it follows that \((1+\lambda\alpha)\|u-\hat{u}\|_{q} \leq \|f-\tilde{f}\|_{q}\) if \(f, \tilde{f} \in (1+\lambda A_{q})u\), \(\tilde{f} \in (1+\lambda A_{q})\hat{u}\), \(\lambda > 0\). Hence \(A_{q}\) is m-accretive.

**Lemma 4.1.** If \(f \in L^{q}(\Omega) \cap L^{r}(\Omega), q \geq 1, r \geq 1, \) then for any \(\lambda > 0\)
\[(1+\lambda L_{q})^{-1}f = (1+\lambda L_{r})^{-1}f.\] (4.13)

**Proof.** The conclusion follows easily from the definition of \(L_{q}, L_{r}\), and Lemma 2.4.

**Proposition 4.2.** For \(p<q \leq p^{*}\) \(A_{q}\) is m-accretive.

**Proof.** Let \(f, \tilde{f} \in L^{p}(\Omega) \cap L^{q}(\Omega)\) and \(\varepsilon > 0\). Put
\[u = (1+\varepsilon A_{p})^{-1}f, \quad \hat{u} = (1+\varepsilon A_{p})^{-1}\tilde{f}.\]

By Sobolev's imbedding theorem
\[W^{1,p}(\Omega) \subset L^{q}(\Omega).\] (4.14)

In view of Lemma 2.5 and (4.14) \(u, \hat{u} \in L^{q}(\Omega)\). Let \(u_{\lambda}, \hat{u}_{\lambda}\) be the solutions of
\[\begin{align*}
(1+\varepsilon L_{p} + \varepsilon M_{\rho})u_{\lambda} &= f, \\
(1+\varepsilon L_{p} + \varepsilon M_{\rho})\hat{u}_{\lambda} &= \tilde{f}.
\end{align*}\] (4.15)
$u_{\lambda}$ is the fixed point of the strictly contractive mapping

$$Tv = \left( 1 + \frac{\lambda \epsilon}{\lambda + \epsilon} L_{p} \right)^{-1} \frac{\lambda f + \epsilon Pv}{\lambda + \epsilon}$$

from $L^{p}(\Omega)$ to itself. Since $f, \Psi \in L^{p}(\Omega) \cap L^{q}(\Omega)$

$$(\lambda f + \epsilon Pv)/(\lambda + \epsilon) \in L^{p}(\Omega) \cap L^{q}(\Omega)$$

if $v \in L^{p}(\Omega) \cap L^{q}(\Omega)$. Hence by Lemma 4.1 $T$ is also a strict contraction from $L^{p}(\Omega) \cap L^{q}(\Omega)$ to itself. Consequently (4.15) may be rewritten as

$$(1+\epsilon L_{q}+\epsilon M_{q,\lambda})u_{\lambda} = f, \quad (1+\epsilon L_{q}+\epsilon M_{q,\lambda})\hat{u}_{\lambda} = \hat{f}.$$  (4.16)

Since $L_{q}$ and $M_{q,\lambda}$ are accretive

$$\|u_{\lambda} - \hat{u}_{\lambda}\|_{q} \leq \|f - \hat{f}\|_{q}.$$  (4.17)

Since $u_{\lambda} - \hat{u}_{\lambda} \rightarrow u - \hat{u}$ in $W^{1,p}(\Omega) \subset L^{q}(\Omega)$ by the proof of Proposition 4.1 we get from (4.17)

$$\|u - \hat{u}\|_{q} \leq \|f - \hat{f}\|_{q}.$$  (4.18)

Once this is established for $f, \hat{f} \in L^{p}(\Omega) \cap L^{q}(\Omega)$ the remaining part of the proof is accomplished in the usual obvious manner.

**PROPOSITION 4.3.** $A_{1}$ is $m$-accretive and $R(A_{1}) = L^{1}(\Omega)$. If for $f, \hat{f} \in L^{1}(\Omega)$

$$A_{1}u + g = f, \quad g \in M_{1}u, \quad A_{1}\hat{u} + \hat{g} = \hat{f}, \quad \hat{g} \in M_{1}\hat{u},$$

then

$$\alpha \|u - \hat{u}\|_{1} + \|g - \hat{g}\|_{1} \leq \|f - \hat{f}\|_{1}.$$  (4.19)

**PROOF.** As is easily seen

$$(g - \hat{g}, \text{sign}^{0}(u - \hat{u})) \geq 0$$

if $g \in M_{1}u$ and $\hat{g} \in M_{1}\hat{u}$. Combining this with (2.21) the accretivity of $A_{1}$ follows. Suppose $f \in L^{1}(\Omega) \cap L^{p}(\Omega)$. Let $u_{\lambda}$ be the solution of

$$L_{1}u_{\lambda} + M_{1,\lambda}u_{\lambda} = f.$$  (4.20)

$u_{\lambda}$ is the fixed point of the mapping

$$Tv = (1 + \lambda L_{1})^{-1}(\lambda f + Pv).$$

In view of Proposition 2.3, Lemma 4.1 and the fact $f, \Psi \in L^{1}(\Omega) \cap L^{p}(\Omega)$ $T$ is a strict contraction from $L^{1}(\Omega) \cap L^{p}(\Omega)$ to itself. Hence (4.20) may be rewritten as

$$L_{p}u_{\lambda} + M_{p,\lambda}u_{\lambda} = f.$$  (4.21)

Multiply both sides of (4.20) by $\text{sign}^{+}(\Psi - u_{\lambda})$ and integrate over $\Omega$. Noting (3.2) we get...
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\[ (L_{1}u_{\lambda}, \text{sign}^{+}(\Psi-u_{\lambda})) - \|M_{1,\lambda}u_{\lambda}\|_{1} = (f, \text{sign}^{+}(\Psi-u_{\lambda})) \].

Using Lemma 3.3

\[ \|M_{1,\lambda}u_{\lambda}\|_{1} \leqq - (f, \text{sign}^{+}(\Psi-u_{\lambda})) + (\mathcal{L}\Psi, \text{sign}^{+}(\Psi-u_{\lambda})) \leqq \|f\|_{1} + \|(\mathcal{L}\Psi)^{+}\|_{1} \].

From (2.22), (4.20), (4.22) it follows that

\[ \alpha \|u_{\lambda}\|_{1} \leqq \|L_{1}u_{\lambda}\|_{1} \leqq 2\|f\|_{1} + \|(\mathcal{L}\Psi)^{+}\|_{1} \].

By the proof of Proposition 4.1 \( u_{\lambda} \rightharpoonup u \) in \( W^{1,p}(\Omega) \), \( L_{p}u_{\lambda} \rightharpoonup L_{p}u \), \( M_{p,\lambda}u_{\lambda} \rightharpoonup g \) \( = f - L_{p}u \) in \( L^{p}(\Omega) \) and \( g \in M_{p}u \). Since \( u_{\lambda}, \ L_{p}u_{\lambda}=L_{1}u_{\lambda}, \ M_{p,\lambda}u_{\lambda}=M_{p,\lambda}u_{\lambda} \) and \( \|M_{1,\lambda}u_{\lambda}\|_{1} \leqq \|f\|_{1} + \|(\mathcal{L}\Psi)^{+}\|_{1} \), (4.22) \( u_{\lambda}, \ L_{p}u_{\lambda}, \ g \) all belong to \( L^{1}(\Omega) \). Since

\[ u + L_{p}u = f - g + u \in L^{1}(\Omega) \cap L^{p}(\Omega) \]

it follows from Lemma 4.1 that

\[ u = (1+L_{p})^{-1}(f-g+u) = (1+L_{1})^{-1}(f-g+u) \],

\[ g = M_{1}u \].

Thus we have proved \( R(A_{1}) \supset L^{1}(\Omega) \cap L^{p}(\Omega) \).

Suppose next \( f, \ f \in L^{1}(\Omega) \cap L^{p}(\Omega) \) and

\[ L_{1}u + g = f, \quad g \in M_{1}u, \quad L_{1}\hat{u} + \hat{g} = f, \quad \hat{g} \in M_{1}\hat{u} \].

In view of Proposition 2.3 \( u, \ \hat{u} \) are uniquely determined by \( f, \ f \). Let \( u_{\lambda}, \ \hat{u}_{\lambda} \) be the solutions of

\[ L_{1}u_{\lambda} + M_{1,\lambda}u_{\lambda} = f, \quad L_{1}\hat{u}_{\lambda} + M_{1,\lambda}\hat{u}_{\lambda} = f \].

