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Isometry of Kaehlerian manifolds to complex projective spaces

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§1. Introduction.

Let M be a complex n-dimensional connected Kaehlerian manifold covered by a system of real coordinate neighborhoods $\{U; x^h\}$, where, here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$ and let g_{ji}, $F_i{}^h$, $\{_j{}^h{}_i\}$, ∇_i, $K_{kji}{}^h$, K_{ji} and K be respectively the Hermitian metric tensor, the complex structure tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to $\{_j{}^h{}_i\}$, the curvature tensor, the Ricci tensor and the scalar curvature of M.

A vector field v^h is called a *holomorphically projective* (or *H-projective*, for brevity) vector field [**2**, **3**, **5**, **7**] if it satisfies

$$L_v\{_j{}^h{}_i\} = \nabla_j\nabla_i v^h + v^k K_{kji}{}^h = \delta_j^h \rho_i + \delta_i^h \rho_j - \rho_t F_j{}^t F_i{}^h - \rho_t F_i{}^t F_j{}^h \tag{1.1}$$

for a certain covariant vector field ρ_i on M, called the associated covariant vector field of v^h, where L_v denotes the operator of Lie derivation with respect to v^h. In particular, if ρ_i is zero vector field then v^h is called an affine vector field. When we refer in the sequel to an H-projective vector field v^h, we always mean by ρ_i the associated covariant vector field appearing in (1.1).

Recently, the present authors [**9**, **10**] and one of the present authors [**1**] proved a series of integral inequalities in a compact Kaehlerian manifold with constant scalar curvature admitting an H-projective vector field and then obtained necessary and sufficient conditions for such a Kaehlerian manifold to be isometric to a complex projective space with Fubini-Study metric.

The purpose of the present paper is to continue the joint work [**9**, **10**] of the present authors and to prove the following theorem.

THEOREM A. *If a complex $n>1$ dimensional, compact, connected and simply connected Kaehlerian manifold M with constant scalar curvature K admits a non-affine H-projective vector field v^h, then M is isometric to a complex projective space CP^n with Fubini-Study metric and of constant holomorphic sectional cur-*

vature $\frac{K}{n(n+1)}$.

In the sequel, we need the following theorem due to Obata [4]. (See also [6].)

THEOREM B. *Let M be a complete, connected and simply connected Kaehlerian manifold. In order for M to admit a non-trivial solution φ of a system of partial differential equations*

$$\nabla_j\nabla_i\varphi_h+\frac{c}{4}(2\varphi_j g_{ih}+\varphi_i g_{jh}+\varphi_h g_{ji}-F_{ji}F_h{}^t\varphi_t-F_{jh}F_i{}^t\varphi_t)=0\,, \tag{1.2}$$

where $\varphi_h=\nabla_h\varphi$ and $F_{ji}=F_j{}^t g_{ti}$, c being a positive constant, it is necessary and sufficient that M is isometric to a complex projective space CP^n with Fubini-Study metric and of constant holomorphic sectional curvature c.

We assume in this paper that Kaehlerian manifolds under consideration are connected.

§ 2. Preliminaries.

Let M be a complex n-dimensional Kaehlerian manifold. The complex structure tensor $F_i{}^h$ and the Hermitian metric tensor g_{ji} of M satisfy

$$F_t{}^h F_i{}^t=-\delta_i^h\,,\quad \nabla_j F_i{}^h=0\,,\quad \nabla_j F_{ih}=0\,, \tag{2.1}$$

$$g_{jt}F_i{}^t+g_{ti}F_j{}^t=0 \tag{2.2}$$

and

$$g_{ji}-g_{ts}F_j{}^t F_i{}^s=0\,. \tag{2.3}$$

We have [7, 9, 10], for the curvature tensor $K_{kji}{}^h$,

$$K_{kji}{}^t F_t{}^h-K_{kjt}{}^h F_i{}^t=0\,, \tag{2.4}$$

$$K_{kji}{}^h+K_{kjt}{}^s F_i{}^t F_s{}^h=0\,, \tag{2.5}$$

$$K_{kjit}F_h{}^t+K_{kjth}F_i{}^t=0 \tag{2.6}$$

and

$$K_{kjih}-K_{kjts}F_i{}^t F_h{}^s=0\,, \tag{2.7}$$

where $K_{kjih}=K_{kji}{}^t g_{th}$.

