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On defining equations of symmetric submanifolds in complex projective spaces

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Introduction.

Let M be a compact complex manifold and L a holomorphic line bundle on M. Let $\Gamma(L)$ denote the vector space of all holomorphic sections of L and let $P(\Gamma(L))$ denote the projective space of hyperplanes of $\Gamma(L)$. A holomorphic line bundle L on M is said to be very ample if we can define a map $j_L: M\rightarrow P(\Gamma(L))$ by $j_L(x)=\{s\in\Gamma(L)\mid s(x)=0\}$ for $x\in M$ and furthermore j_L is a holomorphic imbedding.

A compact simply connected homogeneous complex manifold M is called a C-space. If M has a Kähler metric it is said to be a kählerian C-space. Let L be a very ample holomorphic line bundle on a kählerian C-space M. Consider the homogeneous ideal of the projective submanifold $j_L(M)$ in $P(\Gamma(L))$. For example, for a complex Grassmann manifold M imbedded into a projective space by the Plücker coordinates, it is known that the homogeneous ideal of M is generated by quadrics. Moreover E. Cartan has realized in his Thèse some exceptional complex simple Lie groups as the projective automorphism groups of projective submanifolds defined by some quadrics — these projective submanifolds are all kählerian C-spaces. (See [4] pp. 272-276.)

Motivated by these facts, we ask whether the homogeneous ideal of $j_L(M)$ of a kählerian C-space M is generated by quadrics or not. In this note we shall prove that if M is a Hermitian symmetric space of compact type the answer is affirmative for each L (Corollary of Main Theorem). We give also a sufficient condition for a general kählerian C-space in order that the question is affirmative (Main Theorem).

For a compact projective manifold M and a very ample holomorphic line bundle L on M, Mumford [7] has given a cohomological condition in order that the homogeneous ideal of $j_L(M)$ is generated by quadrics. Our basic formulation in section 1 is due to Mumford [7], while our condition for kählerian C-spaces is not for the cohomologies of L but for the Chern class of L.

After having finished this work, the authors learned that our Corollary to

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Main Theorem follows from the standard monomial theory [6] developed recently by Lakshmibai, Musili and Seshadri. They use the cellular decomposition of M effectively in studying the relations on $j_L(M)$ between monomials of homogeneous coordinates of $P(\Gamma(L))$, while our proof depends on the representation theory of semi-simple Lie algebra.

1. Preliminaries.

In this section we recall the basic formulation due to Mumford [7]. Let M be a compact complex manifold and L a holomorphic line bundle on M. We denote by $\Gamma(L)$ the vector space of all holomorphic sections of L. The *base points* of $\Gamma(L)$ are the points $x\in M$ such that $s(x)=0$ for all $s\in\Gamma(L)$. If $\Gamma(L)$ has no base points, L defines a canonical holomorphic map j_L of M into a projective space in the following way; For a complex vector space V, let $P(V)$ denote the projective space of hyperplanes of V. We define a holomorphic map $j_L: M\to P(\Gamma(L))$ by $j_L(x)=\{s\in\Gamma(L)\mid s(x)=0\}$ for $x\in M$.

A holomorphic line bundle L on M is called *very ample* if $\Gamma(L)$ has no base points and $j_L: M\to P(\Gamma(L))$ is an imbedding. Note that the vector space $\Gamma(L)$ is canonically isomorphic to the space of homogeneous coordinate functions on the projective space $P(\Gamma(L))$. The k^{th} symmetric power of $\Gamma(L)$, which we denote by $S^k\Gamma(L)$, is canonically isomorphic to the space of homogeneous polynomials of degree k in the homogeneous coordinates of $P(\Gamma(L))$. Thus if L is very ample the vector space of homogeneous polynomials of degree k which vanish on $j_L(M)$ is nothing but the kernel of the canonical map

$$S^k\Gamma(L)\longrightarrow\Gamma(L^k).$$

Now our problem is whether the homogeneous ideal of $j_L(M)$ is generated by quadrics or not. This is the same as asking whether the canonical map

$$S^{k-2}\Gamma(L)\otimes\operatorname{Ker}(S^2\Gamma(L)\longrightarrow\Gamma(L^2))\longrightarrow\operatorname{Ker}(S^k\Gamma(L)\longrightarrow\Gamma(L^k)) \tag{1.1}$$

is surjective for all $k\geqq 2$.

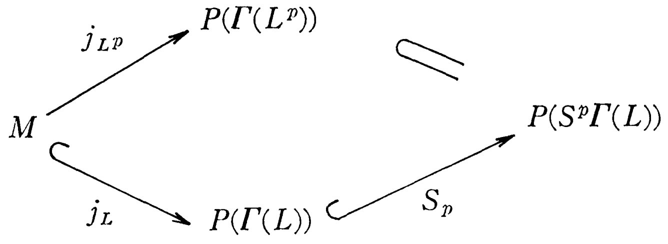
Let L, N be holomorphic line bundles on M and $\varphi:\Gamma(L)\otimes\Gamma(N)\to\Gamma(L\otimes N)$ the canonical map. Let $\mathcal{R}(L, N)$, $\mathcal{S}(L, N)$ denote the kernel and the cokernel of φ respectively.

