Bounds on the degree of the equations
defining Kummer varieties

By Ryuji Sasaki

Let k be an algebraically closed field whose characteristic is not equal to two. Let K be the Kummer variety of an abelian variety X over k, i.e., the quotient of X by the inverse morphism $\iota: X \rightarrow X$, and let M be an ample invertible sheaf on K. For any positive integer a, we denote by $\Phi_{M^a}: K \rightarrow P(\Gamma(K, M^a))$ the mapping defined by the linear system $\Gamma(K, M^a)$. In Section 1, we shall prove the projective normality of Kummer varieties (Corollary 1.5):

The image $\Phi_{M^a}(K)$ is projectively normal for any $a \geq 2$; moreover if the canonical mapping

$$\Gamma(K, M) \otimes \Gamma(K, M) \rightarrow \Gamma(K, M^2)$$

is surjective, then the image $\Phi_M(K)$ is also projectively normal.

In the last section 2, we shall prove the main result (Theorem 2.1) in the present paper, which asserts:

The image $\Phi_{M^a}(K)$ is (set-theoretically) an intersection of cubics when $a=2$ and quadrics when $a \geq 3$.

This gives a partial answer to a problem proposed by D. Mumford.

I want to thank Professor S. Koizumi and Dr. T. Sekiguchi for their encouragement and suggestion during the preparation of this paper, especially the latter's suggestion was very useful for proving the main result.

NOTATION AND TERMINOLOGY

Throughout this paper k is an algebraically closed field of characteristic $p \neq 2$. X will denote an abelian variety over k of dimension g. For an integer n, X_n is the kernel of the homomorphism $n_X: X \rightarrow X$ defined by $x \mapsto nx$. Let L be an invertible sheaf on X. Then we denote by $K(L)$ the kernel of the homomorphism $\phi_L: X \rightarrow \hat{X}$ of X to the dual of X defined by $x \mapsto T_x^* L \otimes L^{-1}$, and denote by $\mathcal{G}(L)$ the theta group of L. As usual the action of $\mathcal{G}(L)$ is denoted by U. $P = P_X$ will denote the Poincaré invertible sheaf on $X \times \hat{X}$, and P_α is the restriction of P to $X \times \{\alpha\}$ for any point α of $\hat{X}$. For a linear form f on a finite dimensional vector space V over k, we denote by $[f]$ the point in the projective space $P(V)$ determined by f.

§1. Projective normality of Kummer varieties.

We start with the following Proposition, which is a slight modification of Proposition 1.5 in [6].

PROPOSITION 1.1. *Let $f\colon X\to Y$ be an isogeny of abelian varieties, and let L and M be ample invertible sheaves on X and Y such that $f^*M\cong L$. Let $\Gamma(X, L^2)^{\mathrm{Ker}(f)}$ be the image of the induced mapping $f^*\colon \Gamma(Y, M^2)\to\Gamma(X, L^2)$. Then the canonical mapping*

$$\tau\colon \Gamma(X, L^2)^{\mathrm{Ker}(f)}\otimes\Gamma(X, L^n)\to\Gamma(X, L^{n+2})$$

is surjective for all $n\geq 3$.

PROOF. By virtue of results of Sekiguchi (Main Theorem in [7] and Corollary 1.3 in [6]), we see that for any closed points α and β of $\hat Y$,

$$\Gamma(Y, M^2\otimes P_{Y,\alpha})\otimes\Gamma(Y, M^n\otimes P_{Y,\beta})\to\Gamma(Y, M^{n+2}\otimes P_{Y,\alpha+\beta})$$

is surjective. We denote by W the image of the canonical map τ. To prove our assertion, it suffices to show that

(*) $\left\{\begin{array}{l}\text{for any local }k\text{-algebra }(R, \mathfrak{m})\text{ with residue field }k\\ \text{and any }R\text{-valued point }\lambda\text{ of }\mathcal{G}(L^{n+1}),\\ U_\lambda(f^*(\Gamma(Y, M^{n+2})\otimes R))\subset W\otimes R\end{array}\right.$

(cf. [7] Corollary).