Using (2.6) and the first Bianchi identity

$$K_{kjih}+K_{ikjh}+K_{jikh}=0\,,$$

we have

$$2K_{kt}F_j{}^t = 2g^{ut}K_{kuts}F_j{}^s = -2g^{ut}K_{kusj}F_t{}^s$$
$$= -2K_{ktsj}F^{ts} = -(K_{ktsj} - K_{kstj})F^{ts}$$
$$= K_{tskj}F^{ts} = K_{kjts}F^{ts} ,$$

from which

(2.8) $$K_{kjts}F^{ts} = 2K_{kt}F_j{}^t ,$$

(2.9) $$K_{tjis}F^{ts} = -K_{jt}F_i{}^t$$

and

(2.10) $$K_{ktsh}F^{ts} = -K_{kt}F_h{}^t ,$$

g^{ji} being contravariant components of g_{ji} and $F^{kj} = g^{kt}F_t{}^j$.

From (2.8), we have, for the Ricci tensor $K_{ji} = K_{tji}{}^t$,

(2.11) $$K_{jt}F_i{}^t + K_{ti}F_j{}^t = 0 ,$$

(2.12) $$K_{ji} - K_{ts}F_j{}^tF_i{}^s = 0 ,$$

(2.13) $$K_t{}^hF_i{}^t - K_i{}^tF_t{}^h = 0$$

and

(2.14) $$K_i{}^h + K_t{}^sF_i{}^tF_s{}^h = 0 ,$$

where $K_i{}^h = K_{it}g^{th}$.

A Kaehlerian manifold M has the constant holomorphic sectional curvature k if and only if

(2.15) $$K_{kji}{}^h = \frac{k}{4}(\delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k{}^hF_{ji} - F_j{}^hF_{ki} - 2F_{kj}F_i{}^h) .$$

We define tensor fields G_{ji} and $Z_{kji}{}^h$ [**9**, **10**] on M by

(2.16) $$G_{ji} = K_{ji} - \frac{K}{2n} g_{ji}$$

and

(2.17) $$Z_{kji}{}^h = K_{kji}{}^h - \frac{K}{4n(n+1)}(\delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k{}^hF_{ji} - F_j{}^hF_{ki} - 2F_{kj}F_i{}^h)$$

respectively. If $G_{ji} = 0$ for $n > 1$ then M is a Kaehler-Einstein manifold and K is a constant and if $Z_{kji}{}^h = 0$ for $n > 1$ then M is of constant holomorphic sectional curvature $\frac{K}{n(n+1)}$.

We easily see that the tensor fields G_{ji} and $Z_{kji}{}^h$ satisfy

(2.18) $$G_{ji} = G_{ij} , \quad G_{ji}g^{ji} = 0 , \quad Z_{tji}{}^t = G_{ji} ,$$

(2.19) $$Z_{kjih}=-Z_{jkih}, \quad Z_{kjih}=Z_{ihkj}$$

and

(2.20) $$Z_{kji}{}^{h}+Z_{ikj}{}^{h}+Z_{jik}{}^{h}=0,$$

where $Z_{kjih}=Z_{kji}{}^{t}g_{th}$.

The tensor fields G_{ji} and $Z_{kji}{}^{h}$ also satisfy

(2.21) $$G_{jt}F_{i}{}^{t}+G_{ti}F_{j}{}^{t}=0,$$

(2.22) $$G_{ji}-G_{ts}F_{j}{}^{t}F_{i}{}^{s}=0,$$

(2.23) $$G_{t}{}^{h}F_{i}{}^{t}-G_{i}{}^{t}F_{t}{}^{h}=0,$$

(2.24) $$G_{i}{}^{h}+G_{t}{}^{s}F_{i}{}^{t}F_{s}{}^{h}=0,$$

(2.25) $$Z_{kji}{}^{t}F_{t}{}^{h}-Z_{kjt}{}^{h}F_{i}{}^{t}=0,$$

(2.26) $$Z_{kji}{}^{h}+Z_{kjt}{}^{s}F_{i}{}^{t}F_{s}{}^{h}=0,$$

(2.27) $$Z_{kjit}F_{h}{}^{t}+Z_{kjth}F_{i}{}^{t}=0,$$

(2.28) $$Z_{kjih}-Z_{kjts}F_{i}{}^{t}F_{h}{}^{s}=0,$$

(2.29) $$Z_{kjts}F^{ts}=2G_{kt}F_{j}{}^{t},$$

(2.30) $$Z_{tjis}F^{ts}=-G_{jt}F_{i}{}^{t}$$

and

(2.31) $$Z_{ktsh}F^{ts}=-G_{kt}F_{h}{}^{t},$$

where $G_{i}{}^{h}=G_{it}g^{th}$.