Let L be a very ample holomorphic line bundle on M. Then L is said to be *normally generated* if the canonical map

$$\varphi:\Gamma(L)^{\otimes k}=\underbrace{\Gamma(L)\otimes\cdots\otimes\Gamma(L)}_{k}\longrightarrow\Gamma(L^k)$$

is surjective for every $k\geqq 1$. Note that L is normally generated if and only if $\mathcal{S}(L^i, L^j)=(0)$ for all i, $j\geqq 1$. Note also that if L is normally generated, so is L^p for $p\geqq 1$. In fact, via the surjection $S^p\Gamma(L)\to\Gamma(L^p)$ we can identify $P(\Gamma(L^p))$

canonically with a linear subspace of $P(S^p\Gamma(L))$. Then we get a commutative diagram



where S_p denotes the p^{th} Veronese imbedding. Thus L^p is also very ample. This together with $\mathcal{S}(L^{pi}, L^{pj})=(0)$ for $i, j\geqq 1$ implies the normal generation of L^p.

LEMMA 1.1. (Mumford [7], p. 39) *Let L be a normally generated holomorphic line bundle on M. Then the canonical map* (1.1) *is surjective for every $k\geqq 2$, that is, the homogeneous ideal of $j_L(M)$ is generated by quadrics, if and only if the canonical map*

$$\text{(1.2)} \qquad \mathrm{id}\otimes\varphi : \mathcal{R}(L^i, L^j)\otimes\Gamma(L^k) \longrightarrow \mathcal{R}(L^i, L^{j+k})$$

is surjective for every $i, j, k\geqq 1$.

PROPOSITION 1.2. (Mumford [7], p. 49) *Let L, N, F be holomorphic line bundles on M. If*

(a) *the linear map*

$$\psi : \mathcal{R}(N, L)\otimes\Gamma(F) \longrightarrow \mathcal{R}(N\otimes F, L)$$

defined by $\psi((\sum a_i\otimes b_i)\otimes c)=\sum(a_ic)\otimes b_i$ $(a_i\in\Gamma(N),\ b_i\in\Gamma(L),\ c\in\Gamma(F))$ is surjective, and

$$\text{(b)} \qquad \mathcal{S}(N, L)=(0),$$

then the linear map

$$\psi' : \mathcal{R}(N, F)\otimes\Gamma(L) \longrightarrow \mathcal{R}(N\otimes L, F)$$

defined by $\psi'((\sum a_i\otimes c_i)\otimes b)=\sum(a_ib)\otimes c_i$ $(a_i\in\Gamma(N), b\in\Gamma(L), c_i\in\Gamma(F))$ is surjective.

LEMMA 1.3. *Let L be a normally generated holomorphic line bundle on M and let $p\geqq 1$. If the linear map*

$$\text{(1.3)} \qquad \psi : \mathcal{R}(L^i, L)\otimes\Gamma(L) \longrightarrow \mathcal{R}(L^{i+1}, L)$$

defined by $\psi((\sum a_j\otimes b_j)\otimes c)=\sum(a_jc)\otimes b_j$ $(a_j\in\Gamma(L^i),\ b_j,\ c\in\Gamma(L))$ is surjective for every $i\geqq p$, then the map (1.2) *is surjective for every $i, j, k\geqq p$, and hence* (*by Lemma* 1.1) *the homogeneous ideal of $j_{L^p}(M)$ is generated by quadrics.*

PROOF. (cf. Mumford [7], p. 51) Interating, we see that

$$\mathfrak{R}(L^i, L)\otimes\Gamma(L^j) \longrightarrow \mathfrak{R}(L^{i+j}, L)$$

is surjective for every $i\geqq p$, $j\geqq 1$. Since $\mathcal{S}(L^i, L)=(0)$ for every $i\geqq 1$,

$$\mathfrak{R}(L^i, L^j)\otimes\Gamma(L) \longrightarrow \mathfrak{R}(L^{i+1}, L^j)$$

is surjective for every $i\geqq p$, $j\geqq 1$ by Proposition 1.2. Interating again, we find that

$$\mathfrak{R}(L^i, L^j)\otimes\Gamma(L^k) \longrightarrow \mathfrak{R}(L^{i+k}, L^j)$$

is surjective for every $i\geqq p$ and j, $k\geqq 1$. Thus we get the required assertion. q. e. d.

2. Kählerian C-spaces.

A compact simply connected homogeneous complex manifold is called a *C-space*. A C-space is said to be *kählerian* if it carries a Kähler metric. In this section we summarize some known results on kählerian C-spaces and holomorphic line bundles on these manifolds (cf. Borel-Hirzebruch [1], Bott [3], Sakane-Takeuchi [8], Takeuchi [9]).