Let $j\colon \mathcal{G}(L^{n+2})\to K(L^{n+2})$ be the canonical surjection. We put $j(\lambda)=u\in K(L^{n+2})(R)$. Then we have the following commutative diagram:

$$(A)\quad\begin{array}{ccc}\Gamma(X_S,(L_S)^{n+2})\cong\Gamma(X, L^{n+2})\otimes R & \xrightarrow{\ T_u^*\ } & \Gamma(X_S,T_u(L_S)^{n+2})\cong\Gamma(X_S,(L_S)^{n+2})\\[4pt] f^*\uparrow & & \uparrow f^*\\[4pt] \Gamma(Y_S,(M_S)^{n+2})\cong\Gamma(Y, M^{n+2})\otimes R & \xrightarrow[\ T_{f(u)}^*\ ]{} & \Gamma(Y_S,T_{f(u)}(M_S)^{n+2})\cong\Gamma(Y_S,(M_S)^{n+2}\otimes P_{Y,\gamma})\end{array}$$

where $\gamma=\phi_{M^{n+2}}(f(u))$, $S=\mathrm{Spec}\,R$, $X_S=X\times S$ and so on. On the other hand, we have the following diagram:

$$\begin{array}{ccc}\Gamma(Y_S,(M_S)^2)\otimes\Gamma(Y_S,(M_S)^n\otimes P_{Y,\gamma}) & \longrightarrow & \Gamma(Y_S,(M_S)^{n+2}\otimes P_{Y,\gamma})\\[4pt] \downarrow & & \downarrow\\[4pt] \Gamma(Y, M^2)\otimes\Gamma(Y, M^n\otimes P_{Y,\bar\gamma}) & \longrightarrow & \Gamma(Y, M^{n+2}\otimes P_{Y,\bar\gamma}),\end{array}$$

where the vertical arrows are reductions modulo $\mathfrak{m}$ and $\bar\gamma$ is the composition $\mathrm{Spec}(R/\mathfrak{m})\hookrightarrow\mathrm{Spec}(R)\xrightarrow{\ \gamma\ }Y$. As mentioned above, the bottom arrow is surjective; hence by Nakayama's lemma, we see that the top arrow is also surjective. Since $f^*((M_S)^n\otimes P_{Y,\gamma})\cong(L_S)^n$, we obtain the following commutative diagram:

$$(\Gamma(X, L^2)^{\mathrm{Ker}(f)} \otimes R) \otimes (\Gamma(X, L^n) \otimes R) \longrightarrow \Gamma(X, L^{n+2}) \otimes R$$

(B) $f^* \otimes f^* \wr \uparrow \qquad\qquad\qquad\qquad\qquad \uparrow$

$$\Gamma(Y_S,(M_S)^2) \otimes \Gamma(Y_S,(M_S)^n \otimes P_{Y,\tau}) \longrightarrow \Gamma(Y_S,(M_S)^{n+2} \otimes P_{Y,\tau}).$$

The surjectivity of the bottom arrow in (B) and the commutativity of the diagrams (A) and (B) show the statement (*).   Q. E. D.

From now on, we assume $\mathrm{char}(k) \neq 2$. Following Mumford [2], we give two definitions.

DEFINITON 1.1. If X is an abelian variety, then the quotient of X by the inverse morphism $\iota: X \to X$ will be denoted by K_X, the Kummer variety of X.

DEFINITION 1.2. An invertible sheaf L on an abelian variety X is said to be totally symmetric if L is of the form $\pi_* M$ for some invertible sheaf M on K_X, where $\pi: X \to K_X$ is the canonical projection.