If the scalar curvature K is a constant, then, from (2.1), (2.17) and the second Bianchi identity

$$\nabla_{l}K_{kji}{}^{h}+\nabla_{j}K_{lki}{}^{h}+\nabla_{k}K_{jli}{}^{h}=0,$$

we have

(2.32) $$\nabla_{l}Z_{kji}{}^{h}+\nabla_{j}Z_{lki}{}^{h}+\nabla_{k}Z_{jli}{}^{h}=0,$$

from which and (2.18) and (2.19),

(2.33) $$\nabla_{t}Z_{kji}{}^{t}=\nabla_{k}G_{ji}-\nabla_{j}G_{ki}.$$

A vector field u^{h} on M is said to be *contravariant analytic* if

(2.34) $$(\nabla_{j}u_{t})F_{i}{}^{t}+(\nabla_{t}u_{i})F_{j}{}^{t}=0$$

or equivalently if

(2.35) $$\nabla_{j}u_{i}-(\nabla_{t}u_{s})F_{j}{}^{t}F_{i}{}^{s}=0,$$

where $u_i=g_{ih}u^h$. A vector field u^h on M is contravariant analytic if and only if

(2.36) $$L_u F_i{}^h=0\,,$$

where L_u denotes the operator of Lie derivation with respect to u^h. It is known [7] that if M is compact then a necessary and sufficient condition for a vector field u^h on M to be contravariant analytic is that

(2.37) $$\nabla^i\nabla_i u^h+K_i{}^h u^i=0$$

holds, where $\nabla^i=g^{ih}\nabla_h$.

For an H-projective vector field v^h on M defined by (1.1), we have

(2.38) $$\nabla_j\nabla_t v^t=2(n+1)\rho_j$$

and

(2.39) $$\nabla^i\nabla_i v^h+K_i{}^h v^i=0\,.$$

(2.38) shows that the associated covariant vector field ρ_j is gradient. Putting

(2.40) $$\rho=\frac{1}{2(n+1)}\nabla_t v^t\,,$$

we have

(2.41) $$\rho_j=\nabla_j\rho\,.$$

We have, for the H-projective vector field v^h, from (1.1),

(2.42) $$\nabla_j L_v g_{ih}=2\rho_j g_{ih}+\rho_i g_{jh}+\rho_h g_{ji}-F_{ji}F_h{}^t\rho_t-F_{jh}F_i{}^t\rho_t\,,$$

from which

(2.43) $$\nabla^j L_v g_{ih}=2\rho^j g_{ih}+\rho_i\delta_h^j+\rho_h\delta_i^j+F_i{}^jF_h{}^t\rho_t+F_h{}^jF_i{}^t\rho_t\,,$$

(2.44) $$\nabla_j L_v g^{ih}=-2\rho_j g^{ih}-\rho^i\delta_j^h-\rho^h\delta_j^i+F_j{}^iF^{ht}\rho_t+F_j{}^hF^{it}\rho_t$$

and

(2.45) $$\nabla^j L_v g^{ih}=-2\rho^j g^{ih}-\rho^i g^{jh}-\rho^h g^{ji}+F^{ji}F^{ht}\rho_t+F^{jh}F^{it}\rho_t\,,$$

where $\rho^h=\rho_i g^{ih}$.

Substituting (1.1) into the well known formula [7, 8]

$$L_v K_{kji}{}^h=\nabla_k L_v\{_j{}^h{}_i\}-\nabla_j L_v\{_k{}^h{}_i\}\,,$$

we find

(2.46) $$L_v K_{kji}{}^h=-\delta_k^h\nabla_j\rho_i+\delta_j^h\nabla_k\rho_i$$
$$+(F_k{}^h\nabla_j\rho_t-F_j{}^h\nabla_k\rho_t)F_i{}^t+(F_k{}^t\nabla_j\rho_t-F_j{}^t\nabla_k\rho_t)F_i{}^h\,,$$

from which, contracting with respect to h and k, (2.47) $$L_v K_{ji} = -2n\nabla_j \rho_i - 2(\nabla_t \rho_s) F_j{}^t F_i{}^s .$$

Suppose that an H-projective vector field v^h on M is contravariant analytic. Then, applying the operator L_v of Lie derivation with respect to v^h to both sides of (2.12), we have

$$L_v K_{ji} = (L_v K_{ts}) F_j{}^t F_i{}^s ,$$

from which and (2.47), we see that if $n>1$ then ρ^h is also contravariant analytic and

(2.48) $$L_v K_{ji} = -2(n+1)\nabla_j \rho_i$$

holds.