We recall first the basic construction of kählerian C-spaces. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote the real part of $\mathfrak{h}$ by $\mathfrak{h}_R$. A weight of a $\mathfrak{g}$-module relative to $\mathfrak{h}$ will be identified with an element of $\mathfrak{h}_R$ by means of the duality defined by the Killing form $(\,,)$ of $\mathfrak{g}$. In particular, the root system Σ of $\mathfrak{g}$ relative to $\mathfrak{h}$ is a subset of $\mathfrak{h}_R$. Choose a lexicographic order on $\mathfrak{h}_R$ and let Π denote the fundamental root system of Σ. Take a subsystem Π_1 of Π and set $\Sigma_1=\Sigma\cap \boldsymbol{Z}\Pi_1$. We define a subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ by

$$\mathfrak{u}=\mathfrak{h}+\sum_{\alpha\in\Sigma_1\cup\Sigma^+}\mathfrak{g}_\alpha\,,$$

where $\mathfrak{g}_\alpha$ is the root space for α and Σ^+ is the set of positive roots. Now let G be the simply connected complex Lie group with the Lie algebra $\mathfrak{g}$ and U the (closed) connected complex Lie subgroup of G generated by $\mathfrak{u}$. Then the quotient complex manifold

$$M=G/U$$

is a kählerian C-space. Conversely any kählerian C-space M is obtained in this way.

It is known that the group Z of weights of $\mathfrak{g}$-modules is given by

$$Z=\left\{\Lambda\in\mathfrak{h}_R \,\middle|\, \frac{2(\Lambda,\alpha)}{(\alpha,\alpha)}\in \boldsymbol{Z} \text{ for each } \alpha\in\Sigma\right\}.$$

It is a lattice of $\mathfrak{h}_R$ generated by the fundamental weights $\{\Lambda_\alpha|\alpha\in\Pi\}$ corresponding to Π. We put

$$Z_1=\{\Lambda\in Z|(\Lambda, \Pi_1)=(0)\},$$

which is a subgroup of Z generated by $\{\Lambda_\alpha|\alpha\in\Pi-\Pi_1\}$. We define further

$$Z_1^+=\{\Lambda\in Z_1|(\Lambda, \alpha)>0 \quad \text{for each } \alpha\in\Pi-\Pi_1\}.$$

Then we have

$$Z_1^+=\sum_{\alpha\in\Pi-\Pi_1} \boldsymbol{Z}^+\Lambda_\alpha,$$

where $\boldsymbol{Z}^+$ denotes the set of positive integers. For each $\Lambda\in Z_1$, there exists a unique holomorphic character χ_Λ of U such that $\chi_\Lambda(\exp H)=\exp(\Lambda, H)$ for each $H\in\mathfrak{h}$. Let L_Λ denote the holomorphic line bundle on M associated to the principal bundle $U\to G\to M$ by χ_Λ. The correspondence $\Lambda\mapsto L_\Lambda$ induces a homomorphism of Z_1 to the group $H^1(M, \mathcal{O}^*)$ of isomorphism classes of holomorphic line bundles on M.

(I) *The above homomorphism $Z_1\to H^1(M, \mathcal{O}^*)$ is an isomorphism. In particular, under this isomorphism the subset $-Z_1^+$ corresponds to the set of isomorphism classes of very ample holomorphic line bundles on M.*

Thus the group G acts on each holomorphic line bundle L on M, and hence $\Gamma(L)$ is a G-module in the canonical way. In particular, if L is very ample the canonical imbedding $j_L: M\to P(\Gamma(L))$ is G-equivariant.

(II) *For each $\Lambda\in Z_1^+$, $\Gamma(L_{-\Lambda})$ is an irreducible G-module with the lowest weight $-\Lambda$, that is, the G-module $\Gamma(L_{-\Lambda})$ is contragredient to an irreducible G-module with the highest weight Λ.*

LEMMA 2.1. *Let L be a very ample holomorphic line bundle on a kählerian C-space M. Then L is normally generated.*

PROOF. We may assume by (I) that $M=G/U$ and $L=L_{-\Lambda}$ for some $\Lambda\in Z_1^+$. Since the canonical map $\varphi: \Gamma(L_{-\Lambda})^{\otimes k}\to\Gamma(L_{-\Lambda}{}^k)$ $(k\geqq 1)$ is a G-homomorphism and $\Gamma(L_{-\Lambda}{}^k)=\Gamma(L_{-k\Lambda})$ is an irreducible G-module by (II), it is enough to show that φ is not trivial. We claim that $\varphi(s\otimes\cdots\otimes s)\neq 0$ for $s\in\Gamma(L_{-\Lambda})$, $s\neq 0$. Suppose that $\varphi(s\otimes\cdots\otimes s)=0$, then the homogeneous polynomial $s^k\in S^k\Gamma(L_{-\Lambda})$ vanishes on $j_{L_{-\Lambda}}(M)$ and hence s vanishes on $j_{L_{-\Lambda}}(M)$. This is a contradiction.

q. e. d.

3. The decomposition of tensor products of irreducible modules.

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and let Π be the fundamental root system of $\mathfrak{g}$ as in section 2. Take a subsystem Π_1 of Π. Let W be the Weyl group for Π and W_1 the subgroup of W generated by reflections relative to the roots of Π_1. We put

$$D=\{\lambda\in Z|(\lambda, \alpha)\geqq 0 \quad \text{for each } \alpha\in\Pi\}$$

and

$$D_1=\{\lambda\in Z \mid (\lambda, \alpha)\geqq 0 \quad \text{for each } \alpha\in\Pi_1\}.$$

Note that the set $Z_1^{\dagger}$ defined in section 2 for Π_1 is a subset of D. We define a subset W^1 of W by

$$W^1=\{w\in W \mid wD\subset D_1\}.$$

LEMMA 3.1. (Borel-Hirzebruch [1]) *Every element $w\in W$ can be uniquely written as $w=w_1w^1$ where $w_1\in W_1$ and $w^1\in W^1$.*

