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Approximation of certain classes of periodic functions with many variables

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§ 0. Introduction.

Let Π_n be the class of all trigonometric polynomials of degree n or less. If f is a continuous 2π-periodic function, the n-th degree of approximation for f is defined by

$$E_n(f)=\inf_{T\in\Pi_n}\|f-T\|=\inf_{T\in\Pi_n}\sup_{|x|\leq\pi}|f(x)-T(x)|.$$

Let the class $W^{(p)}$ $(p\geqq 1)$ consist of all the 2π-periodic functions for which there exists a $(p-1)$-th absolutely continuous derivative $f^{(p-1)}(x)$, and $|f^{(p)}(x)|\leqq 1$ almost everywhere. The exact value of $E_n(W^{(p)})=\sup_{f\in W^{(p)}}E_n(f)$ is well-known.

THEOREM A. (Favard [1], Akhiezer and Krein [2]) *The degree of approximation of the classes* $W^{(p)}$, $p=1, 2, \cdots$ *is given by*

$$E_{n-1}(W^{(p)})=K_p n^{-p}, \qquad n=1, 2, \cdots,$$

where

$$K_p=(4/\pi)\sum_{k=0}^{\infty}(-1)^{k(p+1)}(2k+1)^{-p-1}. \tag{0.1}$$

The class $\tilde{W}^{(p)}$ conjugate to $W^{(p)}$ consists of all conjugate functions $\tilde{f}$ of $f\in W^{(p)}$, that is,

$$\tilde{W}^{(p)}=\{\tilde{f}\,;\,\tilde{f}(x)=(-2/\pi)\int_0^{\pi}[f(x+t)-f(x-t)]\cot(t/2)dt,\ f\in W^{(p)}\}.$$

The exact value of $E_n(\tilde{W}^{(p)})$ is also known.

THEOREM B. (Akhiezer and Krein [2]) *The degree of approximation of the classes* $\tilde{W}^{(p)}$, $p=1, 2, \cdots$ *is given by*

$$E_{n-1}(\tilde{W}^{(p)})=\tilde{K}_p n^{-p}, \qquad n=1, 2, \cdots,$$

where

$$\tilde{K}_p=(4/\pi)\sum_{k=0}^{\infty}(-1)^{kp}(2k+1)^{-p-1}. \tag{0.2}$$

We can consider more general classes. Let w be a modulus function of continuity, and let $w(f, \cdot)$ be the modulus of continuity of a function f. Let p be a certain non-negative integer. Then $W_w^{(p)}$ is the set of all 2π-periodic function f with the property $w(f^{(p)}, h) \leqq w(h)$. We denote the class conjugate to $W_w^{(p)}$ by $\tilde{W}_w^{(p)}$. The following result was given by Korneĭčuk [5].

THEOREM C. *If w is a concave modulus function of continuity we have*

$$E_{n-1}(W_w^{(0)}) = \Phi_{n1}(\pi/n)w(\pi/n),$$

$$E_{n-1}(W_w^{(p)}) = (1/2)\int_0^{\pi/n} w(t)\Phi_{np}(\pi/n - t)dt \qquad (p=1, 2, \cdots),$$

where

$$\Phi_{n1}(x) = 1/2, \ \Phi_{np}(x) = (1/2)\int_0^{\pi/n - x} \Phi_{np-1}(t)dt \qquad (p=2, 3, \cdots).$$

The expression in Theorem C is not so simple. However, we know a simple form relating to the exact order of decrease of $E_{n-1}(W_w^{(p)})$. Also, we have the exact order of decrease of $E_{n-1}(\tilde{W}_w^{(p)})$.

THEOREM D. *Let w be a modulus function of continuity. Then we have*

$$C_p n^{-p} w(1/n) \leqq E_{n-1}(W_w^{(p)}) \leqq 3n^{-p} w(1/n),$$

where C_p is a constant depending on p such that

$$C_p w(1/n) \leqq w(\pi/4n) \int_{|t| \leqq \pi/2} \sum_{k=0}^{\infty} (-1)^{k(p+1)} (2k+1)^{-p} \sin(2k+1)t \, dt .$$

Also, if $p \geqq 1$ we have

$$C_p n^{-p} w(1/n) \leqq E_n(\tilde{W}_w^{(p)}) \leqq 3n^{-p} w(1/n)$$

(see Akhiezer [3], [4] and Timan [7, p. 507]).