If a Kaehlerian manifold M is compact, then we see, from (2.39), *that an H-projective vector field v^h on M is contravariant analytic or equivalently $L_v F_i{}^h = 0$ holds and moreover if $n>1$ then the associated vector field ρ^h is also contravariant analytic and* (2.48) *holds.*

For a contravariant analytic H-projective vector field v^h on a complex $n>1$ dimensional Kaehlerian manifold M with constant scalar curvature K, we have [**9**, **10**], for the tensor field G_{ji},

(2.49) $$L_v G_{ji} = -\nabla_j w_i - \nabla_i w_j ,$$

where we have put

(2.50) $$w^h = (n+1)\rho^h + \frac{K}{2n} v^h$$

and $w_i = g_{ih} w^h$, and, for the tensor field $Z_{kji}{}^h$,

(2.51) $$L_v Z_{kji}{}^h = \frac{1}{2(n+1)} \{\delta_k^h L_v G_{ji} - \delta_j^h L_v G_{ki} - F_k{}^h (L_v G_{jt}) F_i{}^t + F_j{}^h (L_v G_{kt}) F_i{}^t - F_k{}^t (L_v G_{jt}) F_i{}^h + F_j{}^t (L_v G_{kt}) F_i{}^h\} .$$

§ 3. Proof of Theorem A.

In this section, we prove Theorem A. For this purpose, we need a series of lemmas. We use freely formulas (2.1)~(2.51) in the proofs of all lemmas and Theorem A in this section.

LEMMA 1 (Yano and Hiramatu [**9**]). *If, in a compact Kaehlerian manifold M, a non-constant function φ satisfies*

(3.1) $$\nabla_j \nabla_i \varphi_h + \frac{c}{4}(2\varphi_j g_{ih} + \varphi_i g_{jh} + \varphi_h g_{ji} - F_{ji} F_h{}^t \varphi_t - F_{jh} F_i{}^t \varphi_t) = 0 ,$$

where $\varphi_h = \nabla_h \varphi$, c being a constant, then the constant c is necessarily positive.

PROOF. Transvecting (3.1) with g^{ih}, we have

$$\nabla_j \Delta\varphi + (n+1)c\varphi_j = 0,$$

from which

$$c\int_M \varphi_j \varphi^j dV = -\frac{1}{n+1}\int_M (\nabla_j \Delta\varphi)\varphi^j dV = \frac{1}{n+1}\int_M (\Delta\varphi)^2 dV,$$

where $\Delta = g^{ji}\nabla_j\nabla_i$, $\varphi^j = g^{ji}\varphi_i$ and dV denotes the volume element of M. Since φ is a non-constant function, two inequalities

$$\int_M \varphi_j\varphi^j dV > 0, \quad \int_M (\Delta\varphi)^2 dV > 0$$

hold and consequently c is necessarily positive.

LEMMA 2 (Yano and Hiramatu [**9**]). *If a complete and simply connected Kaehlerian manifold M with positive constant scalar curvature K admits a non-affine H-projective vector field v^h and if the vector field w^h defined by* (2.50) *is a Killing vector field, then M is isometric to a complex projective space CP^n with Fubini-Study metric of constant holomorphic sectional curvature* $\dfrac{K}{n(n+1)}$.

PROOF. We have, from (1.1),

$$\nabla_j(\nabla_i v_h + \nabla_h v_i) = 2\rho_j g_{ih} + \rho_i g_{jh} + \rho_h g_{ji} - F_{ji}F_h{}^t\rho_t - F_{jh}F_i{}^t\rho_t. \tag{3.2}$$

If w^h is a Killing vector field then

$$\nabla_i w_h + \nabla_h w_i = 0$$

holds and consequently

$$2(n+1)\nabla_i\rho_h + \frac{K}{2n}(\nabla_i v_h + \nabla_h v_i) = 0,$$

from which and (3.2), we find

$$\nabla_j\nabla_i\rho_h + \frac{K}{4n(n+1)}(2\rho_j g_{ih} + \rho_i g_{jh} + \rho_h g_{ji} - F_{ji}F_h{}^t\rho_t - F_{jh}F_i{}^t\rho_t) = 0.$$

Thus the lemma follows from Theorem B.