Let M be an ample invertible sheaf on the Kummer variety K_X of an abelian variety X. We denote by $[-1]$ the canonical automorphism of $\Gamma(X, L)$, where $L = \pi^* M$, induced by the inverse morphism ι and by $\Gamma(X, L)_+$ the subspace of $\Gamma(X, L)$ consisting of elements invariant under the action $[-1]$. Then the image of the canonical map $\pi^*: \Gamma(K_X, M) \to \Gamma(X, L)$ is $\Gamma(X, L)_+$. Moreover if $K(L) = X_2$, then $\Gamma(X, L)_+ = \Gamma(X, L)$ (cf. [2] §3 Inverse Formula). Before proving the projective normality of Kummer varieties, we shall give two lemmas.

LEMMA 1.2. *Let L be an ample totally symmetric invertible sheaf on an abelian variety X. Then there exist a finite subgroup scheme H of X and an ample totally symmetric invertible sheaf L' on the quotient $Y = X/H$ such that $K(L') = Y_2$ and $p^* L' \simeq L$, where $p: X \to Y$ is the canonical surjection.*

PROOF. Since L is ample and totally symmetric, $K(L)$ contains the group X_2; hence $L \simeq (L_1)^2$ for some invertible sheaf L_1 (cf. [2] §2 Corollary 4 to Proposition 6 and [4] §23 Theorem 4). Let H be a maximal subgroup of $K(L_1)$ satisfying $e^{L_1}|_{H \times H} \equiv 1$, and let $p: X \to Y = X/H$ be the canonical surjection. Then there exists a principal invertible sheaf M_1 on Y such that $p^* M_1 \simeq L_1$. Let M_2 be a symmetric invertible sheaf on Y, which is algebraically equivalent to M_1. Put $L' = (M_2)^2$. Then L' is totally symmetric and $p^* L'$ is algebraically equivalent to L. Both $p^* L'$ and L are totally symmetric. Therefore $p^* L$ is isomorphic to L (cf. [2] p. 307).   Q. E. D.

LEMMA 1.3. *Let $\pi: X \to K_X$ be the canonical surjection of an abelian variety X to its Kummer variety. Then the group homomorphism $\pi^*: \mathrm{Pic}(K_X) \to \mathrm{Pic}(X)$ is injective.*

PROOF. Let M be an element of the kernel of π^*. Then $\mathcal{O}_X$ has a structure of G-sheaf induced by an isomorphism $\pi^* M \simeq \mathcal{O}_X$, where $G = \{1_X, \iota\}$, and there is a natural injection $M \to \pi_*(\mathcal{O}_X)^G$. Suppose M is non-trivial. Then for any open subset U of K_X,

$$\Gamma(U, M) \subset \Gamma(U, \pi_*(\mathcal{O}_X)^G) = \{f \in \Gamma(\pi^{-1}(U), \mathcal{O}_X) \mid \iota^* f = -f\}.$$

Let U be a small open subset of K_X containing $\pi(x_0)$ with $x_0 \in X_2$ and let f be an element of $\Gamma(U, M)$ such that the image of f in the fibre $M(\pi(x_0))$ at $\pi(x_0)$ is not zero. Put $\pi^* f = g \in \Gamma(X, \mathcal{O}_X)$. Let $\pi^*(x_0): M(\pi(x_0)) \to \mathcal{O}_X(x_0)$ be the homomorphism induced by π. Then $\pi^*(x_0)(f(\pi(x_0))) = g(x_0) = -g(x_0)$, so $g(x_0) = 0$. This contradicts to $f(\pi(x_0)) \neq 0$; hence we have $M \simeq \mathcal{O}_{K_X}$.   Q. E. D.

Now we shall prove the normal generation of ample invertible sheaves on Kummer varieties in the following style.

THEOREM 1.4. *Let K_X be the Kummer variety of an abelian variety X, and let M be an ample invertible sheaf on K_X. Then the canonical map*

$$\Gamma(K_X, M) \otimes \Gamma(K_X, M^a) \to \Gamma(K_X, M^{a+1})$$

is surjective for all $a \geq 2$.