Then by the above argument \( u_{\lambda} \rightharpoonup u, \ \hat{u}_{\lambda} \rightharpoonup \hat{u} \) in \( W^{1,p}(\Omega) \), \( M_{1,\lambda}u_{\lambda} \rightarrow g, \ M_{1,\lambda}\hat{u}_{\lambda} \rightarrow \hat{g} \) in \( L^{p}(\Omega) \). Multiplying both sides of

\[ L_{1}u_{\lambda} - L_{1}\hat{u}_{\lambda} + M_{1,\lambda}u_{\lambda} - M_{1,\lambda}\hat{u}_{\lambda} = f - \hat{f} \]

by \( \text{sign}^{\theta}(u_{\lambda}-\hat{u}_{\lambda}) \) and noting

\[ \langle (u_{\lambda}-Pu_{\lambda}) - (\hat{u}_{\lambda}-P\hat{u}_{\lambda}), \ \text{sign}^{\theta}(u_{\lambda}-\hat{u}_{\lambda}) \rangle = \| (u_{\lambda}-Pu_{\lambda}) - (\hat{u}_{\lambda}-P\hat{u}_{\lambda}) \|_{1} \]

we get

\[ (L_{1}u_{\lambda} - L_{1}\hat{u}_{\lambda}, \ \text{sign}^{\theta}(u_{\lambda}-\hat{u}_{\lambda})) + \| M_{1,\lambda}u_{\lambda} - M_{1,\lambda}\hat{u}_{\lambda} \|_{1} \leqq \| f - \hat{f} \|_{1} \].

From this equality and Proposition 2.3 it follows that

\[ \alpha \| u_{\lambda} - \hat{u}_{\lambda} \|_{1} + \| M_{1,\lambda}u_{\lambda} - M_{1,\lambda}\hat{u}_{\lambda} \|_{1} \leqq \| f - \hat{f} \|_{1} \].

(4.25)
In view of Fatou's lemma
\[
\|u - \hat{u}\|_1 \leq \liminf \|u_\lambda - \hat{u}_\lambda\|_1. \quad (4.26)
\]
Let \( \Omega_r = \{ x \in \Omega, |x| < r \} \) for \( r > 0 \). Then \( M_{1,\lambda}u_\lambda \to g, M_{1,\lambda}\hat{u}_\lambda \to \beta \) in \( L^1(\Omega_r) \).
Hence
\[
\int_{\Omega_r} |g - \beta| \, dx \leq \liminf \int_{\Omega_r} |M_{1,\lambda}u_\lambda - M_{1,\lambda}\hat{u}_\lambda| \, dx
\]
\[
\leq \liminf \|M_{1,\lambda}u_\lambda - M_{1,\lambda}\hat{u}_\lambda\|_1.
\]
Since \( r > 0 \) is arbitrary
\[
\|g - \beta\|_1 \leq \liminf \|M_{1,\lambda}u_\lambda - M_{1,\lambda}\hat{u}_\lambda\|_1. \quad (4.27)
\]
From (4.25), (4.26) and (4.27) it follows that
\[
\alpha \|u - \hat{u}\|_1 + \|g - \beta\|_1 \leq \|f - \beta\|_1. \quad (4.28)
\]
Finally suppose \( f, \beta \) are arbitrary elements of \( L^1(\Omega) \). Let \( \{f_n\}, \{\beta_n\} \) be sequences of \( L^1(\Omega) \cap L^p(\Omega) \) tending to \( f, \beta \) respectively in \( L^1(\Omega) \), and \( u_n, \hat{u}_n, g_n, \hat{g}_n \) be such that
\[
L_1u_n + g_n = f_n, \quad g_n \in M_1u_n, \quad \hat{g}_n \in M_1\hat{u}_n.
\]
An application of (4.28) yields the existence of the elements \( u, g \in L^1(\Omega) \) such that \( u_n \to u, g_n \to g \) in \( L^1(\Omega) \). Replacing by a subsequence if necessary we may assume \( u_n(x) \to u(x), g_n(x) \to g(x) \) at almost every \( x \in \Omega \). Since \( g_n \leq 0 \) a.e. in \( \Omega \) the same is true of \( g \). If \( u(x) > \Psi(x) \), then \( u_n(x) > \Psi(x) \) if \( n \) is sufficiently large, and hence \( g_n(x) = 0 \) for these values of \( n \), which implies \( g(x) = 0 \). Consequently we have proved \( g \in M_1u \). Since \( L_1 \) is a closed operator \( u \in D(L_1) \) and \( L_1u + g = f \). Hence we have established \( R(A_1) = L^1(\Omega) \). Letting \( n \to \infty \) in
\[
\alpha \|u_n - \hat{u}_n\|_1 + \|g_n - \beta_n\|_1 \leq \|f_n - \beta_n\|_1
\]
we obtain (4.19).

**Lemma 4.2.** \( j(\Psi^+|_\Gamma) \in L^1(\Gamma) \).

**Proof.** By (4.4) and (4.8)
\[
0 \leq j(x, \Psi^+(x)) \leq \beta^-(x, \Psi^+(x)) \Psi^+(x) \leq -\partial \Psi(x)/\partial n \cdot \Psi^+(x). \quad (4.29)
\]
By the assumption and Sobolev's imbedding theorem
\[
\frac{\partial \Psi}{\partial n} \in L^{p(N-1)/(N-p)}(\Gamma), \quad \Psi^+|_\Gamma \in H^{1/2}(\Gamma) \subset L^{2(N-1)/(N-2)}(\Gamma).
\]
If we choose \( q \) and \( r \) so that (3.10) holds, then \( \partial \Psi/\partial n \in L^q(\Gamma), \Psi^+|_\Gamma \in L^r(\Gamma), \) and hence \( \partial \Psi/\partial n \cdot \Psi^+|_\Gamma \in L^1(\Gamma) \). Combining this and (4.29) we get the desired result.

Let \( \phi : L^2(\Omega) \to [0, \infty] \) be the function defined by
\[\psi(u) = \begin{cases} \frac{1}{2} \int_{\Omega} \left( \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + au^2 \right) \, dx + \int_{\Gamma} j(u |_{\Gamma}) \, d\Gamma & \text{if } \Psi \leq u \in H^1(\Omega), \quad j(u |_{\Gamma}) \in L^1(\Gamma), \\ \infty & \text{otherwise.} \end{cases} \] (4.30)

In view of Lemma 4.2 \( \Psi^+ |_{\Gamma} \in D(\psi) \), and hence \( \phi \) is proper convex. Let \( B \) be the linear differential operator

\[ B = \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i} + c - \alpha. \]

**Lemma 4.3.** Let \( f \in L^2(\Omega), u \in D(\psi) \). Then \( f \in \partial \psi(u) + Bu \) if and only if

\[ a(u, v-u) + \Phi(v |_{\Gamma}) - \Phi(u |_{\Gamma}) \geq \langle f, v-u \rangle \] (4.31)

for every \( v \) satisfying \( \Psi \leq v \in H^1(\Omega), j(v |_{\Gamma}) \in L^1(\Gamma) \), where \( \Phi \) is the function defined by (2.9). \( \partial \psi + B \) is demiclosed.

**Proof.** The proof of the first part is straightforward. The demiclosedness of \( \partial \psi + B \) is verified without difficulty with the aid of the first part of the lemma and noting that \( a(u, u)^{1/2} \) is a norm of \( H^1(\Omega) \).

By (1.9) and Proposition 4.2 the mapping \( A_2 \) is defined and \( m \)-accretive in \( L^2(\Omega) \).

**Lemma 4.4.** \( A_2 = \partial \psi + B \).

**Proof.** Suppose first that \( f \in A_2 u, f, u \in L^2(\Omega) \). Let \( u_\lambda \) be the solution of

\[ L_p u_\lambda + M_{p, \lambda} u_\lambda = f = L_2 u_\lambda + M_{2, \lambda} u_\lambda, \] (4.32)

where we used Lemma 4.1 as in the proof of Proposition 4.2. Let \( v \in D(\psi) \). From (4.32) and the definition of \( L_2 \) it follows that

\[ a(u_\lambda, v-u_\lambda) + \Phi(v |_{\Gamma}) - \Phi(u_\lambda |_{\Gamma}) \geq \langle f, v-u_\lambda \rangle \] (4.33)

If \( u_\lambda(x) - Pu_\lambda(x) < 0 \) at some point \( x \), then \( u_\lambda(x) < \Psi(x) \leq v(x) \) there. Consequently \( M_{2, \lambda} u_\lambda \cdot (v-u_\lambda) \leq 0 \) a.e. Hence from (4.33) it follows that

\[ a(u_\lambda, v-u_\lambda) + \Phi(v |_{\Gamma}) - \Phi(u_\lambda |_{\Gamma}) \geq \langle f, v-u_\lambda \rangle \] (4.34)

By the proof of Proposition 4.1 \( u_\lambda \rightarrow u \) in \( W^{1,p}(\Omega) \subset L^q(\Omega) \). It is easily shown that \( u \) satisfies (4.31). The remaining part of the proof is omitted.