We put

$$\rho=\frac{1}{2}\sum_{\gamma\in\Sigma^+}\gamma.$$

LEMMA 3.2. *For $\alpha\in\Pi-\Pi_1$ and $w_1\in W_1$,*

$$(w_1\rho-\rho, \alpha)\geqq 0.$$

PROOF. Put

$$\rho_1=\frac{1}{2}\sum_{\gamma\in\Sigma_1\cap\Sigma^+}\gamma, \qquad \rho_2=\frac{1}{2}\sum_{\gamma\in\Sigma^+-\Sigma_1}\gamma,$$

so that $\rho=\rho_1+\rho_2$. Since $w_1\rho_2=\rho_2$ for $w_1\in W_1$, $w_1\rho-\rho=w_1\rho_1-\rho_1$. On the other hand, $w_1\rho_1-\rho_1=-\sum_{\alpha\in\Pi_1}n_\alpha\alpha$ ($n_\alpha\in\boldsymbol{Z}$, $n_\alpha\geqq 0$) and $(\alpha, \beta)\leqq 0$ for $\alpha\in\Pi_1$, $\beta\in\Pi-\Pi_1$. Thus we get our assertion. q.e.d.

For $\lambda\in D$ let $[\lambda]$ denote the character of an irreducible $\mathfrak{g}$-module with the highest weight λ. Now take elements λ, $\mu\in D$. Suppose that in the character ring of $\mathfrak{g}$ we have

$$[\mu][\lambda]=\sum_{\nu\in D}M_{\mu,\lambda}(\nu)[\nu],$$

where $M_{\mu,\lambda}(\nu)$ are non-negative integers with $M_{\mu,\lambda}(\mu+\lambda)=1$. Let $\Delta(\lambda)$ denote the set of all weights of an irreducible $\mathfrak{g}$-module with the highest weight λ and $m(\tau)$ the multiplicity of a weight $\tau\in\Delta(\lambda)$.

LEMMA 3.3. (Brauer-Weyl) *For $\nu\in D$,*

$$M_{\mu,\lambda}(\nu)=\sum_{w\in W}\det(w)m(\nu+\rho-w(\mu+\rho)).$$

PROOF. See the proof of [2], Ch. VIII, § 9, Proposition 2.

For $\lambda\in D$ we define non-negative integers $k_\alpha(\lambda)$ ($\alpha\in\Pi$) by

$$k_\alpha(\lambda)=\operatorname{Max}\left\{-\frac{2(\tau, \alpha)}{(\alpha, \alpha)}\,\middle|\,\tau\in\Delta(\lambda)\right\}.$$

THEOREM 3.4. *Let $\mu\in Z_1^{\dagger}$, $\lambda\in D$ and suppose that*

$$[\mu][\lambda]=\sum_{\nu\in D}M_{\mu,\lambda}(\nu)[\nu].$$

If $\mu\in Z_1^+$ satisfies the inequalities

$$\frac{2(\mu, \alpha)}{(\alpha, \alpha)}\geqq k_\alpha(\lambda) \tag{3.1}$$

for all $\alpha\in\Pi-\Pi_1$, then

$$[\mu+\Lambda_{\alpha'}][\lambda]=\sum_{\nu\in D} M_{\mu,\lambda}(\nu)[\nu+\Lambda_{\alpha'}]$$

for $\alpha'\in\Pi-\Pi_1$.

PROOF. At first we claim that if $\nu\in D$ satisfies $M_{\mu,\lambda}(\nu)\neq 0$ then

$$M_{\mu,\lambda}(\nu)=\sum_{w_1\in W_1}\det(w_1)m(\nu-\mu+\rho-w_1\rho). \tag{3.2}$$

Since $M_{\mu,\lambda}(\nu)\neq 0$, we have $m(\nu+\rho-w(\mu+\rho))\neq 0$ for some $w\in W$ by Lemma 3.3. We put $\tau=\nu+\rho-w(\mu+\rho)\in\Delta(\lambda)$. Decomposing $w^{-1}=w_1w^1$ as in Lemma 3.1, we get

$$w^1(\nu+\rho)=w_1^{-1}(\mu+\rho)+w_1^{-1}w^{-1}\tau.$$

Since $\mu\in Z_1^+$, $w_1^{-1}\mu=\mu$ and thus

$$w^1(\nu+\rho)=\mu+w_1^{-1}\rho+w_1^{-1}w^{-1}\tau.$$

Note that $w^1(\nu+\rho)\in D_1$ since $\nu+\rho\in D$. For $\alpha\in\Pi-\Pi_1$

$$\frac{2(w^1(\nu+\rho), \alpha)}{(\alpha, \alpha)}=\frac{2(\mu+w_1^{-1}\rho+w_1^{-1}w^{-1}\tau, \alpha)}{(\alpha, \alpha)}$$

$$=\frac{2(\mu, \alpha)}{(\alpha, \alpha)}+\frac{2(w_1^{-1}w^{-1}\tau, \alpha)}{(\alpha, \alpha)}+\frac{2(w_1^{-1}\rho-\rho, \alpha)}{(\alpha, \alpha)}+\frac{2(\rho, \alpha)}{(\alpha, \alpha)}.$$