REMARK. Using Lemma 1, we see that if, in Lemma 2, M is compact then we can remove the positiveness of the scalar curvature K from the assumption.

LEMMA 3 (Yano and Hiramatu [**9**]). *For an H-projective vector field v^h on a complex $n>1$ dimensional compact Kaehlerian manifold M with constant scalar curvature K, we have*

$$\int_M (\nabla_t w^t)^2 dV = \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV. \tag{3.3}$$

PROOF. By using a well known formula [7, 8] on a compact orientable Riemannian manifold, we have

$$\int_M (\nabla^i\nabla_i w^h + K_i{}^h w^i)w_h dV - \int_M (\nabla_t w^t)^2 dV + \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV = 0.$$

On the other hand, as was stated in Section 2, the associated vector field ρ^h is contravariant analytic and hence satisfies

$$\nabla^i\nabla_i\rho^h+K_i{}^h\rho^i=0\,.$$

Consequently (3.3) follows from (2.39), (2.50) and the above relations since K is a constant.

LEMMA 4 (Yano and Hiramatu [9]). *For an H-projective vector field v^h on a complex $n>1$ dimensional compact Kaehlerian manifold M with constant scalar curvature K, we have*

$$\int_M G_{ji}\rho^j w^i dV=\frac{1}{4(n+1)}\int_M(\nabla_j w_i+\nabla_i w_j)(\nabla^j w^i+\nabla^i w^j)dV\,. \tag{3.4}$$

PROOF. The associated vector field ρ^h is contravariant analytic and hence satisfies

$$\nabla^j\nabla_j\rho^i+K_j{}^i\rho^j=0\,,$$

from which and the equality

$$\nabla_i\nabla_t\rho^t=\nabla^t\nabla_t\rho_i-K_{ji}\rho^j$$

we find

$$\nabla_i\nabla_t\rho^t=-2K_{ji}\rho^j\,.$$

Using the above relation, (2.40), (2.41), (2.50) and Lemma 3, we have

$$\begin{aligned}\int_M G_{ji}\rho^j w^i dV&=\int_M K_{ji}\rho^j w^i dV-\frac{K}{2n}\int_M\rho_t w^t dV\\&=-\frac{1}{2}\int_M(\nabla_i\nabla_t\rho^t)w^i dV-\frac{K}{2n}\int_M\rho_t w^t dV\\&=-\frac{1}{2}\int_M(\nabla_i\nabla_t\rho^t)w^i dV-\frac{K}{4n(n+1)}\int_M(\nabla_i\nabla_t v^t)w^i dV\\&=-\frac{1}{2(n+1)}\int_M\left[\nabla_i\nabla_t\left\{(n+1)\rho^t+\frac{K}{2n}v^t\right\}\right]w^i dV\\&=-\frac{1}{2(n+1)}\int_M(\nabla_i\nabla_t w^t)w^i dV=\frac{1}{2(n+1)}\int_M(\nabla_t w^t)^2 dV\\&=\frac{1}{4(n+1)}\int_M(\nabla_j w_i+\nabla_i w_j)(\nabla^j w^i+\nabla^i w^j)dV\,.\end{aligned}$$

LEMMA 5 (Yano and Hiramatu [9]). *For an H-projective vector field v^h on a complex $n>1$ dimensional compact Kaehlerian manifold M with constant scalar curvature K, we have*

$$\int_M(\nabla^j L_v G_{ji})w^i dV=\frac{1}{2}\int_M(\nabla_j w_i+\nabla_i w_j)(\nabla^j w^i+\nabla^i w^j)dV\,. \tag{3.5}$$

PROOF. Integrating

$$\nabla^j\{(L_vG_{ji})w^i\}=(\nabla^jL_vG_{ji})w^i+(L_vG_{ji})\nabla^jw^i$$
$$=(\nabla^jL_vG_{ji})w^i+\frac{1}{2}(L_vG_{ji})(\nabla^jw^i+\nabla^iw^j)$$

over M and using (2.49), we have (3.5).