**Lemma 4.5.** Suppose \( f \in L^q(\Omega) \cap L^q(\Omega) \). Then for \( \epsilon > 0, 1 \leq q \leq 2 \)

\[ (1 + \epsilon A_1)^{-1} f = (1 + \epsilon A_q)^{-1} f = (1 + \epsilon A_q)^{-1} f \] (4.35)

**Proof.** In case \( p < q \leq 2 \) (4.35) is an immediate consequence of the definition of \( A_q \). In case \( 1 \leq q \leq p \) (4.35) is easily established with the aid of Proposition 4.1 and 4.3.
REMARK. It follows from Lemma 4.3 that if $f \in L^q(\Omega) \cap L^r(\Omega)$, $1 \leq q < r \leq 2$, then $(1+\varepsilon A_q)^{-1}f = (1+\varepsilon A_r)^{-1}f$ for $\varepsilon > 0$.

PROPOSITION 4.4. For $1 \leq q \leq 2$

$$D(A_q) := \{u \in L^q(\Omega) : u \geq \Psi \text{ a.e.} \} \quad (4.36)$$

where the left side of (4.36) is the closure of $D(A_q)$ in $L^q(\Omega)$.

PROOF. It is obvious that the left side of (4.36) is contained in the right side.

(i) We first prove (4.36) for $1 < q \leq 2$. Let $\Psi \leq u \in L^q(\Omega)$. We set

$$u_n = (1 + n^{-1}L_q)^{-1}(u + n^{-1}\mathcal{L}\Psi) \quad (4.37)$$

Then

$$\Psi - u_n + n^{-1}\mathcal{L}\Psi - n^{-1}L_q u_n = \Psi - u \leq 0 \quad (4.38)$$

Form the inner product of (4.37) and $((\Psi - u_n)^{+})^{q-1}$. This yields

$$\|\Psi - u_n^{+}\|_{q}^{q} + n^{-1}(\mathcal{L}\Psi - L_q u_n, ((\Psi - u_n)^{+})^{q-1}) \leq 0 \quad (4.39)$$

By Lemma 3.2 and (3.3)

$$L_q u_n, ((\Psi - u_n)^{+})^{q-1} \leq (L\Psi - L_q u_n, ((\Psi - u_n)^{+})^{q-1}) \quad (4.40)$$

Combining (4.38) and (4.39) we get $\Psi \leq u_n$. Hence $u_n \in D(L_q) \cap D(M_q) = D(A_q)$.

Since $C^{0}_{c}(\Omega) \subseteq D(L_q)$, $D(L_q)$ is dense in $L^q(\Omega)$. Hence $\|v - (1 + n^{-1}L_q)^{-1}v\|_{q} \to 0$ as $n \to \infty$ for any $v \in L^q(\Omega)$. Thus it follows easily that $u_n \to u$ in $L^q(\Omega)$, and hence $u \in D(A_q)$.

(ii) In case $q = 1$ the proof is almost identical with that of (i). Form the inner product of (4.37) and $\text{sign}^+_0((\Psi - u_n)^{+})$, and use Lemma 3.3.

(iii) In this step we consider the case $q = 2$. Noting Lemma 4.4 and $D(\partial \psi + B) = D(\partial \tilde{\psi}) = D(\tilde{\psi})$ it suffices to show

$$D(\tilde{\psi}) \supset \{u \in L^q(\Omega) : u \geq \Psi \text{ a.e.} \} \quad (4.41)$$

Let $\chi$ be a smooth function such that $\chi(0) = 0$, $\chi(t) > 0$ for $t > 0$, $\chi(t) = 1$ for $t \geq 1$ and $0 \leq \chi(t) \leq 1$ for all $t \geq 0$. Set $\rho(x) = \text{dist}(x, \partial \Omega)$ and $\chi_n(x) = \chi(n \rho(x))$. Then $\chi_n \in C^{0}_{c}(\tilde{\Omega})$ if $n$ is sufficiently large. Let $u$ be an arbitrary element such that $\Psi \leq u \in L^q(\Omega)$. Let $v_n$ be a sequence in $H^1(\Omega)$ such that $v_n \to u$ in $L^q(\Omega)$ and $w_n(x) = \max\{v_n(x), \Psi(x)\}$. Then $\Psi \leq w_n \in H^1(\Omega)$ and

$$\int_{\Omega} (u - w_n)^2 dx = \int_{\Omega} \left(\max\{u, \Psi\} - \max\{v_n, \Psi\}\right)^2 dx \quad (4.42)$$

$$\leq \int_{\Omega} (u - v_n)^2 dx \to 0 \quad \text{as} \quad n \to \infty.$$ 

Put $u_n = (1 - \chi_n)\Psi^{+} + \chi_n w_n$. Then $\Psi \leq u_n \in H^1(\Omega)$ and $j(u_n, \Gamma) = j(\Psi^{+}, \Gamma) \in L^1(\Gamma)$.
by Lemma 4.2. Hence \( u_n \in D(\phi) \). Now,
\[
\int_\Omega (u-u_n)^2 \, dx = \int_{\rho<1/n} (u-u_n)^2 \, dx + \int_{\rho \geq 1/n} (u-w_n)^2 \, dx .
\] (4.42)

By (4.41) the second term on the right of (4.42) tends to 0 as \( n \to \infty \), while as for the first term
\[
\int_{\rho<1/n} (u-u_n)^2 \, dx = \int_{\rho<1/n} (u-\Psi^*-\chi_n(w_n-\Psi^*))^2 \, dx
\leq 2\int_{\rho<1/n} (u-\Psi^*)^2 \, dx + 2\int_{\rho<1/n} (w_n-\Psi^*)^2 \, dx \to 0
\]
since
\[
\int_{\rho<1/n} w_n^2 \, dx \leq 2\int_{|x|>n} (w_n-u)^2 \, dx + 2\int_{\rho<1/n} u^2 \, dx
\]
Hence \( u_n \to u \) in \( L^2(\Omega) \) which implies \( u \in \overline{D(\psi)} \).

(iv) In the final step we consider the case \( p < q < 2 \). Suppose that \( \Psi \leq u \in L^q(\Omega) \). If we define
\[
\Psi \leq u_n \in L^1(\Omega) \cap L^2(\Omega)
\] (4.43)
then \( \Psi \leq u_n \in L^1(\Omega) \cap L^2(\Omega) \), and
\[
\int_\Omega |u-u_n|^q \, dx \leq \int_{|x| > n} |u-\Psi|^q \, dx + \int_{u > n} |u-\Psi|^q \, dx \to 0
\]
as \( n \to \infty \). Thus it suffices to show that any element \( u \) satisfying \( \Psi \leq u \in L^q(\Omega) \cap L^q(\Omega) \) belongs to \( \overline{D(A_q)} \). Let
\[
u_n=(1+n^{-1}A_p)^{-1}u=(1+n^{-1}A_2)^{-1}u=(1+n^{-1}A_q)^{-1}u.
\]
Here we recall Lemma 4.5. By (i) and (iii) \( u \in \overline{D(A_p)} \cap \overline{D(A_2)} \). Hence as \( n \to \infty \), \( u_n \to u \) in \( L^q(\Omega) \cap L^q(\Omega) \subset L^q(\Omega) \). Since \( u_n \in D(A_q) \) it follows that \( u \in \overline{D(A_q)} \).

REMARK. From the proof of (iv) of Proposition 4.4 it follows that for \( \Psi \leq u \in L^q(\Omega) \) there exists a sequence \( \{u_n\} \subset D(A_p) \cap L^q(\Omega) \) such that \( u_n \to u \) in \( L^q(\Omega) \).

\S 5. \( L^2 \)-estimate of solutions.

For \( f \in W^{1,1}(0, T; L^q(\Omega)) \), \( 1 \leq q \leq 2 \), \( u_0 \in \overline{D(A_q)} \) and \( 0 \leq s \leq t \leq T \), set
\[
U_q(t, s; f)u_0=\lim_{n \to \infty} \prod_{i=1}^{n} \left\{ 1 + \frac{t-s}{n} \left( A_q - f \left( s + \frac{i}{n} (t-s) \right) \right) \right\}^{-1} u_0 .
\] (5.1)
The convergence of the right side of (5.1) was established by M.G. Crandall and A. Pazy (Theorem 5.1 of [7]). If $1 < q \leq 2$ and $u_0 \in D(A_q)$, then $u(t) = U_q(t, 0; f)u_0$ is the unique strong solution of
\[ du(t)/dt + A_q u(t) \ni f(t), \quad 0 \leq t \leq T, \tag{5.2} \]
\[ u(0) = u_0, \tag{5.3} \]
i.e. $u(t)$ is an absolutely continuous (actually Lipschitz continuous) function in $[0, T]$ with values in $L^q(\Omega)$, $u(t) \in D(A_q)$ and (5.2) holds a.e. in $[0, T]$, and (5.3) holds.