Since $2(\rho, \alpha)/(\alpha, \alpha)=1$, we have by Lemma 3.2 and (3.1)

$$\frac{2(w^1(\nu+\rho), \alpha)}{(\alpha, \alpha)}\geqq 1+\frac{2(\mu, \alpha)}{(\alpha, \alpha)}+\frac{2(w_1^{-1}w^{-1}\tau, \alpha)}{(\alpha, \alpha)}\geqq 1.$$

Hence $w^1(\nu+\rho)\in D$. Because $\nu+\rho$ is regular, that is, $(\nu+\rho, \alpha)\neq 0$ for all $\alpha\in\Pi$, we have $w^1=\mathrm{id}$ and we get our assertion.

Now we shall show that if $\nu'\in D$ satisfies $M_{\mu+\Lambda_{\alpha'},\lambda}(\nu')\neq 0$ then $\nu'=\nu+\Lambda_{\alpha'}$ for some $\nu\in D$. Since $\mu'=\mu+\Lambda_{\alpha'}$ also satisfies (3.1), by (3.2) we have

$$M_{\mu+\Lambda_{\alpha'},\lambda}(\nu')=\sum_{w_1\in W_1}\det(w_1)m(\nu'-\mu-\Lambda_{\alpha'}+\rho-w_1\rho).$$

Thus $\tau'=\nu'-\mu-\Lambda_{\alpha'}+\rho-w_1\rho\in\Delta(\lambda)$ for some $w_1\in W_1$. Obviously $(\nu'-\Lambda_{\alpha'}, \alpha)=(\nu', \alpha)\geqq 0$ for $\alpha\in\Pi_1$. For $\alpha\in\Pi-\Pi_1$ we have

$$\frac{2(\nu'-\Lambda_{\alpha'}, \alpha)}{(\alpha, \alpha)}=\frac{2(\mu+w_1\rho-\rho+\tau', \alpha)}{(\alpha, \alpha)}$$

$$\geqq\frac{2(\mu, \alpha)}{(\alpha, \alpha)}+\frac{2(\tau', \alpha)}{(\alpha, \alpha)}\geqq 0$$

by Lemma 3.2 and (3.1). Therefore $\nu'-\Lambda_{\alpha'}\in D$.

Now we see that

$$\begin{aligned} M_{\mu+\Lambda_{\alpha'},\lambda}(\nu') &= M_{\mu+\Lambda_{\alpha'},\lambda}(\nu+\Lambda_{\alpha'}) \\ &= \sum_{w_1\in W_1}\det(w_1)m(\nu+\Lambda_{\alpha'}-\mu-\Lambda_{\alpha'}+\rho-w_1\rho) \\ &= \sum_{w_1\in W_1}\det(w_1)m(\nu-\mu+\rho-w_1\rho) \\ &= M_{\mu,\lambda}(\nu). \end{aligned}$$

q. e. d.

COROLLARY 3.5. *Let* $\lambda\in Z_1^+$, $l\in \boldsymbol{Z}^+$ *and suppose that*

$$[l\lambda][\lambda]=\sum_{\nu\in D}M_{l\lambda,\lambda}(\nu)[\nu].$$

If the integer l *satisfies the inequalities*

$$l\geqq k_\alpha(\lambda)\Big/\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \tag{3.3}$$

for all $\alpha\in\Pi-\Pi_1$, *then*

$$[(l+1)\lambda][\lambda]=\sum_{\nu\in D}M_{l\lambda,\lambda}(\nu)[\nu+\lambda].$$

PROOF. Since $\lambda=\sum_{\alpha\in\Pi-\Pi_1}n_\alpha\Lambda_\alpha\ (n_\alpha\in\boldsymbol{Z}^+)$, our assertion follows from Theorem 3.4.

q. e. d.

For $\alpha\in\Pi$ let α_* be the unique element in $W\alpha\cap D$. we define an involutive automorphism π of Π by $\pi=-w_0$, where $w_0\in W$ is the unique element in W such that $w_0D=-D$. We set $\alpha^*=\pi(\alpha_*)\in D$.

LEMMA 3.6. *For* $\lambda\in D$ *and* $\alpha\in\Pi$,

$$k_\alpha(\lambda)=\frac{2(\lambda,\alpha^*)}{(\alpha^*,\alpha^*)}.$$

PROOF. Take $\tau\in\Delta(\lambda)$. Then $w\tau\in-D$ for some $w\in W$. Since $w\alpha=\alpha_*-\sum_{\beta\in\Pi}n_\beta\beta$ $(\boldsymbol{Z}\ni n_\beta\geqq 0)$, we have

$$\begin{aligned} -\frac{2(\tau,\alpha)}{(\alpha,\alpha)} &= -\frac{2(w\tau,w\alpha)}{(\alpha,\alpha)} = -\frac{2(w\tau,\alpha_*)}{(\alpha,\alpha)}+\sum n_\beta\frac{2(w\tau,\beta)}{(\alpha,\alpha)} \\ &\leqq -\frac{2(w\tau,\alpha_*)}{(\alpha,\alpha)} = -\frac{2(w\tau,\alpha_*)}{(\alpha_*,\alpha_*)}. \end{aligned}$$

Since $w_0\lambda$ is the lowest weight in $\Delta(\lambda)$, we have $w\tau=w_0\lambda+\sum_{\beta\in\Pi}m_\beta\beta$ $(\boldsymbol{Z}\ni m_\beta\geqq 0)$.