PROOF. Put $L = \pi^* M$. Then L, L^a and L^{a+1} are totally symmetric and we have a commutative diagram:

$$
\begin{array}{ccc}
\Gamma(X, L) \otimes \Gamma(X, L^a) & \longrightarrow & \Gamma(X, L^{a+1}) \\
\cup & & \cup \\
\Gamma(X, L)_+ \otimes \Gamma(X, L^a)_+ & \longrightarrow & \Gamma(X, L^{a+1})_+ \\
\pi^* \otimes \pi^* \wr \uparrow & & \uparrow \wr \pi^* \\
\Gamma(K_X, M) \otimes \Gamma(K_X, M^a) & \longrightarrow & \Gamma(K_X, M^{a+1}).
\end{array}
$$

To prove the surjectivity of the map in the statement, it suffices to show that the canonical map

$$\varphi: \Gamma(X, L)_+ \otimes \Gamma(X, L^a) \to \Gamma(X, L^{a+1})$$

is surjective. For if t is an element of $\Gamma(X, L^{a+1})_+$ then t is of the form $\sum_i \varphi(r_i \otimes s_i)$ with $r_i \in \Gamma(X, L)_+$ and $s_i \in \Gamma(X, L^a)$. Since t and r_i are even, it follows that

$$t = \sum \varphi(r_i \otimes [-1] s_i).$$

Therefore

$$t = \sum \varphi(r_i \otimes ([-1] s_i + s_i)/2),$$

where $([-1] s_i + s_i)/2$ is contained in $\Gamma(X, L^a)_+$. Now we shall show the surjectivity of φ. By Lemma 1.2, we see that there exist an isogeny $f: X \to Y$ of abelian varieties and an ample totally symmetric invertible sheaf L' on Y such that $K(L') = Y_2$ and $f^* L' \simeq L$. On the other hand we have a commutative diagram:

$$X \xrightarrow{\ f\ } Y$$
$$\pi \downarrow \qquad\qquad \downarrow \pi'$$
$$K_X \xrightarrow[\ f'\ ]{} K_Y$$

where π' is the canonical surjection and f' is the morphism induced by f. Since L' is totally symmetric, there exists an ample invertible sheaf M' on K_Y such that $(\pi')^* M' \simeq L'$. By virtue of Lemma 1.3, we get $(f')^* M' \simeq M$. Therefore we have the following commutative diagram:

$$\Gamma(X,\ L) \xleftarrow{\ f^*\ } \Gamma(Y,\ L')$$
$$\cup \qquad\qquad\qquad \|$$
$$\Gamma(X,\ L)_+ \qquad \Gamma(Y,\ L')_+$$
$$\pi^* \wr \uparrow \qquad\qquad \wr \uparrow (\pi')^*$$
$$\Gamma(K_X,\ M) \xleftarrow{\quad\quad} \Gamma(K_Y,\ M') .$$

Thus we have $\Gamma(X,\ L)_+ \supset f^*(\Gamma(Y,\ L'))$. By Proposition 1.1, we see that the map φ is surjective. Q. E. D.

For an ample invertible sheaf M on the Kummer variety K_X of an abelian variety X and a positive integer a, let $\Phi_{M^a} : K_X \to P(\Gamma(K_X,\ M^a))$ be the canonical map defined by the linear system $\Gamma(K_X,\ M^a)$. Then, as a direct consequence of Theorem 1.4, we have the following:

COROLLARY 1.5. *The image of Φ_{M^a} is projectively normal for $a \geq 2$. Moreover if the canonical map $\Gamma(K_X,\ M) \otimes \Gamma(K_X,\ M) \to \Gamma(K_X,\ M^2)$ is surjective, then the image of Φ_M is also projectively normal.*

§ 2. Estimation of the bound on degree of equations defining Kummer varieties.

This section is devoted to proving our main theorem, which is obtained by the theory of equations defining abelian varieties.