**Lemma 5.1.** If $u \in D(L_p)$, $0 \leq v \in W^{2,p}(\Omega)$, $\partial v / \partial n = (\partial \Psi / \partial n)^+$ on $\Gamma$, then
\[ (L_p u - \mathcal{L} v, ((u-v)^+)^{p-1}) \geq 0. \tag{5.4} \]

**Proof.** First we note $(\partial \Psi / \partial n)^+ \in W^{1-1/p,p}(\Gamma)$. Let us begin with the case
\[ u \in D(L_2) \cap L^p(\Omega), \quad L_2 u \in L^p(\Omega). \]
Let $\phi_n$ be the function defined by (3.24) with $p$ in place of $q$. Let $u_\epsilon$ be the solution of $L_{2,\epsilon} u_\epsilon = L_2 u$. Noting $v \in H^1(\Omega)$ and (2.11)
\[ (L_2 u, \phi_n(u_\epsilon - v)) = (L_{2,\epsilon} u_\epsilon, \phi_n(u_\epsilon - v)) \]
\[ = \int_{\Gamma} \beta_\epsilon(u_\epsilon) \phi_n(u_\epsilon - v) d\Gamma + a(u_\epsilon, \phi_n(u_\epsilon - v)). \tag{5.5} \]
If $u_\epsilon(x) > v(x)$ at some point $x$, then $u_\epsilon(x) > 0$, which implies $\beta_\epsilon(x, u_\epsilon(x)) \geq 0$. Hence
\[ \int_{\Gamma} \beta_\epsilon(u_\epsilon) \phi_n(u_\epsilon - v) d\Gamma \geq 0. \]
Combining this and (5.5)
\[ (L_2 u, \phi_n(u_\epsilon - v)) \geq a(u_\epsilon, \phi_n(u_\epsilon - v)). \tag{5.6} \]
Repeating the arguments running from (3.8) to (3.12) and using the hypothesis we get
\[ (\mathcal{L} v, \phi_n(u_\epsilon - v)) = - \int_{\Gamma} (\frac{\partial \Psi}{\partial n})^+ \phi_n(u_\epsilon - v) d\Gamma + a(v, \phi_n(u_\epsilon - v)) \]
\[ \leq a(v, \phi_n(u_\epsilon - v)). \tag{5.7} \]
Combining (5.6) and (5.7), and using Lemma 2.1
\[ (L_2 u - \mathcal{L} v, \phi_n(u_\epsilon - v)) \geq a(u_\epsilon - v, \phi_n(u_\epsilon - v)) \]
\[ \geq a(u_\epsilon - v, \phi_n(u_\epsilon - v)) \geq 0. \tag{5.8} \]
In view of Proposition 2.1 and (3.1) $u_\epsilon - v \rightarrow u - v$ in $L^{p'}(\Omega)$ as $\epsilon \rightarrow 0$, and so $\phi_n(u_\epsilon - v) \rightarrow \phi_n(u - v)$ in $L^{p'}(\Omega)$. Hence first letting $\epsilon \rightarrow 0$ and then $n \rightarrow \infty$ in (5.8) we get (5.4) in this special case. The conclusion in the general case is
easily obtained by noting
\[ |(r^+)^{p-1}-(s^+)^{p-1}| \leq K|r-s|^{p-1}, \quad r \geq 0, \ s \geq 0. \]

Let \( G_q(t), 1 \leq q < \infty \), be the semigroup generated by the realization of \(-\mathcal{L}\) under the Neumann boundary condition \( \partial u/\partial n = 0 \) on \( \Gamma \). \( G_q(t) \) is an integral operator with kernel \( G(t, x, y) \) satisfying
\[ 0 \leq G(t, x, y) \leq C t^{-N/2} H(t, x-y), \]
(5.9)
\[ |(\partial/\partial x_i)G(t, x, y)| \leq C t^{-(N+1)/2} H(t, x-y), \]
(5.10)
\[ |(\partial/\partial t)G(t, x, y)| \leq C t^{-N/2-1} H(t, x-y), \]
(5.11)
where \( H(t, x) = \exp(-c|x|^2/t) \), and \( C \) and \( c \) are some positive constants. Part of the above estimates were established in [12]. \( G(t, x, y) \) does not depend on \( q \), and we write simply \( G(t) \) instead of \( G_q(t) \).

**Lemma 5.2.** Let \( f \in W^{1,1}(0, T ; L^p(\Omega)) \) and \( \Psi \leq \Psi^+ \leq v_0 \in L^q(\Omega) \). Let \( v \) be such that
\[ v \in C([0, T]; L^p(\Omega)) \cap C((0, T]; W^{2,p}(\Omega)), \]
(5.12)
\[ \partial v/\partial t + \mathcal{L}v = f^+ + (\mathcal{L}\Psi)^+ \quad \text{in} \quad \Omega \times (0, T), \]
(5.13)
\[ \partial v/\partial n = (\partial \Psi/\partial n)^+ \quad \text{on} \quad \Gamma \times (0, T), \]
(5.14)
\[ v(x, 0) = v_0(x) \quad \text{in} \quad \Omega. \]
(5.15)
Then \( v(x, t) \geq \Psi^+(x) \ a.e. \ in \ \Omega \times (0, T) \).

**Proof.** The conclusion is easily established by integrating by part in
\[ (\partial v/\partial t + \mathcal{L}v, v^-) \leq 0, \quad ((\partial/\partial t + \mathcal{L})(\Psi-v), (\Psi-v)^+) \leq 0, \]
where \( v^- = \min \{ v, 0 \} \). Here we note \( u(t) \in H^1(\Omega) \subset L^p(\Omega) \) for \( t > 0 \).

**Lemma 5.3.** Let \( u \) be the strong solution of (5.2) and (5.3) with \( f \in W^{1,1}(0, T ; L^p(\Omega)), u_0 \in D(A_p) \) and \( q = p \). Let \( v \) be the function satisfying (5.12)-(5.15) with \( v_0 \) replaced by \( u_0^+ \). Then
\[ \Psi \leq u \leq \Psi^+, \ a.e. \ in \ \Omega \times (0, T). \]
(5.16)

**Proof.** Let \( g \) be such that
\[ du(t)/dt + L_p u(t) + g(t) = f(t), \quad g(t) \in M_p u(t) \ a.e. \]
(5.17)
Then
\[ (\partial(u-v)/\partial t + L_p u - L_p v + g = f - f^+ - (\mathcal{L}\Psi)^+ \leq 0. \]
(5.18)
Hence
\[ (\partial(u-v)/\partial t, ((u-v)^{p-1}) + (L_p u - L_p v, ((u-v)^{p-1}) + (g, ((u-v)^{p-1}) \leq 0. \]
(5.19)
In view of Lemma 5.2 \( v \geq 0 \ a.e. \ in \ \Omega \times (0, T) \), and hence Lemma 5.1 implies
If \( u > v \) somewhere, then by Lemma 5.2 \( u > \Psi \) and hence \( g = 0 \) there. Consequently

\[
(g, ((u-v)^{+})^{p-1})=0.
\]

Combining (5.19), (5.20), and (5.21) we get

\[
\|u(t)-v(t)\|_{p} \leq \|u_{0}-u_{0}^{+}\|_{p} = 0.
\]

Thus we conclude \( u \leq v \) a.e. in \( \Omega \times (0, T) \). \( \Psi \leq u \) is clear since \( u(t) \in D(A_{p}) \) for every \( t \in [0, T] \), and the proof of the lemma is complete.

**Proposition 5.1.** Suppose that \( f \in W^{1,1}(0, T; L^{q}(\Omega)) \), \( 1 \leq q \leq 2 \), and \( \Psi \leq u_{0} \in L^{q}(\Omega) \). Let \( u(t) = U_{q}(t, 0; f)u_{0} \) and \( v \) be the solution of (5.13), (5.14), (5.15) with \( u_{0}^{+} \) in place of \( v_{0} \). Then

\[
\Psi \leq u \leq v \quad \text{a.e. in} \; \Omega \times (0, T).
\]

**Proof.** Let \( f_{n} \in W^{1,1}(0, T; L^{q}(\Omega) \cap L^{p}(\Omega)) \) and \( u_{0n} \in D(A_{p}) \cap L^{q}(\Omega) \) be such that \( f_{n} \rightarrow f, \; u_{0n} \rightarrow u_{0} \) in \( W^{1,1}(0, T; L^{q}(\Omega), L^{q}(\Omega)) \) respectively. Here we recall the remark after Proposition 4.4. Let

\[
u_n(t) = U_{q}(t, 0; f_{n})u_{0n} \]

where the second equality is due to the remark after Lemma 4.5, and \( v_{n} \) be the solution of (5.13), (5.14), (5.15) with \( f_{n}^{+}, u_{0n}^{+} \) in place of \( f^{+}, v_{0} \) respectively. Then for each fixed \( t > 0 \)

\[
v_{n}(t) - v(t) = G(t)(u_{0n}^{+} - u_{0}^{+}) + \int_{0}^{t} G(t-s)(f_{n}^{+}(s) - f^{+}(s))ds \rightarrow 0
\]

in \( L^{q}(\Omega) \) as \( n \to \infty \). In view of Lemma 5.3

\[
\Psi \leq u_{n} \leq v_{n} \quad \text{a.e. in} \; \Omega \times (0, T).
\]

By the fact that \( U_{q}(t, 0; f_{n}) \) is a contraction and Theorem 4.1 of M.G. Crandall and A. Pazy \( \Pi \) \( u_{n}(t) \rightarrow u(t) \) in \( L^{q}(\Omega) \). Going to the limit in (5.24) we conclude (5.23).