Thus we get

$$-\frac{2(\tau,\ \alpha)}{(\alpha,\ \alpha)} \leqq -\frac{2(w_0\lambda,\ \alpha_*)}{(\alpha_*,\ \alpha_*)} = \frac{2(\pi(\lambda),\ \alpha_*)}{(\alpha_*,\ \alpha_*)} = \frac{2(\lambda,\ \alpha^*)}{(\alpha^*,\ \alpha^*)}\,.$$

On the other hand, since $\alpha_* = w'\alpha$ for some $w' \in W$, we have

$$\frac{2(\lambda,\ \alpha^*)}{(\alpha^*,\ \alpha^*)} = -\frac{2(w_0\lambda,\ \alpha_*)}{(\alpha_*,\ \alpha_*)} = -\frac{2(w'^{-1}w_0\lambda,\ \alpha)}{(\alpha,\ \alpha)}\,,$$

where $w'^{-1}w_0\lambda \in \Delta(\lambda)$. Hence $k_\alpha(\lambda) = 2(\lambda,\ \alpha^*)/(\alpha^*,\ \alpha^*)$. q. e. d.

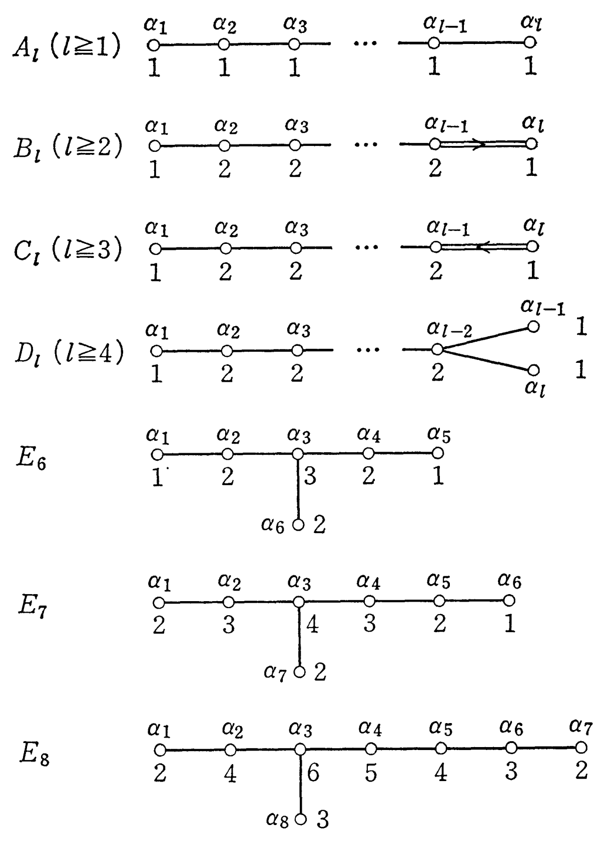
COROLLARY 3.7. *For $\alpha \in \Pi$ and $\lambda,\ \mu \in D$,*

$$k_\alpha(\lambda+\mu) = k_\alpha(\lambda) + k_\alpha(\mu)\,.$$

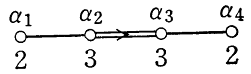
REMARK. Note that if Π is irreducible and $\alpha \in \Pi$ has the same length as the highest root α_0 of Σ, which is the same as that α has the largest length in Π, then $\alpha^* = \alpha_0$. In particular, in this case the integer $k_\alpha(\Lambda_\alpha)$ for the corresponding fundamental weight Λ_α is given as follows; Let $\alpha_0 = \sum_{\alpha \in \Pi} a_\alpha \alpha$ $(a_\alpha \in \boldsymbol{Z}^+)$. If α has the same length as the highest root α_0, then $k_\alpha(\Lambda_\alpha) = a_\alpha$.

We give here the table of the integers $k_\alpha(\Lambda_\alpha)$ $(\alpha \in \Pi)$ for each irreducible Dynkin diagram Π.

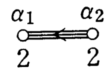
Table 1.



F_4



G_2



In the table the integer attached to $\alpha_i \in \Pi$ denotes $k_{\alpha_i}(\Lambda_{\alpha_i})$ for the corresponding fundamental weight Λ_{α_i}.

4. Main theorem.

MAIN THEOREM. *Let $L_{-\Lambda}$ ($\Lambda \in Z_1^+$) be a very ample line bundle on a kählerian C-space $M=G/U$. If a positive integer l satisfies the inequalities*

$$l \geqq k_\alpha(\Lambda) \Big/ \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \tag{4.1}$$

for all $\alpha \in \Pi - \Pi_1$, then the homogeneous ideal of $j_{L_{-\Lambda}^l}(M)$ is generated by quadrics.

PROOF. We shall show that the linear map

$$\psi:\ \mathfrak{R}(L_{-\Lambda}^i, L_{-\Lambda}) \otimes \Gamma(L_{-\Lambda}) \longrightarrow \mathfrak{R}(L_{-\Lambda}^{i+1}, L_{-\Lambda})$$

in Lemma 1.3 is surjective for all $i \geqq l$. Then our assertion will follow from Lemmas 1.3 and 2.1.