THEOREM 2.1. *Let K_X be the Kummer variety of an abelian variety X, and M an ample invertible sheaf on K_X. Then the image $\Phi_{M^a}(K_X)$ via the canonical mapping $\Phi_{M^a} : K_X \to P(\Gamma(K_X,\ M^a))$ is an intersection of cubics when $a=2$ and quadrics when $a \geq 3$.*

PROOF. First of all, we shall prove the theorem in the case of $a=2$. Let $\pi : X \to K_X$ be the canonical surjection. By the proof of Lemma 1.2, we see that $\pi^* M$ is of the form L^2 for some symmetric ample invertible sheaf on X. Moreover there exists an isomorphism $\varphi : L \to \iota^* L$ such that $\varphi^{\otimes 2}$ is the canonical isomorphism of L^2 to $\iota^* L^2$. As in § 1, we denote by $\Gamma(X,\ L^b)_+$ the subspace of

$\Gamma(X, L^b)$ consisting of elements invariant under the automorphism $[-1]$ of $\Gamma(X, L^b)$ induced by

$$L^b \xrightarrow{\quad \varphi^{\otimes b} \quad} \iota^* L^b \simeq L^b.$$

Through the canonical injection $\pi^* : \Gamma(K_X, M^b) \to \Gamma(X, L^{2b})$, we identify $\Gamma(K_X, M^b)$ with $\Gamma(X, L^{2b})_+$. Now let

$$l : \Gamma(X, L^4)_+ \longrightarrow k$$

be a non-trivial linear form such that there exists a linear form $l^{(3)} : \Gamma(X, L^{12})_+ \to k$ which fits into the commutative diagram:

$$
\begin{array}{ccc}
(\Gamma(X, L^4)_+)^{\otimes 3} & \longrightarrow & \Gamma(X, L^{12})_+ \\
& l^{\otimes 3} \searrow \quad \swarrow l^{(3)} & \\
& k &
\end{array}
,
$$

where the horizontal arrow is the canonical mapping. Then we have the following:

LEMMA. *There exists a linear form* $n : \Gamma(X, L^6)_+ \to k$ *which satisfies the following commutative diagram:*

$$
\begin{array}{ccc}
(\Gamma(X, L^6)_+)^{\otimes 2} & \xrightarrow{\text{canonical}} & \Gamma(X, L^{12})_+ \\
n^{\otimes 2} \searrow \quad & \swarrow l^{(3)} & \\
& k &
\end{array}
.
$$

PROOF OF LEMMA. By the assumption on l, we have a linear form $l^{(2)} : \Gamma(X, L^8)_+ \to k$ such that the diagram

$$
\begin{array}{ccc}
(\Gamma(X, L^4)_+)^{\otimes 2} & \xrightarrow{\text{canonical}} & \Gamma(X, L^8)_+ \\
l^{\otimes 2} \searrow \quad & \swarrow l^{(2)} & \\
& k &
\end{array}
$$

is commutative. Therefore, for any a, b, c and $d \in \Gamma(X, L^2)_+$, we have $l(a \cdot b) \cdot l(c \cdot d) = l(a \cdot d) \cdot l(b \cdot c)$, where $a \cdot b$ is the image of $a \otimes b$ in $\Gamma(X, L^4)_+$ and so on. Hence by an elementary linear algebra we have a linear form $m : \Gamma(X, L^2)_+ \to k$ which fits into the commutative diagram:

$$(\Gamma(X, L^2)_+)^{\otimes 2} \longrightarrow \Gamma(X, L^4)_+$$

$$m^{\otimes 2} \searrow \qquad \swarrow l$$

$$k$$

To prove the Lemma, it suffices to show that

(*)
$$\begin{cases} l^{(3)}(a \cdot b) \cdot l^{(3)}(c \cdot d) = l^{(3)}(a \cdot d) \cdot l^{(3)}(b \cdot c) \\ \text{for all } a, b, c, d \in \Gamma(X, L^6)_+. \end{cases}$$

Since the canonical map

$$\Gamma(X, L^2)_+ \otimes \Gamma(X, L^4)_+ \to \Gamma(X, L^6)_+$$

is surjective, the above assertion (*) comes from the following:

(**)
$$\begin{cases} l^{(3)}((a_1 \cdot a_2) \cdot (b_1 \cdot b_2)) \cdot l^{(3)}((c_1 \cdot c_2) \cdot (d_1 \cdot d_2)) \\ = l^{(3)}((a_1 \cdot a_2) \cdot (d_1 \cdot d_2)) \cdot l^{(3)}((b_1 \cdot b_2) \cdot (c_1 \cdot c_2)) \\ \text{for all } a_1, b_1, c_1, d_1 \in \Gamma(X, L^2)_+ \text{ and all } a_2, b_2, c_2, d_2 \in \Gamma(X, L^4)_+. \end{cases}$$

Since we have

$$l^{(3)}((a_1 \cdot a_2) \cdot (b_1 \cdot b_2)) = l^{(3)}((a_1 \cdot b_1) \cdot a_2 \cdot b_2)$$

$$= l(a_1 \cdot b_1) \cdot l(a_2) \cdot l(b_2)$$

$$= m(a_1) \cdot m(b_1) \cdot l(a_2) \cdot l(b_2),$$

it follows that the left side of the equation in (**) becomes

$$m(a_1) \cdot m(b_1) \cdot l(a_2) \cdot l(b_2) \cdot m(c_1) \cdot m(d_1) \cdot l(c_2) \cdot l(d_2)$$

and this is equal to the right hand side of the equation. Thus the lemma is proved.

Now we continue the proof of the theorem. We put

$$\Gamma(X, L^3)_- = \{ f \in \Gamma(X, L^3) \mid [-1]f = -f \}.$$

Then we easily see that $\Gamma(X, L^3) = \Gamma(X, L^3)_+ \oplus \Gamma(X, L^3)_-$. By the same way as the proof of the preceding lemma, we have linear forms

$$p : \Gamma(X, L^3)_+ \longrightarrow k$$

and

$$q : \Gamma(X, L^3)_- \longrightarrow k$$

such that the diagram

$$(\Gamma(X,\ L^3)_+)^{\otimes 2} \longrightarrow \Gamma(X,\ L^6)_+ \longleftarrow (\Gamma(X,\ L^3)_-)^{\otimes 2}$$

$$p^{\otimes 2} \searrow \qquad \downarrow n \qquad \swarrow q^{\otimes 2}$$

$$k$$

is commutative. Since $(\Gamma(X,\ L^3)_+)^{\otimes 2}+(\Gamma(X,\ L^3)_-)^{\otimes 2}\to\Gamma(X,\ L^6)_+$ is surjective, it follows that $p\oplus q:\Gamma(X,\ L^3)\to k$ is non-trivial. Then $p\oplus q$ satisfies the following:

(C)   For any $F\in\mathrm{Ker}[\Gamma(X,\ L^3)^{\otimes 3} \longrightarrow \Gamma(X,\ L^9)]$,

$$(p\oplus q)^{\otimes 3}(F)=0.$$

In fact, suppose F is of the form

$$\sum_i f_i\otimes g_i\otimes h_i.$$

We denote by f^+ (resp. f^-) the even (resp. odd) part of $f\in\Gamma(X,\ L^3)$. Then the even and odd part of F are the following:
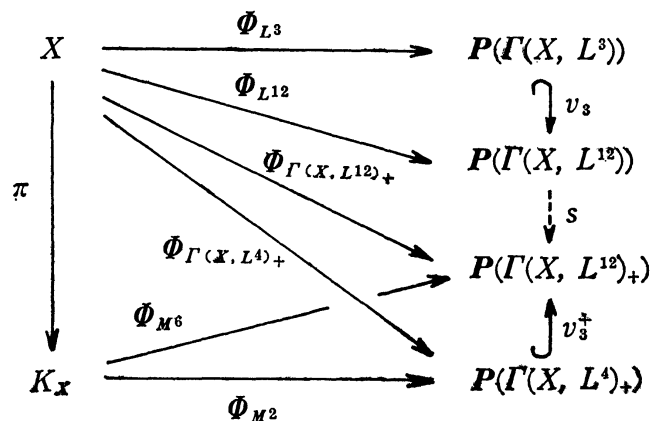
$$F^+=\sum f_i^+\otimes g_i^-\otimes h_i^- +\sum f_i^-\otimes g_i^+\otimes h_i^- +\sum f_i^-\otimes g_i^-\otimes h_i^+ +\sum f_i^+\otimes g_i^+\otimes h_i^+,$$

$$F^-=\sum f_i^-\otimes g_i^+\otimes h_i^+ +\sum f_i^+\otimes g_i^-\otimes h_i^+ +\sum f_i^+\otimes g_i^+\otimes h_i^- +\sum f_i^-\otimes g_i^-\otimes h_i^-.$$