Let \( w \) be the solution of the boundary value problem

\[
w = 0 \quad \text{in} \; \Omega, \quad \frac{\partial w}{\partial n} = (\frac{\partial \Psi}{\partial n})^{+} \quad \text{on} \; \Gamma.
\]

In view of the a priori estimate of the elliptic boundary value problem

\[
\|w\|_{2, p} \leq C \left( \left( \frac{\partial \Psi}{\partial n} \right)^{+} \right)_{1-1/p, p}.
\]

\[\]
The function \( v \) in Proposition 5.1 is expressed as
\[
v(t) = w - G(t)w + G(t)u_0^+ + \int_0^t G(t-s)(f^+(s) + (\mathcal{L}\Psi)^+)ds .
\] (5.26)

In order to estimate the right side of (5.26) we use the following lemma, a proof of which is found in Lemma 2.6.1 of [10].

**Lemma 5.4.** Let \( G(x, y) \) be a kernel which is measurable in \( X \times Y \) where \( X \) and \( Y \) are open subsets of \( \mathbb{R}^N \). Suppose
\[
\int_X |G(x, y)|^q dx \leq K^q \quad \text{for all } y \in Y ,
\]
and
\[
\int_Y |G(x, y)|^q dy \leq K^q \quad \text{for all } x \in X .
\]

Let \( 1 \leq p, q, r \leq \infty, 1/r = 1/p+1/q-1 \), and set
\[
(Gf)(x) = \int_Y G(x, y)f(y)dy
\]
for \( f \in L^p(Y) \). Then \( \|Gf\|_r \leq K\|f\|_p \).

Suppose \( f \in W^{1,1}(0, T; L^q(\Omega) \cap L^2(\Omega)) \), \( 1 \leq q \leq 2 \), and \( \Psi \leq u_0 \in L^q(\Omega) \). In view of (3.1) and (5.25)
\[
\|w-G(t)w\|_2 \leq 2\|w\|_2 \leq c\bigg(\frac{\partial\Psi}{\partial n}\bigg)^{+}_{1-1/p.p} .
\] (5.27)

(5.9) implies
\[
\int_\Omega G(t, x, y)^q dx \leq Ct^{N(1-q)/2} ,
\] (5.28)
\[
\int_\Omega G(t, x, y)^q dy \leq Ct^{N(1-q)/2} ,
\] (5.29)
with some constant \( C \). Hence with the aid of Lemma 5.4
\[
\|G(t)u_0^+\|_2 \leq Ct^{N(1-q)/2} \|u_0^+\|_q ,
\] (5.30)
\[
\left\| \int_0^t G(t-s)(\mathcal{L}\Psi)^+ds \right\|_2 \leq Ct^{N(1-q)/2+1} \|G(t)^+\|_p .
\] (5.31)

We used \( N(2^{-1}-p^{-1})/2+1>1/2>0 \) in the derivation of (5.31). Hence \( v(t) \in L^q(\Omega) \) if \( t>0 \) and
\[
\|v(t)\|_2 \leq C\bigg(\frac{\partial\Psi}{\partial n}\bigg)^{+}_{1-1/p.p} + Ct^{N(1-q)/2} \|u_0^+\|_q ,
\] (5.32)
In view of Proposition 5.1, $u(t) \in L^2(\Omega)$ if $t > 0$ and
\[ \|u(t)\|_2 \leq \|\Psi\|_2 + \text{the right side of (5.32)}. \] (5.33)
Furthermore applying Proposition 4.4 and noting the remark after Lemma 4.5 we conclude that for $0 < \tau \leq t \leq T$,
\[ u(t) = U_2(t, \tau; f)u(\tau). \] (5.34)


In order to derive the differentiability of the right side of (5.32) and establish some estimates of the derivative we investigate a certain differential equation in Hilbert space in this section.

Let $H$ and $V$ be Hilbert spaces such that $V \subset H$ algebraically and topologically, and $V$ is dense in $H$. The norm and inner product of $H$ are denoted by $| \cdot |$ and $(\ ,\ )$ respectively, and those of $V$ are by $\| \cdot \|$ and $(\ ,\ )$. Identifying $H$ with its dual we consider $V \subset H \subset V^*$. The norm of $V^*$ is denoted by $\| \cdot \|_*$. The pairing between $V$ and $V^*$ is also denoted by $(\ ,\ )$.

Let $\Phi$ be a proper convex, lower semicontinuous function defined on $V$. Let $\Phi$ be a convex function on $L^2(0, T; V)$ defined by
\[ \Phi(u) = \begin{cases} \int_0^T \phi(u(t)) dt & \text{if } \phi(u) \in L^1(0, T), \\ \infty & \text{otherwise}. \end{cases} \] (6.3)

Following H. Brézis [2], we say $f \in M_{u_0}(u)$ for a fixed $u_0 \in H$ if $u \in D(\Phi)$, $f \in L^2(0, T; V^*)$ and
\[ \int_0^T (v', v - u) dt + \Phi(v) - \Phi(u) \geq \int_0^T (f, v - u) dt - \frac{1}{2} |v(0) - u_0|^2. \]
for each $v \in D(\Phi)$, $v' \in L^q(0, T; V^*)$ where $v' = dv/dt$. Let $A$ be the mapping defined by

$$Au = (Lu + \partial \Phi(u))|H.$$  \hspace{1cm} (6.4)

By Theorem 2 of F. E. Browder [4] $L + \partial \Phi$ is maximal monotone in $V \times V^*$, and so is $A$ in $H \times H$. Furthermore by Theorem 4 of [4] $R(L + \partial \Phi) = V^*$, and hence $R(A) = H$.

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For $f \in W^{1,1}(0, T; H)$ and $u_0 \in \overline{D(A)}$ we set

$$U(t, 0; f)u_0 = \lim_{n \to \infty} \prod_{i=1}^{n} \{1 + \frac{t}{n}\left(A - f\left(\frac{i}{n}t\right)\right)\}^{-1}u_0.$$  \hspace{1cm} (6.5)

The main result of this section is as follows (cf. Theorem 3.2 of [11]).

**Theorem 6.1.** Suppose $f \in W^{1,1}(0, T; H)$ and $u_0 \in \overline{D(A)}$. Then $u(t) = U(t, 0; f)u_0$ is the strong solution of

$$du(t)/dt + Au(t) \ni f(t),$$  \hspace{1cm} (6.6)

$$u(0) = u_0,$$  \hspace{1cm} (6.7)

and there exists a constant $K$ such that

$$|tD^+ u(t)| \leq K(|u_0 - v| + t|A^*v| + \int_0^t |f(s)| ds + \int_0^t |sf'(s) + f(s)| ds)$$  \hspace{1cm} (6.8)

where $D^+$ is the right derivative, $A^*$ is the minimal cross section of $A$, and $v$ is an arbitrary element of $D(A)$.

**Lemma 6.1.** If $u_0 \in D(A)$, then $u(t) = U(t, 0; f)u_0$ is a function belonging to $L^q(0, T; V)$ and satisfies the variational inequality

$$\int_0^T (v' + Lu, v - u) dt + \Phi(v) - \Phi(u) \geq \int_0^T (f, v - u) dt - \frac{1}{2} |v(0) - u_0|^2$$  \hspace{1cm} (6.9)

for all $v \in D(\Phi)$, $v' \in L^q(0, T; V^*)$, i.e. $f - Lu \in M_{u_0}(u)$.

**Proof.** Under the hypothesis of the lemma $u(t)$ is the strong solution of (6.6), (6.7). Let $M = L + \partial \Phi$. $M^{-1}$ is an everywhere defined single valued mapping on $V^*$ to $V$ satisfying a uniform Lipschitz condition. Hence $u(t) = M^{-1}(f(t) - u'(t))$ is a measurable function with values in $V$. Let $h \in V^*$ and $\gamma$ be such that

$$\phi(u) \geq (h, u) + \gamma$$  \hspace{1cm} (6.10)

for any $u \in V$. Let $v \in D(\phi)$. Then
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\[ \phi(v) \geq (f(t) - u'(t) - Lu(t), v - u(t)) + \phi(u(t)) \]
\[ \geq (f(t), v - u(t)) + \frac{1}{2} \frac{d}{dt} |u(t) - v|^2 - a(u(t), v) \]
\[ + \gamma \]
\[ \geq (f(t), v-u(t)) + \frac{1}{2} \frac{d}{dt} |u(t) - v|^2 + \alpha \| u(t) \|^2 \]
\[ - C \| u(t) \| \| v \| - \| h \|_* \| u(t) \| + \gamma. \]

Hence \( u \in L^2(0, T; V) \).