In this paper we extend Theorem D to certain classes of functions with many variables. If we apply our method to the case of one variable, we have the following.

THEOREM D′. *If w is a concave modulus function of continuity, we have*

$$(K_{p+1}/\pi) n^{-p} w(\pi/n) \leqq E_{n-1}(W_w^{(p)}) \leqq (K_p/2) n^{-p} w(\pi/n),$$

for each $p=1, 2, \cdots$, $n=1, 2, \cdots$, and

$$(\tilde{K}_{p+1}/\pi) n^{-p} w(\pi/n) \leqq E_{n-1}(\tilde{W}_w^{(p)}) \leqq (\tilde{K}_p/2) n^{-p} w(\pi/n),$$

for each $p=1, 2, \cdots$, $n=1, 2, \cdots$, where K_p and $\tilde{K}_p$ are the constants mentioned in (0.1) *or* (0.2).

Some of the results relating to the approximation of functions of several variables are found in the books [7] or [8]. We need the following theorem in the next section.

THEOREM E. ([8, Chapter 6, Theorem 7]) *Let $f(x_1, \cdots, x_s)$ be a continuous 2π-periodic function that has continuous partial derivatives $\partial^k f/\partial x_i^k$, $0 \leqq k \leqq p$, $i=1, 2, \cdots, s$ $(p \geqq 0)$. Assume that the modulus of continuity of $\partial^p f/\partial x_i^p$ with respect to x_i does not exceed a given modulus function of continuity $w(h)$ for $i=1, 2, \cdots, s$. Then*

$$E_{n-1;s}(f) \leqq Mn^{-p}w(1/n), \tag{0.3}$$

where $E_{n-1;s}(f)$ is the degree of approximation for f by the trigonometric polynomials, with s variables $x_1, \cdots, x_s$, of degree $n-1$ or less, and M is a constant depending only on p.

(0.3) cannot be improved, that is, we are able to find a certain function f_0, with the property $w(\partial^p f_0/\partial x_i^p, h) \leqq w(h)$ for each $i=1, 2, \cdots, s$, such that

$$M'_p n^{-p} w(1/n) \leqq E_{n-1}(f_0),$$

where M'_p is some positive constant depending only on p (see [8, Chapter 9, Theorem 1]).

§ 1. Notations and lemmas.

Let $f(x_1, \cdots, x_s)$ be a continuous function of s variables on an s-dimensional torus $K^{(s)}$, the product of s circles $K=(-\pi, \pi]$. The space $C[K^{(s)}]$ consists of such continuous functions, and $f \in C[K^{(S)}]$ has a norm

$$\|f\| = \max_{(x_1, \cdots, x_s) \in K^{(s)}} |f(x_1, \cdots, x_s)|.$$

Let $L^1[K^{(s)}]$ consist of all functions f with a norm

$$\|f\|_1 = \int_K \cdots \int_K |f(x_1, \cdots, x_s)| dx_1 \cdots dx_s < \infty.$$

We put

$$C_k(x) = \cos kx \quad (k=0, 1, \cdots), \qquad S_k(x) = \sin kx \quad (k=1, 2, \cdots).$$

We define the trigonometric polynomials of degrees $n_1, \cdots, n_s$ (n_i non-negative integers, $i=1, 2, \cdots, s$) by

$$T_{n_1, \cdots, n_s}(x_1, \cdots, x_s) = \sum_{k_i \leqq n_i,\, i=1, \cdots, s} a_{k_1, \cdots, k_s} U_{k_1}(x_1) \cdots U_{k_s}(x_s),$$

where U_k denotes C_k or S_k. $\Pi_{n_1, \cdots, n_s}$ is a subspace of $C[K^{(s)}]$ consisting of all trigonometric polynomials of degrees $n_1, \cdots, n_s$. The degree of approximation for $f \in C[K^{(s)}]$ by $\Pi_{n_1, \cdots, n_s}$ is