LEMMA 6 (Yano and Hiramatu [9]). *For an H-projective vector field v^h on a complex $n>1$ dimensional compact Kaehlerian manifold M with constant scalar curvature K, we have*

$$\int_M g^{kj}(L_v\nabla_kG_{ji})w^i dV=\frac{n}{2(n+1)}\int_M(\nabla_jw_i+\nabla_iw_j)(\nabla^jw^i+\nabla^iw^j)dV. \tag{3.6}$$

PROOF. Substituting (1.1) into

$$g^{kj}(L_v\nabla_kG_{ji})w^i=(\nabla^jL_vG_{ji})w^i-g^{kj}(L_v\{{}_k{}^t{}_j\})G_{ti}w^i-g^{kj}(L_v\{{}_k{}^t{}_i\})G_{jt}w^i,$$

we have

$$g^{kj}(L_v\nabla_kG_{ji})w^i=(\nabla^jL_vG_{ji})w^i-2G_{ji}\rho^jw^i.$$

Integrating this over M and using Lemmas 4 and 5, we have (3.6).

LEMMA 7 (Yano and Hiramatu [10]). *For an H-projective vector field v^h on a complex $n>1$ dimensional compact Kaehlerian manifold M with constant scalar curvature K, we have*

$$\int_M(\nabla^kL_vZ_{kji}{}^h)g^{ji}w_h dV=\frac{1}{n+1}\int_M(\nabla_jw_i+\nabla_iw_j)(\nabla^jw^i+\nabla^iw^j)dV. \tag{3.7}$$

PROOF. By using (2.51), we have

$$\begin{aligned}&(\nabla^kL_vZ_{kji}{}^h)g^{ji}w_h\\&=\frac{1}{2(n+1)}\{(\nabla^kL_vG_{ji})g^{ji}w_k-(\nabla^jL_vG_{ji})w^i\\&\qquad-F_k{}^h(\nabla^kL_vG_{jt})F_i{}^tg^{ji}w_h+F_j{}^h(\nabla^kL_vG_{kt})F_i{}^tg^{ji}w_h\\&\qquad-F_k{}^t(\nabla^kL_vG_{jt})F_i{}^hg^{ji}w_h+F_j{}^t(\nabla^kL_vG_{kt})F_i{}^hg^{ji}w_h\}.\end{aligned}$$

Here we notice that

$$-F_k{}^h(\nabla^kL_vG_{jt})F_i{}^tg^{ji}w_h=-F_k{}^h(\nabla^kL_vG_{jt})F^{jt}w_h=0,$$
$$F_j{}^h(\nabla^kL_vG_{kt})F_i{}^tg^{ji}w_h=(\nabla^jL_vG_{ji})w^i,$$
$$-F_k{}^t(\nabla^kL_vG_{jt})F_i{}^hg^{ji}w_h=F_j{}^t(\nabla^kL_vG_{tk})F_i{}^hg^{ji}w_h=(\nabla^jL_vG_{ji})w^i$$

and

$$F_j{}^t(\nabla^kL_vG_{kt})F_i{}^hg^{ji}w_h=(\nabla^jL_vG_{ji})w^i$$

hold. Therefore we have

$$(\nabla^k L_v Z_{kji}{}^h) g^{ji} w_h = \frac{1}{2(n+1)} \{(\nabla^k L_v G_{ji}) g^{ji} w_k + 2(\nabla^j L_v G_{ji}) w^i\}.$$

Integrating this over M and using (2.49) and Lemmas 3 and 5, we find

$$\begin{aligned}\int_M (\nabla^k L_v Z_{kji}{}^h) g^{ji} w_h dV &= \frac{1}{2(n+1)} \left\{\int_M (\nabla^k L_v G_{ji}) g^{ji} w_k dV + 2\int_M (\nabla^j L_v G_{ji}) w^i dV\right\} \\ &= \frac{1}{2(n+1)} \left\{-\int_M (L_v G_{ji}) g^{ji} \nabla_t w^t dV + 2\int_M (\nabla^j L_v G_{ji}) w^i dV\right\} \\ &= \frac{1}{n+1} \left\{\int_M (\nabla_t w^t)^2 dV + \int_M (\nabla^j L_v G_{ji}) w^i dV\right\} \\ &= \frac{1}{n+1} \int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV.\end{aligned}$$

LEMMA 8 (Yano and Hiramatu [**10**]). *For an H-projective vector field v^h on a complex $n>1$ dimensional compact Kaehlerian manifold M with constant scalar curvature K, we have*

$$\int_M (\nabla^k L_v Z_{kjih}) g^{ji} w^h dV = \frac{3}{2(n+1)} \int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dV. \tag{3.8}$$