Let \( v \in D(\Phi), \) \( v' \in L^2(0, T; V^\ast). \) Then

\[ \phi(v(t)) \geq (f(t) - u'(t) - Lu(t), v(t) - u(t)) + \phi(u(t)) \]
\[ = (f(t), v(t) - u(t)) + (v'(t) - u'(t), v(t) - u(t)) \]
\[ - (v'(t), v(t) - u(t)) - (Lu(t), v(t) - u(t)) + \phi(u(t)), \]

which implies \( u \in D(\Phi). \) Integrating this inequality over \([0, T]\) we get \([6.9]\).

Let \( A \) be the operator defined by

\[ \langle u, v \rangle = \langle Au, v \rangle \quad \text{for} \quad u, v \in V. \quad (6.11) \]

\( A \) is a linear bounded operator from \( V \) onto \( V^\ast, \) and \( \| Au \|_* = \| u \| \) for any \( u \in V. \)

Since \( A^{-1} \partial \phi \) is the subdifferential of \( \phi \) when \( V \) is identified with \( V^\ast \) by Riesz' theorem,

\[ \phi_{\epsilon}(u) = \frac{1}{2\epsilon} \| u - J_{\epsilon} u \|^2 + \phi(J_{\epsilon} u) \]

(6.12)
is the Yosida approximation of \( \phi, \)
where

\[ J_{\epsilon} = (1 + \epsilon A^{-1} \partial \phi)^{-1}. \]

We denote by \( \Phi_{\epsilon} \) the function defined by \((6.3)\) with \( \phi_{\epsilon} \) in place of \( \phi. \) Set

\[ A_{\epsilon} u = (Lu + \partial \phi_{\epsilon}(u)) \cap H. \]

(6.14)
The operator defined by \((6.5)\) with \( A \) replaced by \( A_{\epsilon} \) is denoted by \( U_{\epsilon}(t, 0; f). \)

**Lemma 6.2.** Let \( u(t) = U(t, 0; f)u_0, \) \( u_{\epsilon}(t) = U_{\epsilon}(t, 0; f)u_{0\epsilon}, \) \( u_0 \in D(A), \) \( u_{0\epsilon} \in D(A_{\epsilon}). \) If \( u_{0\epsilon} \rightarrow u_0 \) in \( H, \) then \( u_{\epsilon} \rightarrow u \) in \( L^2(0, T; V). \)

**Proof.** Let \( v \in D(\Phi), \) \( v' \in L^2(0, T; V^\ast). \) In view of \([\text{Lemma 6.1}]\)

\[ \Phi_{\epsilon}(v) \geq \int_0^T (f, v - u_{\epsilon}) dt - \frac{1}{2} \| v(0) - u_{0\epsilon} \|^2 \]
\[ - \int_0^T (v' + Lu_{\epsilon}, v - u_{\epsilon}) dt + \Phi_{\epsilon}(u_{\epsilon}). \]

(6.15)

In view of \((6.12)\) and \([6.10]\)
Differentiability of solutions

\[ \Phi_{\epsilon}(u_{\epsilon}) = \frac{1}{2\epsilon} \int_{0}^{T} \| u_{\epsilon} - J_{\epsilon}u_{\epsilon} \|^2 dt + \int_{0}^{T} \phi(J_{\epsilon}u_{\epsilon}) dt \]

\[ \geq \frac{1}{2\epsilon} \int_{0}^{T} \| u_{\epsilon} - J_{\epsilon}u_{\text{e}} \|^2 dt + \int_{0}^{T} (h, u_{\epsilon}) dt \]

(6.16)

\[-\Vert h\Vert_{*} \int_{0}^{T} \| J_{\epsilon}u_{\epsilon} - u_{\epsilon} \| dt + T\gamma . \]

Combining (6.15) and (6.16)

\[ \Phi_{\epsilon}(v) \geq \int_{0}^{T} (f, v - u_{\epsilon}) dt - \frac{1}{2} |v(0) - u_{0\epsilon}|^2 - \int_{0}^{T} (v', v - u_{\epsilon}) dt \]

\[-\int_{0}^{T} a(u_{\epsilon}, v) dt + \int_{0}^{T} a(u_{\epsilon}, u_{\epsilon}) dt + \frac{1}{2\epsilon} \int_{0}^{T} \| u_{\epsilon} - J_{\epsilon}u_{\epsilon} \|^2 dt \]

\[ + \int_{0}^{T} (h, u_{\epsilon}) dt - \Vert h\Vert_{*} \int_{0}^{T} \| J_{\epsilon}u_{\epsilon} - u_{\epsilon} \| dt + T\gamma . \]

Hence \( \int_{0}^{T} \| u_{\epsilon} \|^2 dt \) and \( \epsilon^{-1} \int_{0}^{T} \| u_{\epsilon} - J_{\epsilon}u_{\epsilon} \|^2 dt \) is bounded as \( \epsilon \to 0 \). Let \( \{u_{\epsilon_{n}}\} \) be a subsequence such that \( u_{\epsilon_{n}} - u^{*} \) in \( L^{2}(0, T ; V) \). Then \( J_{\epsilon_{n}}u_{\epsilon_{n}} - u^{*} \) in \( L^{2}(0, T ; V) \). Letting \( \epsilon = \epsilon_{n} \to 0 \) in

\[ \Phi_{\epsilon}(v) \geq \int_{0}^{T} (f, v - u_{\epsilon}) dt - \frac{1}{2} |v(0) - u_{0\epsilon}|^2 - \int_{0}^{T} (v', v - u_{\epsilon}) dt \]

\[-\int_{0}^{T} (Lu_{\epsilon}, v) dt + \int_{0}^{T} a(u_{\epsilon}, u_{\epsilon}) dt + \Phi(J_{\epsilon}u_{\epsilon}) \]

(6.17)

we get

\[ \Phi(v) \geq \int_{0}^{T} (f, v - u^{*}) dt - \frac{1}{2} |v(0) - u_{0\epsilon}|^2 - \int_{0}^{T} (v', v - u^{*}) dt \]

\[-\int_{0}^{T} (Lu^{*}, v) dt + \int_{0}^{T} a(u^{*}, u^{*}) dt + \Phi(u^{*}) , \]

or \( f - Lu^{*} \in M_{u_{0}}(u^{*}) \). Here we used that \( a(u, u)^{1/2} \) is a norm of \( V \) as was indicated in the proof of \[ \text{Proposition 2.1} \]. By \[ \text{Lemma 6.1} \] \( f - Lu \in M_{u_{0}}(u) \). By virtue of Theorem II.3 of H. Brézis \[ \text{[2]} \] \( u^{*} \in C([0, T] ; H) \) and

\[ \frac{1}{2} |u(t) - u^{*}(t)|^2 \leq -\int_{0}^{t} a(u - u^{*}, u - u^{*}) ds \leq 0 , \]

which implies \( u = u^{*} \) and \( u_{\epsilon} - u \) in \( L^{2}(0, T ; V) \). Noting \( u' \in L^{\infty}(0, T ; H) \subset L^{2}(0, T ; V^{*}) \) and \( u \in D(\Phi) \) let \( \epsilon \to 0 \) in (6.17) with \( v = u \). Then we get

\[ \int_{0}^{T} a(u, u) dt \geq \limsup_{\epsilon \to 0} \int_{0}^{T} a(u_{\epsilon}, u_{\epsilon}) dt . \]
Thus we conclude $u_{\epsilon} \to u$ in $L^2(0, T; V)$ since $\left\{ \int_0^T a(u, u) \, dt \right\}^{1/2}$ is a norm of $L^2(0, T; V)$.

The following lemma is proved in a routine manner and the proof is omitted.

**Lemma 6.3.** Let $u_{\epsilon} = (1 + \epsilon A_{\epsilon})^{-1} u_0$ for $u_0 \in D(A)$. The $u_{\epsilon} \to u_0$ in $H$ as $\epsilon \to 0$.

**Proof of Theorem 6.1.** It suffices to show the theorem in the special case

$$\min_{u} \phi(u) = \phi(0) = 0 \quad (6.18)$$

since the general case is easily reduced to this case. Hence in what follows we assume (6.18). Next suppose that the theorem was established when $u_0 \in D(A)$ and $f \in W^{1,1}(0, T; H)$. For $u_0 \in D(A)$ and $f \in W^{1,1}(0, T; H)$ let $u_{\epsilon} \in D(A)$ and $f_{\epsilon} \in W^{1,1}(0, T; H)$ be such that $u_{\epsilon} \to u_0$ in $H$ and $f_{\epsilon} \to f$ in $W^{1,1}(0, T; H)$. Then with the aid of Theorem 4.1 of [7] $U(t, 0; f_{\epsilon}) u_{\epsilon} \to U(t, 0; f) u_0$ in $C([0, T]; H)$. Hence (6.8) for $u(t) = U(t, 0; f) u_0$ follows. Finally by virtue of Lemmas 6.2 and 5.3 it suffices to prove the theorem for $A_{\epsilon}$ in place of $A$ with constant $K$ in (6.8) independent of $\epsilon$. Thus in what follows we assume (6.18), $u_0 \in D(A_{\epsilon})$ and $f \in W^{1,1}(0, T; H)$, and set $u(t) = U_{\epsilon}(t, 0; f) u_0$.