$$E_{n_1,\cdots,n_s}(f)=\inf_{T\in\Pi_{n_1,\cdots,n_s}}\|f-T\|.$$

Similarly,

$$E_{n_1,\cdots,n_s}(f)_1=\inf_{T\in\Pi_{n_1,\cdots,n_s}}\|f-T\|_1$$

is the degree of approximation for $f\in L^1[K^{(s)}]$ by $\Pi_{n_1,\cdots,n_s}$. Let $p_1, \cdots, p_s$ be s non-negative integers. We write $(r_1, \cdots, r_s)\leqq(p_1, \cdots, p_s)$ if s non-negative integers $r_1, \cdots, r_s$ satisfy $r_i\leqq p_i$ for $i=1, \cdots, s$. For the given integers $p_1, \cdots, p_s$ we define the class of functions

$$F^{(p_1\cdots p_s)}$$
$$=\Big\{f;\, f\in C[K^{(s)}],\ \int_K\cdots\int_K|(\partial^{r_s}/\partial x_s^{r_s}\cdots\partial^{r_1}/\partial x_1^{r_1})f(x_1, \cdots, x_s)|\,dx_1\cdots dx_s<\infty$$
$$\text{for all } (r_1, \cdots, r_s) \text{ with } (r_1, \cdots, r_s)\leqq(p_1, \cdots, p_s)\Big\}.$$

We consider the subclasses of $F^{(p_1\cdots p_s)}$. Let

$$P(\{p_j\};s)=\{(r_1, \cdots, r_s);\, r_i=0 \text{ or } p_i,\ i=1, \cdots, s,\ r_1+\cdots+r_s\neq 0\}.$$

Put

$$W^{(p_1\cdots p_s)}=\{f;\, f\in F^{(p_1\cdots p_s)},\ |(\partial^{r_s}/\partial x_s^{r_s}\cdots\partial^{r_1}/\partial x_1^{r_1})f(x_1, \cdots, x_s)|\leqq 1 \text{ a.e.},$$
$$\text{for all } (r_1, \cdots, r_s)\in P(\{p_j\};s)\}.$$

Let w be a modulus function of continuity, then we define

$$W_w^{(p_1\cdots p_s)}=\{f;\, (\partial^{p_s}/\partial x_s^{p_s}\cdots\partial^{p_1}/\partial x_1^{p_1})f\in C[K^{(s)}],$$
$$w((\partial^{r_s}/\partial x_s^{r_s}\cdots\partial^{r_1}/\partial x_1^{r_1})f,\, h)\leqq w(h)$$
$$\text{for all } (r_1, \cdots, r_s) \text{ with } (r_1, \cdots, r_s)\in P(\{p_j\};s)\},$$

where $w(g, h)$ is the modulus of continuity of g, that is,

$$w(g, h)=\max_{|t_i|\leqq h,\, i=1,\cdots,s}|g(x_1+t_1, \cdots, x_s+t_s)-g(x_1, \cdots, x_s)|.$$

Let W be a class of functions, then we denote the degree of approximation of W by

$$E_{n_1,\cdots,n_s}(W)=\sup_{f\in W}E_{n_1,\cdots,n_s}(f).$$

Our main purpose is to estimate the exact order of $E_{n_1,\cdots,n_s}(W)$, where W is one of the classes

$$W^{(p_1\cdots p_s)} \quad\text{and}\quad W_w^{(p_1\cdots p_s)}.$$

We use the following notations:

$$U_{k;p}=\begin{cases} C_k & \text{if } p \text{ is even}, \\ S_k & \text{if } p \text{ is odd}, \end{cases}$$

$$T_{n;s}=T_{n,\cdots,n}, \qquad E_{n;s}(W)=E_{n,\cdots,n}(W),$$

$$f^{[r_1\cdots r_s]}(x_1, \cdots, x_s)=(\partial^{r_s}/\partial x_s^{r_s} \cdots \partial^{r_1}/\partial x_1^{r_1})f(x_1, \cdots, x_s),$$

$$I(\{r_j\}\,;\, s)=\{i\,;\, r_i\neq 0 \text{ and such that } 1\leqq i\leqq s\}.$$

We need several lemmas. Certain results in L^1-approximation are also necessary for uniform approximation. The following is well-known in the case $s=1$ (see [8, Chapter 8]).