Note that the canonical map

$$\varphi:\ \Gamma(L_{-\Lambda}^i) \otimes \Gamma(L_{-\Lambda}) \longrightarrow \Gamma(L_{-\Lambda}^{i+1})$$

is a G-homomorphism so that the kernel $\mathfrak{R}(L_{-\Lambda}^i, L_{-\Lambda})$ of φ is a G-module and ψ is a G-homomorphism. By the same argument as in the proof of Lemma 2.1, we see that φ is surjective. In general, for a $\mathfrak{g}$-module V, the character of V is denoted by $[V]$, and for a character χ of $\mathfrak{g}$, the character contragredient to χ is denoted by χ^*. Suppose that

$$[i\Lambda][\Lambda] = \sum_{\nu \in D} M_{i\Lambda, \Lambda}(\nu)[\nu],$$

so that

$$[i\Lambda]^*[\Lambda]^* = \sum_{\nu \in D} M_{i\Lambda, \Lambda}(\nu)[\nu]^*.$$

Then it follows from section 2 (II) that

$$[\Gamma(L_{-\Lambda}^i) \otimes \Gamma(L_{-\Lambda})] = \sum_{\nu \in D} M_{i\Lambda, \Lambda}(\nu)[\nu]^*.$$

Recalling that $M_{i\Lambda, \Lambda}((i+1)\Lambda)=1$ and $[\Gamma(L_{-\Lambda}^{i+1})]=[(i+1)\Lambda]^*$ and the surjectivity of φ, we get

$$[\mathcal{R}(L_{-\Lambda}^{i}, L_{-\Lambda})]=\sum_{\substack{\nu\in D\\ \nu\neq(i+1)\Lambda}} M_{i\Lambda,\Lambda}(\nu)[\nu]^{*}.$$

Let $s\in\mathcal{R}(L_{-\Lambda}^{i}, L_{-\Lambda})$ and $v_{-\Lambda}\neq 0$ the element of $\Gamma(L_{-\Lambda})$ corresponding to a lowest weight vector. We claim that if $\phi(s\otimes v_{-\Lambda})=0$ then $s=0$. Taking a basis $\{b_j\}$ of $\Gamma(L_{-\Lambda})$, we can write $s=\sum a_j\otimes b_j$ $(a_j\in\Gamma(L_{-\Lambda}^{i}))$. Then $0=\phi(s\otimes v_{-\Lambda})=\sum(a_j v_{-\Lambda})\otimes b_j$ and hence $a_j v_{-\Lambda}=0$ in $\Gamma(L_{-\Lambda}^{i+1})$. Since the canonical map

$$\phi:\ S^{i}\Gamma(L_{-\Lambda})\longrightarrow\Gamma(L_{-\Lambda}^{i})$$

is surjective by Lemma 2.1, there is an element $A_j\in S^{i}\Gamma(L_{-\Lambda})$ such that $\phi(A_j)=a_j$. Now $a_j v_{-\Lambda}=0$ in $\Gamma(L_{-\Lambda}^{i+1})$ is the same as that the homogeneous polynomial $A_j v_{-\Lambda}$ vanishes on $j_{L_{-\Lambda}}(M)$. Since $v_{-\Lambda}\neq 0$, the homogeneous polynomial A_j vanishes on $j_{L_{-\Lambda}}(M)$. Therefore $a_j=0$ in $\Gamma(L_{-\Lambda}^{i})$ and we get $s=0$.

Let $s_{\nu}^{1},\cdots,s_{\nu}^{M_{i\Lambda,\Lambda}(\nu)}$ be linearly independent lowest weight vectors with weight $-\nu$ in $\mathcal{R}(L_{-\Lambda}^{i}, L_{-\Lambda})$. It follows from the above that then $\phi(s_{\nu}^{1}\otimes v_{-\Lambda}),\cdots,\phi(s_{\nu}^{M_{i\Lambda,\Lambda}(\nu)}\otimes v_{-\Lambda})$ are linearly independent lowest weight vectors with weight $-(\nu+\Lambda)$ in $\mathcal{R}(L_{-\Lambda}^{i+1}, L_{-\Lambda})$. This implies that ϕ is surjective for $i\geqq l$, because

$$[\mathcal{R}(L_{-\Lambda}^{i+1}, L_{-\Lambda})]=\sum_{\substack{\nu\in D\\ \nu\neq(i+1)\Lambda}} M_{i\Lambda,\Lambda}(\nu)[\nu+\Lambda]^{*}$$

for $i\geqq l$ by Corollary 3.5. q. e. d.

REMARK. By an explicit description (Borel-Hirzebruch [**1**]) of the Chern form of L_{Λ} $(\Lambda\in Z_1)$, we see that (4.1) is a condition for the Chern class of $L_{-\Lambda}$.