Form the scalar product of

$$u' + Lu + \partial \phi(u) = f \quad (6.19)$$

and $u$. This and

$$0 \leq \phi(u) \leq \langle \partial \phi(u), u \rangle$$

yield

$$\frac{1}{2} \frac{d}{dt} |u|^2 + (Lu, u) + \phi(u) \leq (f, u).$$

Integrating (6.20) over $[0, t]$ and noting

$$|u(t)| \leq |u_0| + \int_0^t |f(s)| \, ds$$

we get

$$\frac{1}{2} |u(t)|^2 + \int_0^t (Lu, u) \, ds + \int_0^t \phi(u) \, ds \leq \frac{1}{2} \left( |u_0|^2 + \int_0^t |f| \, ds \right)^2. \quad (6.21)$$

Set for $h > 0$

$$u_h(t) = \frac{1}{h} (u(t+h) - u(t)), \quad f_h(t) = \frac{1}{h} (f(t+h) - f(t)).$$

It follows from (6.19) and the monotonicity of $\partial \phi_{\epsilon}$ that
\[
\frac{1}{2} \frac{d}{dt} |u_h|^2 + (Lu_h, u_h) \leq (f_h, u_h).
\]

Using (6.1) and the Schwarz inequality we get
\[
\frac{d}{dt} |u_h|^2 + \alpha \| u_h \|^2 \leq \frac{1}{\alpha} \| f_h \|. 
\]

Integrating this inequality over \([0, T-h]\)
\[
|u_h(T-h)|^2 + \alpha \int_0^{T-h} \| u_h \|^2 dt \leq |u_h(0)|^2 + \frac{1}{\alpha} \int_0^{T-h} \| f_h \| dt.
\]

Since \(u(t)\) is a Lipschitz continuous function with values in \(H\) on \([0, T]\), the right side of the inequality just obtained is bounded as \(h \to 0\). Hence \(u' \in L^2(0, T; V)\). Since
\[
\| \partial \phi_{\epsilon}(u(t)) - \partial \phi_{\epsilon}(u(s)) \| \leq \epsilon^{-1} \| u(t) - u(s) \|
\]
\(\partial \phi_{\epsilon}(u(t))\) is absolutely continuous and \((\partial \phi_{\epsilon}(u))' \in L^2(0, T; V^*)\). Hence \(u'' \in L^2(0, T; V^*)\) and
\[
u'' + Lu' + (\partial \phi_{\epsilon}(u))' = f'.
\]

Multiplying both sides of (6.22) by \(t\)
\[
\frac{d}{dt} (tu') - u' + tLu' + t \frac{d}{dt} \partial \phi_{\epsilon}(u) = tf'.
\]

Forming the scalar product of \(tu'\) and (6.23), noting
\[
(\frac{d}{dt} (\partial \phi_{\epsilon}(u(t)), u'(t))) \leq 0
\]
in view of the monotonicity of \(\partial \phi_{\epsilon}\), and integrating over \([0, t]\), we get
\[
\frac{1}{2} |tu'(t)|^2 - \int_0^t s|u'|^2 ds + \int_0^t (sLu', su') ds
\]
\[
\leq \int_0^t (sf', su') ds.
\]

Note here that \(u' \in C([0, T]; H)\) since \(u' \in L^2(0, T; V)\) and \(u'' \in L^2(0, T; V^*)\). Since \(\phi_{\epsilon}\) is Fréchet differentiable, \(\phi_{\epsilon}(u)\) is absolutely continuous, and as is easily seen at a Lebesgue point of \(u' \in L^2(0, T; V)\)
\[
(d/dt) \phi_{\epsilon}(u(t)) = (\partial \phi_{\epsilon}(u(t)), u'(t)).
\]

Consequently multiplying both sides of (6.19) by \(tu'\) and integrating the equality thus obtained over \([0, t]\) we get
\[
\int_0^t s|u'|^2 ds + \int_0^t (Lu, su') ds
\]
\[ \leq \int_0^t \phi_\epsilon(u)ds + \int_0^t (f, su')ds . \]

Combining (6.24) and (6.25)

\[ \frac{1}{2} |tu'(t)|^2 + \int_0^t (sLu' + Lu, su')ds \leq \int_0^t \phi_\epsilon(u)ds + \int_0^t (f + sf', su')ds . \]  \hspace{1cm} (6.26)

Noting

\[ (Lv + Lu, v) \geq \left( \frac{C}{2\alpha} \right)^2 (Lu, u) \]

for \( u, v \in V \), we get from (6.26)

\[ \frac{1}{2} |tu'(t)|^2 \leq \left( \frac{C}{2\alpha} \right)^2 \int_0^t (Lu, u)ds + \int_0^t \phi_\epsilon(u)ds + \int_0^t (f + sf', su')ds . \]  \hspace{1cm} (6.27)

Combining (6.27) and (6.21) we obtain

\[ \frac{1}{2} |tu'(t)|^2 \leq \frac{1}{2} \max \left\{ \left( \frac{C}{2\alpha} \right)^2, 1 \right\} \left( |u_0| + \int_0^t |f|ds \right)^2 + \int_0^t |f + sf'|s|u'|ds . \]  \hspace{1cm} (6.28)

Applying the following lemma to (6.28) we complete the proof.

**LEMMA 6.4.** Let \( \sigma \) be a real valued continuous function on \([0, T]\) and \( m \) be a nonnegative integrable function on \([0, T]\). Let \( a \) be a nonnegative increasing function on \([0, T]\). If

\[ \sigma(t)^2 \leq a(t)^2 + 2 \int_0^t m(s)\sigma(s)ds \]

in \([0, T]\), then

\[ |\sigma(t)| \leq a(t) + \int_0^t m(s)ds . \]

This lemma is proved in p. 157 of H. Brézis [1] when \( a \) is constant. The case where \( a \) is increasing is easily reduced to the case \( a \) is constant.
§ 7. Final result.

The goal of this paper is the following theorem.

**Theorem 7.1.** Suppose that $\Psi \leq u_0 \in L^q(\Omega)$ and $f \in W^{1,1}(0, T; L^q(\Omega) \cap L^r(\Omega))$, $1 \leq q \leq 2 \leq r$. Then

$$u(t) = \lim_{n \to \infty} \prod_{i=1}^{n} (1 + \frac{t}{u}(A_{q} - f(i/n)))^{-1}u_0,$$

which exists and is continuous in $[0, T]$ with $u(0) = u_0$ in the strong topology of $L^q(\Omega)$, is a strong solution of

$$du(t)/dt + A_2 u(t) \ni f(t)$$

in $(0, T)$. The right derivative $D^+u(t)$, which exists at every $t > 0$ in the strong topology of $L^q(\Omega)$, belongs to $L^r(\Omega)$ and the following inequality holds:

$$\|D^+u(t)\|_r \leq C \left\{ t^{-\beta-1} \left( \|\Psi\|_q + \|u\|_q + t\|A^*_2 v\|_q + \left[ \frac{\partial\Psi}{\partial n} \right]_{1-1/p, p} \right) + t^{-r-1} \|u_0\|_q + t^{-\beta-1} \int_0^t \|f(s)\|_q ds ight\}$$

(7.1)

where $v$ is an arbitrary element of $D(A_2)$, $A^*_2 v$ is the element of $A_2 v$ of the minimal norm, and $\beta = N(2^{-1} - r^{-1})/2$, $\gamma = N(q^{-1} - r^{-1})/2$, $\delta = N(p^{-1} - r^{-1})/2$.

Let $a(u, v)$ again be the bilinear form defined by (1.1), and $\phi$ be the convex function on $H^1(\Omega)$ defined by either

$$\phi(u) = \begin{cases} \int_{\Gamma} j(x, u(x))d\Gamma & \text{if } \Psi \leq u \in H^1(\Omega), \ j(u|_{\Gamma}) \in L^1(\Gamma), \\ \infty & \text{otherwise} \end{cases}$$

(7.2)

or

$$\phi(u) = \phi_1(u) + \phi_2(u),$$

$$\phi_1(u) = \Phi(u|_{\Gamma}), \quad \phi_2(u) = \frac{1}{2\lambda} \|u - Pu\|_2^2, \quad \lambda > 0$$

(7.3)

where $\Phi$ is the function defined by (2.9). The effective domain $D(\phi)$ of $\phi$ defined by [7.1] is not empty since $\Psi^+ \in D(\phi)$ in view of Lemma 4.2.

The following lemma is easily established and the proof is omitted.