LEMMA 1. *Let $T_0\in \Pi_{n_1,\cdots,n_s}$, and let $f\in L^1[K^{(s)}]$. Then T_0 is a polynomial of best approximation of f if*

$$\int_K \cdots \int_K T(x_1, \cdots, x_s) \operatorname{sign}[f(x_1, \cdots, x_s)-T_0(x_1, \cdots, x_s)]dx_1 \cdots dx_s=0$$
$$\text{for all} \quad T\in \Pi_{n_1,\cdots,n_s}.$$

If $f(x_1, \cdots, x_s)-T_0(x_1, \cdots, x_s)$ vanishes only on a set of measure zero, this condition is also necessary.

We consider the kernels

$$D^{[0]}(x)=-1/2, \quad D^{[p]}(x)=(1/\pi)\sum_{k=1}^{\infty} k^{-p}\cos(kx-p\pi/2) \quad (p=1, 2, \cdots),$$

and

$$\tilde{D}^{[p]}(x)=(1/\pi)\sum_{k=1}^{\infty} k^{-p}\sin(kx-p\pi/2) \qquad (p=1, 2, \cdots).$$

If $f^{(p)}$ $(p\geqq 1)$ is integrable, we have a representation

$$f(x)=(1/2\pi)\int_K f(t)dt+\int_K D^{[p]}(x-t)f^{(p)}(t)dt$$

or

$$\tilde{f}(x)=\int_K \tilde{D}^{[p]}(x-t)f^{(p)}(t)dt,$$

where $\tilde{f}$ is the conjugate function of f (see [7, p. 288, p. 315]). Thus we have a representation for $f\in F^{(p_1\cdots p_s)}$ as follows.

LEMMA 2. *Let $p_1, \cdots, p_s$ be the positive integers. For each $f\in F^{(p_1\cdots p_s)}$ we have*

$$f(x_1, \cdots, x_s)=(2\pi)^{-s}\int_K \cdots \int_K f(x_1, \cdots, x_s)dx_1\cdots dx_s$$

$$+\sum_{(r_1,\cdots,r_s)\in P(\{p_j\};s)}\int_K\cdots\int_K\prod_{i=1}^{s}D^{[r_i]}(x_i-t_i)f^{[r_1\cdots r_s]}(t_1,\cdots,t_s)dt_1\cdots dt_s.$$

But, if $r_j=0$ *for some* $j=1,\cdots,s$ *we consider the integral*

$$\int_K\cdots\int_K\prod_{i=1}^{s}D^{[r_i]}(x_i-t_i)f^{[r_1\cdots r_s]}(t_1,\cdots,t_s)dt_1\cdots dt_s$$

as the integral

$$\int_K\cdots\int_K D^{[0]}(x_j-t_j)\prod_{i\neq j}D^{[r_i]}(x_i-t_i)(\partial^{r_s}/\partial x_s^{r_s}\cdots\partial^{r_1}/\partial x_1^{r_1})f(t_1,\cdots,t_s)$$
$$\times dt_1\cdots(-dt_j)\cdots dt_s$$
$$=(1/2\pi)\int_K\cdots\int_K\prod_{i\neq j}D^{[r_i]}(x_i-t_i)(\partial^{r_s}/\partial x_s^{r_s}\cdots\partial^{r_1}/\partial x_1^{r_1})f(t_1,\cdots,t_s)$$
$$\times dt_1\cdots dt_j\cdots dt_s.$$

We can define the "conjugate" class of $W^{(p_1\cdots p_s)}$ or $W_w^{(p_1\cdots p_s)}$, that is, for each positive integers $p_1,\cdots,p_s$

$$\widetilde{W}^{(p_1\cdots p_s)}=\Big\{\tilde{f};f\in W^{(p_1\cdots p_s)},\ \tilde{f}(x_1,\cdots,x_s)$$
$$=\sum_{(r_1,\cdots,r_s)\in P(\{p_j\};s)}\int_K\cdots\int_K\prod_{i=1}^{s}\tilde{D}^{[r_i]}(x_i-t_i)f^{[r_1\cdots r_s]}(t_1,\cdots,t_s)dt_1\cdots dt_s\Big\},$$

and

$$\widetilde{W}_w^{(p_1\cdots p_s)}=\Big\{\tilde{f};f\in W_w^{(p_1\cdots p_s)},\ \tilde{f}(x_1,\cdots,x_s)$$
$$=\sum_{(r_1,\cdots,r_s)\in P(\{p_j\};s)}\int_K\cdots\int_K\prod_{i=1}^{s}\tilde{D}^{[r_i]}(x_i-t_i)f^{[r_1\cdots r_s]}(t_1,\cdots,t_s)dt_1\cdots dt_s\Big\},$$

where $\tilde{D}^{[0]}(t)\equiv-1/2\pi$. The following result is well-known ([6], [7, p. 81]).