Since the image of F^+ (resp. F^-) in $\Gamma(X,\ L^9)$ is even (resp. odd), these are zero. If p is trivial, then $(p\oplus q)^{\otimes 3}(F^+)=0$. So we may assume that $p(f)\neq 0$ for some $f\in\Gamma(X,\ L^3)_+$. Then we have

$$(p\oplus q)^{\otimes 4}(F^+\otimes f)=\sum p(f_i^+)\cdot q(g_i^-)\cdot q(h_i^-)\cdot p(f)+\sum q(f_i^-)\cdot p(g_i^+)\cdot q(h_i^-)\cdot p(f)$$

$$+\sum q(f_i^-)\cdot q(g_i^-)\cdot p(h_i^+)\cdot p(f)+\sum p(f_i^+)\cdot p(g_i^+)\cdot p(h_i^+)\cdot p(f)$$

$$=\sum n(f_i^+\cdot f)\cdot n(g_i^+\cdot h_i^+)+\sum n(f_i^-\cdot h_i^-)\cdot n(g_i^+\cdot f)$$

$$+\sum n(f_i^-\cdot g_i^-)\cdot n(h_i^+\cdot f)+\sum n(f_i^+\cdot g_i^+)\cdot n(h_i^+\cdot f)$$

$$=l^{(3)}(\sum f_i^+\cdot f\cdot g_i^-\cdot h_i^- +\sum f_i^-\cdot h_i^-\cdot g_i^+\cdot f$$

$$+\sum f_i^-\cdot g_i^-\cdot h_i^+\cdot f +\sum f_i^+\cdot g_i^+\cdot h_i^+\cdot f)$$

$$=0.$$

Therefore we have $(p\oplus q)^{\otimes 3}(F^+)=0$. Similarly we have $(p\oplus q)^{\otimes 3}(F^-)=0$; hence $(p\oplus q)^{\otimes 3}(F)=0$. By virtue of a theorem of Sekiguchi ([6], Theorem 3.1), we see that there exists a closed point x of X such that $\varPhi_{L^3}(x)\in P(\Gamma(X,\ L^3))$ gives the linear form $p\oplus q$, where $\varPhi_{L^3}:X\to P(\Gamma(X,\ L^3))$ is the canonical mapping. Now we shall show that $\varPhi_{M^2}(\pi(x))\in P(\Gamma(K_X,\ M^2))$ gives the linear form l on $\Gamma(K_X,\ M^2)\simeq\Gamma(X,\ L^4)_+$. Let $l':\Gamma(X,\ L^4)_+\to k$ be a linear form corresponding with the point $\varPhi_{\Gamma(X,L^4)_+}(x)$, where $\varPhi_{\Gamma(X,L^4)_+}:X\to P(\Gamma(X,\ L^4)_+)$ is the morphism

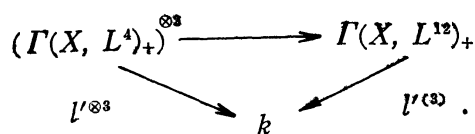defined by the linear system $\Gamma(X, L^4)_+$. We have the following diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\;\Phi_{L^3}\;} & P(\Gamma(X, L^3)) \\
& \Phi_{L^{12}} & \uparrow v_3 \\
\pi \Big\downarrow & \Phi_{\Gamma(X, L^{12})_+} & P(\Gamma(X, L^{12})) \\
& \Phi_{\Gamma(X, L^4)_+} & \Big\downarrow s \\
& \Phi_{M^6} & P(\Gamma(X, L^{12})_+) \\
K_X & \xrightarrow{\;\;\;\;\;\;\;\;\;} & \uparrow v_3^+ \\
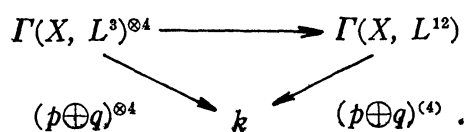& \Phi_{M^2} & P(\Gamma(X, L^4)_+)
\end{array}
$$