**Lemma 7.1.** Let $A$ be the mapping defined by

$$Au = (Lu + \partial\phi(u)) \cap L^q(\Omega).$$

If $\phi$ is the function defined by (7.2), then $A = A_2$. If $\phi$ is defined by (7.3), then
In view of (5.32) and Lemma 7.1 we can apply Theorem 6.1 to $u(t) = U_{q}(t, 0; f)u_{0}$ in $t > \tau > 0$ taking $H = L^{q}(\Omega)$ and $V = H^{1}(\Omega)$. It follows that $u(t)$ is differentiable in $L^{q}(\Omega)$ a.e. in $(0, T]$, and

$$
\| (t-\tau)D^{+}u(t) \|_{2} \leq K \left\{ \| u(\tau) - v \|_{2} + (t-\tau) \| A^{*}v \|_{2} + \int_{0}^{t} \| f(\sigma) \|_{2} d\sigma + \int_{\tau}^{t} \| \sigma f'(\sigma) + f(\sigma) \|_{2} d\sigma \right\},
$$

(7.4)

for $t > \tau > 0$ and $v \in D(A_{q})$.

**Remark.** If we use the expression $A = \partial \psi + B$ and consider $B$ as a perturbation to $\partial \psi$, we get an estimate analogous to (7.8), but with $1 + \sqrt{t-\tau}$ as a factor in the right hand side.

**Lemma 7.2.** If $p < q \leq 2$, then for any $f \in L^{q}(\Omega)$ and $\epsilon > 0$

$$(1 + \epsilon (L_{q} + M_{q, \lambda}))^{-1}f \rightarrow (1 + \epsilon A_{q})^{-1}f$$

in $L^{q}(\Omega)$ as $\lambda \rightarrow 0$.

**Proof.** If $f \in \cap L^{p}(\Omega)$ it follows from the proof of Proposition 4.1 that

$$(1 + \epsilon (L_{q} + M_{q, \lambda}))^{-1}f = (1 + \epsilon (L_{p} + M_{p, \lambda}))^{-1}f \rightarrow (1 + \epsilon A_{p})^{-1}f = (1 + \epsilon A_{q})^{-1}f$$

in $W^{1, p}(\Omega) \subset L^{q}(\Omega)$. The conclusion in the general case follows easily from that in this special case.

For $\lambda > 0$, $t \geq s > \tau > 0$ let

$$u_{\lambda}(t) = \lim_{n \rightarrow \infty} \prod_{i=1}^{n} \left\{ 1 + \frac{t-s}{n} \left( L_{2} + M_{2, \lambda} - f(s + \frac{i}{n}(t-s)) \right) \right\}^{-1} u(s).$$

In view of Theorem 6.1 and Lemma 7.1 $u_{\lambda}(t)$ is the strong solution of

$$du_{\lambda}/dt + L_{2}u_{\lambda} + M_{2, \lambda}u_{\lambda} = f, \quad s < t \leq T,$$

$$u_{\lambda}(s) = u(s).$$

(7.5)

(7.6)

By virtue of Theorem 4.1 of [7] and Lemma 7.2

$$u_{\lambda} \rightarrow u \quad \text{in} \quad C([s, T]; L^{q}(\Omega)), \quad (7.7)$$

as $\lambda \rightarrow 0$.

**Lemma 7.3.** For $w, \tilde{w} \in D(L_{z})$ and $0 \leq v \in H^{2}(\Omega)$, $\partial v / \partial n = 0$ on $\Gamma$,

$$\langle L_{z}w - L_{z}\tilde{w} - Lv, (w - \tilde{w} - v)^{+} \rangle \geq 0. \quad (7.8)$$

**Proof.** (7.8) is shown by approximating $w, \tilde{w}$ by the solutions $w_{\epsilon}, \tilde{w}_{\epsilon}$ of $L_{z, \epsilon}w_{\epsilon} = L_{z}w$, $L_{z, \epsilon}\tilde{w}_{\epsilon} = L_{z}\tilde{w}$, and noting

$$\langle Lv, (w - \tilde{w} - v)^{+} \rangle = a(v, (w - \tilde{w} - v)^{+}).$$
Now we follow the argument of [8, 9, 11] to show $u'(t) \in L^r(\Omega)$ for $t > 0$. For $h > 0$ let $\nu_\pm$ be the solution of

$$\begin{align*}
\partial \nu_\pm/\partial t + \mathcal{L} \nu_\pm &= (f(x, t+h)-f(x, t))^\pm \quad \text{in } \Omega \times (s, T), \\
\partial \nu_\pm/\partial n &= 0 \quad \text{on } \Gamma \times (s, T), \\
v_\pm(x, s) &= (u(x, s+h)-u(x, s))^\pm \quad \text{in } \Omega.
\end{align*}$$

For $h > 0$, $\lambda > 0$ let $\nu_\lambda$ be the solution of

$$\begin{align*}
\partial \nu_\lambda/\partial t + \mathcal{L} \nu_\lambda &= (f(x, t+h)-f(x, t))^+ \quad \text{in } \Omega \times (s, T), \\
\partial \nu_\lambda/\partial n &= 0 \quad \text{on } \Gamma \times (s, T), \\
v_\lambda(x, s) &= (u_\lambda(x, s+h)-u_\lambda(x, s))^+ \quad \text{in } \Omega.
\end{align*}$$

$\nu_\pm$ is expressed as

$$\nu_\pm(t) = G(t-s)(u(s+h)-u(s))^\pm + \int_s^t G(t-\sigma)(f(\sigma+h)-f(\sigma))^\pm d\sigma \quad (7.9)$$

By (5.9) $v_\pm \geq 0$, $v_- \leq 0$ a.e. in $\Omega \times (s, T)$. Similarly $v_\lambda \geq 0$ a.e. in $\Omega \times (s, T)$. By (7.7)

$$v_\lambda \rightarrow v_+ \quad \text{in } C([s, T]; L^2(\Omega)) \quad (7.10)$$

as $\lambda \rightarrow 0$. Set $u_{\lambda, h}(t) = u_\lambda(t+h)-u_\lambda(t)$. With the aid of Lemma 7.3

$$(L_2 u_\lambda(t+h) - L_2 u_\lambda(t) - \mathcal{L} \nu_\lambda(t), (u_{\lambda, h}(t) - v_\lambda(t))^+) \geq 0. \quad (7.11)$$

If $u_{\lambda, h}(x, t) - v_\lambda(x, t) > 0$ at some point $(x, t)$, then $u_\lambda(x, t+h) > u_\lambda(x, t)$ since $v_\lambda \geq 0$, and so $M_{2, \lambda} u_\lambda(x, t+h) \geq M_{2, \lambda} u_\lambda(x, t)$ there as is easily seen by (4.2).

Hence

$$(M_{2, \lambda} u_\lambda(t+h) - M_{2, \lambda} u_\lambda(t), (u_{\lambda, h}(t) - v_\lambda(t))^+) \geq 0. \quad (7.12)$$

In view of (7.10) and (7.11)

$$\frac{1}{2} \frac{d}{dt} \|u_{\lambda, h} - v_\lambda\|^2 = (u_{\lambda, h}'(t) - v_\lambda'(t), (u_{\lambda, h}(t) - v_\lambda(t))^+$$

$$= (u_\lambda'(t+h) - u_\lambda'(t) - v_\lambda'(t), (u_{\lambda, h}(t) - v_\lambda(t))^+)$$

$$\leq (f(t+h) - f(t) - (f(t+h) - f(t))^+, (u_{\lambda, h}(t) - v_\lambda(t))^+) \leq 0.$$ 

Hence

$$\|u_{\lambda, h}(t) - v_\lambda(t)\|^2 \leq \|u_{\lambda, h}(s) - u_{\lambda, h}(s)^+\| = 0,$$

which implies $u_{\lambda, h} \leq v_\lambda$. Letting $\lambda \rightarrow 0$

$$u(t+h) - u(t) \leq v_\lambda(t) \quad (7.13)$$
in view of (7.7) and (7.9). Analogously we can show

$$v_-(t) \leq u(t+h) - u(t).$$

(7.14)

With the aid of Lemma 5.4, (5.28), (5.29), (7.9), (7.13) and (7.14) we get

$$\|\langle u(t+h) - u(t) \rangle / h \|_r \leq \|v_+(t)\|_r + \|v_-(t)\|_r / h$$

$$\leq C(t-s)^{N(r^{-1} - 2^{-1})/2} \|\langle u(s+h) - u(s) \rangle / h \|_2$$

$$+ \int_s^t \|\langle f(\sigma+h) - f(\sigma) \rangle / h \|_r d\sigma.$$  

Letting $h \to 0$

$$\|D^+u(t)\|_r \leq C(t-s)^{N(r^{-1} - 2^{-1})/2} \|D^+u(s)\|_2$$

(7.15)

$$+ \int_s^t \|f'(\sigma)\|_r d\sigma.$$  

Combining (5.33) with $t/3$ in place of $t$, (7.4) with $2t/3$ and $3/t$ in place of $t$ and $\tau$ respectively, and (7.15) with $s=2t/3$, we obtain (7.1).

Bibliography


Differentiability of solutions


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