LEMMA 3. *For each* $p=1,2,\cdots$ *we have*

$$E_{n-1}(D^{[p]})_1=\int_K D^{[p]}(t)\operatorname{sign}U_{n;p}(t)dt=K_pn^{-p},$$

$$E_{n-1}(\tilde{D}^{[p]})_1=\int_K\tilde{D}^{[p]}(t)\operatorname{sign}U_{n;p}(t)dt=\tilde{K}_pn^{-p}.$$

Our main lemma is

LEMMA 4. *For each* $(r_1,\cdots,r_s)\in P(\{p_j\};s)\ (\neq\emptyset)$, *we have*

$$\max_{i\in I(\{r_j\};s)}\prod_{i\neq j}\|D^{[r_j]}\|_1K_{r_i}n_i^{-r_i}\leqq E_{n_1-1,\cdots,n_s-1}\Big(\prod_{i\in I(\{r_j\};s)}D^{[r_i]}\Big)_1$$
$$\leqq\sum_{\emptyset\neq Q\subset I(\{r_j\};s)}\prod_{i\in I(\{r_j\};s)\setminus Q}\|D^{[r_i]}\|_1\prod_{i\in Q}K_{r_i}n_i^{-r_i},$$

where $\prod_{i\in\emptyset}\|D^{[r_i]}\|_1=1$. *Also, we have*

$$\max_{i\in I(\{r_j\};s)}\prod_{i\neq j}\|\tilde{D}^{[r_i]}\|_1\tilde{K}_{r_i}n_i^{-r_i}\leqq E_{n_1-1,\dots,n_s-1}(\prod_{i\in I(\{r_j\};s)}\tilde{D}^{[r_i]})_1$$

$$\leqq\sum_{\emptyset\neq Q\subset I(\{r_j\};s)}\prod_{i\in I(\{r_j\};s)\setminus Q}\|\tilde{D}^{[r_i]}\|_1\prod_{i\in Q}\tilde{K}_{r_i}n_i^{-r_i},$$

where $\prod_{i\in\emptyset}\|\tilde{D}^{[r_i]}\|_1=1$.

PROOF. Let $r_j\neq 0$, and let $I=I(\{r_j\};s)$. By Lemma 1 and Lemma 3 we have

$$E_{n_1-1,\dots,n_s-1}(\prod_{i\in I}D^{[r_i]})_1$$

$$=\|\prod_{i\in I}D^{[r_i]}-T_{n_1-1,\dots,n_s-1}\|_1$$

$$=\int_K\cdots\int_K[\prod_{i\in I}D^{[r_i]}-T_{n_1-1,\dots,n_s-1}]\operatorname{sign}[\prod_{i\in I}D^{[r_i]}-T_{n_1-1,\dots,n_s-1}]dx_1\cdots dx_s$$

$$\geqq\int_K\cdots\int_K[\prod_{i\in I}D^{[r_i]}-T_{n_1-1,\dots,n_s-1}]\operatorname{sign}U_{n_j;r_j}\prod_{i\in I,i\neq j}D^{[r_i]}dx_1\cdots dx_s$$

$$=\int_K\cdots\int_K\prod_{i\in I}D^{[r_i]}\operatorname{sign}U_{n_j;r_j}\prod_{i\in I,i\neq j}D^{[r_i]}dx_1\cdots dx_s$$

$$=\prod_{i\neq j}\|D^{[r_i]}\|_1K_{r_j}n_j^{-r_j}.$$

Conversely, we must have the estimation from above. We put

$$E_{n_i-1}(D^{[r_i]})_1=\|D^{[r_i]}-T_{n_i-1}\|_1\qquad\text{for each } i=1,\cdots,s.$$