PROOF. By using $Z_{kji}{}^h g^{ji} = G_k{}^h$ and

$$(\nabla^k Z_{kji}{}^h) g^{ji} = \nabla^k G_k{}^h = \frac{n-1}{2n} \nabla^h K = 0,$$

we have

$$\begin{aligned}(\nabla^k L_v Z_{kjih}) g^{ji} w^h &= \{\nabla^k L_v (Z_{kji}{}^t g_{th})\} g^{ji} w^h \\ &= (\nabla^k L_v Z_{kji}{}^h) g^{ji} w_h + G_k{}^t (\nabla^k L_v g_{th}) w^h.\end{aligned}$$

Substituting (2.43) into this, we find

$$(\nabla^k L_v Z_{kjih}) g^{ji} w^h = (\nabla^k L_v Z_{kji}{}^h) g^{ji} w_h + 2G_{ji} \rho^j w^i,$$

and consequently, integrating this over M and using Lemmas 4 and 7, we have (3.8).

Now we prove Theorem A. Using

$$\nabla^k Z_{kjih} = \nabla_k Z_{hij}{}^k$$

and

$$(L_v Z_{kjih}) g^{ji} = L_v G_{kh} - Z_{kjih} L_v g^{ji},$$

we have

$$(\nabla^k L_v Z_{kjih}) g^{ji} w^h = (\nabla^k L_v G_{kh}) w^h - (\nabla_k Z_{hij}{}^k)(L_v g^{ji}) w^h - Z_{kjih} (\nabla^k L_v g^{ji}) w^h.$$

Substituting (2.33) and (2.45) into this, we find

$$(\nabla^k L_v Z_{kjih})g^{ji}w^h=(\nabla^j L_v G_{ji})w^i-(\nabla_h G_{ij})(L_v g^{ji})w^h$$
$$+(\nabla_i G_{hj})(L_v g^{ji})w^h+4G_{ji}\rho^j w^i\,,$$

from which, integrating over M,

$$\int_M(\nabla^k L_v Z_{kjih})g^{ji}w^h dV=\int_M(\nabla^j L_v G_{ji})w^i dV-\int_M(\nabla_h G_{ji})(L_v g^{ji})w^h dV$$
$$+\int_M(\nabla_j G_{ih})(L_v g^{ji})w^h dV+4\int_M G_{ji}\rho^j w^i dV\,.$$

Here we notice that we have, using (2.44) and (2.49),

$$-\int_M(\nabla_h G_{ji})(L_v g^{ji})w^h dV=\int_M G_{ji}(\nabla_h L_v g^{ji})w^h dV+\int_M G_{ji}(L_v g^{ji})\nabla_t w^t dV$$
$$=-4\int_M G_{ji}\rho^j w^i dV-\int_M(L_v G_{ji})g^{ji}\nabla_t w^t dV$$
$$=-4\int_M G_{ji}\rho^j w^i dV+2\int_M(\nabla_t w^t)^2 dV$$

and, using

$$(\nabla_j G_{ih})g^{ji}=\frac{n-1}{2n}\nabla_h K=0\,,$$

$$\int_M(\nabla_j G_{ih})(L_v g^{ji})w^h dV=-\int_M g^{ji}(L_v\nabla_j G_{ih})w^h dV\,.$$

Consequently, we have

$$\int_M(\nabla^k L_v Z_{kjih})g^{ji}w^h dV$$
$$=2\int_M(\nabla_t w^t)^2 dV+\int_M(\nabla^j L_v G_{ji})w^i dV-\int_M g^{kj}(L_v\nabla_k G_{ji})w^i dV\,,$$

from which, using Lemmas 3, 5 and 6,

$$\int_M(\nabla^k L_v Z_{kjih})g^{ji}w^h dV=\frac{2n+3}{2(n+1)}\int_M(\nabla_j w_i+\nabla_i w_j)(\nabla^j w^i+\nabla^i w^j)dV\,.$$

From this and Lemma 8, we have

$$\int_M(\nabla_j w_i+\nabla_i w_j)(\nabla^j w^i+\nabla^i w^j)dV=0\,,$$

from which

$$\nabla_j w_i+\nabla_i w_j=0\,,$$

that is, the vector field w^h is a Killing vector field. Thus, Theorem A follows from Lemma 2.

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