Let Π be an irreducible Dynkin diagram and Π_1 a subset of Π such that $\Pi-\Pi_1$ consists of only one root, say α, and that the highest root α_0 of the root system Σ with the fundamental root system Π has an expression as

$$\alpha_0=\alpha+\sum_{\beta\in\Pi_1} a_{\beta}\beta,\qquad a_{\beta}\in\boldsymbol{Z}^{+}.$$

Such a pair (Π, Π_1) is called an *irreducible symmetric pair*. The root α is called a *distinguished root* of Π. Let now (Π, Π_1) be a general pair of Dynkin diagrams. Decompose Π into the sum of irreducible components:

$$\Pi=\Pi^{1}\cup\cdots\cup\Pi^{t},$$

and put

$$\Pi_1^{i}=\Pi^{i}\cup\Pi_1,\qquad i=1,\cdots,t.$$

If each pair (Π^{i}, Π_1^{i}) is irreducible symmetric, the pair (Π, Π_1) is called a *symmetric pair*. It is known that the kählerian C-space corresponding to a symmetric pair is a Hermitian symmetric space of compact type. Conversely any Hermitian symmetric space of compact type is obtained in this way (cf. Takeuchi [**9**], § 4).

Now we give here the table of distinguished roots for each irreducible Dynkin diagram Π.

Table 2.

A_l $(l\geqq 1)$	$\alpha_1, \alpha_2, \cdots, \alpha_l$
B_l $(l\geqq 3)$	α_1
C_l $(l\geqq 3)$	α_l
D_l $(l\geqq 4)$	$\alpha_1, \alpha_{l-1}, \alpha_l$
E_6	α_1, α_5
E_7	α_6
F_4, G_2, E_8	no distinguished roots

In the table the numbering of Π is the same as that in Table 1.

COROLLARY OF MAIN THEOREM. *Let L be a very ample holomorphic line bundle on a Hermitian symmetric space M of compact type. Then the homogeneous ideal of $j_L(M)$ is generated by quadrics.*

PROOF. We may assume that $M=G/U$ is associated to a symmetric pair (Π, Π_1) and $L=L_{-\Lambda}$ for some $\Lambda\in Z_1^+$. Now we claim that

$$k_\alpha(\Lambda)=\frac{2(\Lambda, \alpha)}{(\alpha, \alpha)}$$

for all $\alpha\in\Pi-\Pi_1$. Decompose the Lie algebra $\mathfrak{g}$ into the direct sum of complex simple Lie algebras:

$$\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_t\,.$$

Since then an irreducible $\mathfrak{g}$-module is the tensor product of irreducible $\mathfrak{g}_i$-modules $(i=1, \cdots, t)$, we may assume that (Π, Π_1) is irreducible symmetric. Let α be the distinguished root for (Π, Π_1). Then $\Lambda=p\Lambda_\alpha$ $(p\in \boldsymbol{Z}^+)$, and hence $2(\Lambda, \alpha)/(\alpha, \alpha)=p$ and $k_\alpha(\Lambda)=pk_\alpha(\Lambda_\alpha)$ by Corollary 3.7. Thus it is enough to show that $k_\alpha(\Lambda_\alpha)=1$. But this follows from Tables 1 and 2.

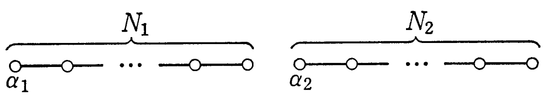
Hence we can take $l=1$ in Main Theorem. q. e. d.

REMARK. The same argument shows that if M is the kählerian C-space corresponding to (Π, Π_1) with

(a) $\Pi=B_l$, $\quad\Pi-\Pi_1=\{\alpha_l\}$, or
(b) $\Pi=C_l$, $\quad\Pi-\Pi_1=\{\alpha_1\}$,

then the same conclusion holds for M. But these kählerian C-spaces are also Hermitian symmetric spaces, so that these cases are included in our Corollary.

EXAMPLE 1. Let Π be



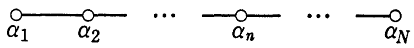
,

$\Pi-\Pi_1=\{\alpha_1, \alpha_2\}$ and $\Lambda=\Lambda_{\alpha_1}+\Lambda_{\alpha_2}$. Then $M=P_{N_1}(\boldsymbol{C})\times P_{N_2}(\boldsymbol{C})$ and the imbedding $j_{L_{-\Lambda}}$: $M\rightarrow P(\Gamma(L_{-\Lambda}))$ is given by

$$[z_i]_{0\leq i\leq N_1}\times[w_j]_{0\leq j\leq N_2}\longmapsto[z_iw_j]_{\substack{0\leq i\leq N_1\\ 0\leq j\leq N_2}}.$$

This imbedding is called the *Segre imbedding* and the homogeneous ideal of $j_{L_{-\Lambda}}(M)$ is generated by quadrics (cf. Hodge-Pedoe [5], vol. 2, p. 98).

EXAMPLE 2. Let Π be



,

$\Pi-\Pi_1=\{\alpha_n\}$ and $\Lambda=\Lambda_{\alpha_n}$. Then M is the complex Grassmann manifold of n-planes in $\boldsymbol{C}^{N+1}$ and the imbedding $j_{L_{-\Lambda}}$: $M\rightarrow P(\Gamma(L_{-\Lambda}))$ is the Plücker imbedding. The homogeneous ideal of $j_{L_{-\Lambda}}(M)$ is generated by quadrics and this is well known (cf. for example, Hodge-Pedoe [5], vol. 1, p. 315).

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