where v_3 and v_3^+ are the Veronese mappings and s is the projection with respect to the inclusion $\Gamma(X, L^{12})_+\hookrightarrow\Gamma(X, L^{12})$. Then $v_3([l'])$ corresponds with the linear form

$$l'^{(3)} : \Gamma(X, L^{12})_+ \longrightarrow k$$

which satisfies the following commutative diagram:

$$
\begin{array}{ccc}
(\Gamma(X, L^4)_+)^{\otimes 3} & \longrightarrow & \Gamma(X, L^{12})_+ \\
& l'^{\otimes 3} \searrow \quad \swarrow l'^{(3)} & \\
& k &
\end{array}
$$

On the other hand $\Phi_{L^3}(x)=[p\oplus q]$ and $\Phi_{L^{12}}(x)=[(p\oplus q)^{(4)}]$, where $(p\oplus q)^{(4)}$ is the linear form satisfying the commutative diagram:

$$
\begin{array}{ccc}
\Gamma(X, L^3)^{\otimes 4} & \longrightarrow & \Gamma(X, L^{12}) \\
& (p\oplus q)^{\otimes 4} \searrow \quad \swarrow (p\oplus q)^{(4)} & \\
& k &
\end{array}
$$

By the definitions, we have $s([(p\oplus q)^{(4)}])=[l^{(3)}]$; hence $[l^{(3)}]=[l'^{(3)}]$. Since v_3^+ is a closed immersion, it follows that we have $[l]=[l']$. So we complete the proof of the theorem for $a=2$. Next we shall prove the case $a=3$. Let

$$l : \Gamma(X, L^6)_+ \longrightarrow k$$

be a linear form such that there exists a linear form

$$l^{(2)} : \Gamma(X, L^{12})_+ \longrightarrow k$$

satisfying the commutative diagram:

$$(\Gamma(X,\ L^6)_+)^{\otimes 2} \longrightarrow \Gamma(X,\ L^{12})_+$$

$$l^{\otimes 2} \searrow \quad \swarrow l^{(2)}$$
$$k$$

Then we have linear forms

$$p:\ \Gamma(X,\ L^3)_+ \longrightarrow k$$

and

$$q:\ \Gamma(X,\ L^3)_- \longrightarrow k$$

such that the diagram

$$(\Gamma(X,\ L^3)_+)^{\otimes 2} \longrightarrow \Gamma(X,\ L^6)_+ \longleftarrow (\Gamma(X,\ L^3)_-)^{\otimes 2}$$

$$p^{\otimes 2} \searrow \quad \downarrow l \quad \swarrow q^{\otimes 2}$$
$$k$$

is commutative. By the same way as the case $a=2$, we see that $p \oplus q$ satisfies the above condition (C). The rest of the proof is similar to the first one. As for the case $a \geq 4$, by a theorem of Mumford ([3], Theorem 10), we have a similar proof.   Q. E. D.

As a direct consequence of the preceding theorem, we have the following:

COROLLARY 2.2. *Let L be a symmetric ample invertible sheaf on an abelian variety X. Assume the canonical map*

$$\Gamma(X,\ L^2)_+ \otimes \Gamma(X,\ L^2)_+ \longrightarrow \Gamma(X,\ L^4)_+$$

is surjective. Then the image of the morphism $\Phi_{\Gamma(X,L^2)_+}: X \to P(\Gamma(X,\ L^2)_+)$ defined by the linear system $\Gamma(X,\ L^2)_+$ is an intersection of hypersurfaces of degree six.

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