Then it is sufficient to show

$$\int_K\cdots\int_K|\prod_{i\in I}D^{[r_i]}-\prod_{i\in I}T_{n_i-1}|dx_1\cdots dx_s$$

(1.2)

$$\leqq\sum_{\emptyset\neq Q\subset I}\prod_{i\in I\setminus Q}\|D^{[r_i]}\|_1\prod_{i\in Q}K_{r_i}n_i^{-r_i},$$

for simplicity we denote by A_s the second hand side. In the case of $s=1$ it follows from Lemma 3. For some $s-1$ ($\geqq 1$) we assume (1.2). Let $r_s\neq 0$, then for s we have

$$\int_K\cdots\int_K|\prod_{i\in I}D^{[r_i]}-\prod_{i\in I}T_{n_i-1}|dx_1\cdots dx_s$$

$$\leqq\int_K\cdots\int_K|\prod_{i\in I}D^{[r_i]}-D^{[r_s]}\prod_{i\in I,i\neq s}T_{n_i-1}|dx_1\cdots dx_s$$

$$+\int_K\cdots\int_K|D^{[r_s]}\prod_{i\in I,i\neq s}T_{n_i-1}-\prod_{i\in I}T_{n_i-1}|dx_1\cdots dx_s$$

$$\leqq \|D^{[r_s]}\|_1 A_{s-1}+B_s ,$$

where $B_s=\|\prod_{i\in I, i\neq s} T_{n_i-1}\|_1 K_{r_s} n_s^{-r_s}$. But, by $\|D^{[r_i]}-T_{n_i-1}\|_1=K_{r_i} n_i^{-r_i}$ we have

$$\|T_{n_i-1}\|_1 \leqq \|D^{[r_i]}\|_1+K_{r_i} n_i^{-r_i} .$$

Therefore we have

$$B_s \leqq \prod_{i\in I, i\neq s} [\|D^{[r_i]}\|_1+K_{r_i} n_i^{-r_i}] K_{r_s} n_s^{-r_s}$$

$$= \prod_{i\in I, i\neq s} \|D^{[r_i]}\|_1 K_{r_s} n_s^{-r_s}+\prod_{i\in I} K_{r_i} n_i^{-r_i} .$$

Thus, we have

$$\|D^{[r_s]}\|_1 A_{s-1}+B_s \leqq A_s .$$

By the inductive method we have (1.2).

Similarly, we have also the estimations for the conjugate type. (q. e. d.)

LEMMA 5. *For each* $j\in I(\{r_j\}\,;\, s)$ *the function*

$$f_j(x_1, \cdots, x_s)=\int_K D^{[r_j]}(t_j)\,\mathrm{sign}\, U_{n_j;r_j}(x_j-t_j)\,dt_j$$

belongs to $W^{(p_1\cdots p_s)}$, *and then*

$$E_{n_1-1,\dots,n_s-1}(f_j)=\|f_j\| \geqq |f_j(0, \cdots, 0)|=K_{r_j} n_j^{-r_j} . \tag{1.3}$$

Similarly, we see that the function

$$\tilde{f}_j(x_1, \cdots, x_s)=\int_K \tilde{D}^{[r_j]}(t_j)\,\mathrm{sign}\, U_{n_j;r_j}(x_j-t_j)\,dt_j$$

belongs to $\widetilde{W}^{(p_1\cdots p_s)}$, *and then*

$$E_{n_1-1,\dots,n_s-1}(\tilde{f}_j)=\|\tilde{f}_j\| \geqq |\tilde{f}_j(0, \cdots, 0)|=\tilde{K}_{r_j} n_j^{-r_j} . \tag{1.4}$$

PROOF. Let

$$E_{n_1-1,\dots,n_s-1}(f_j)=\|f_j-T_{n_1-1,\dots,n_s-1}\| .$$

For each fixed point $(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_s)$ we see

$$E_{n_1-1,\dots,n_s-1}(f_j) \geqq \max_{x_j} |f_j(x_1, \cdots, x_s)-T_{n_1-1,\dots,n_s-1}(x_1, \cdots, x_s)|$$

$$\geqq E_{n_j-1}(f_j(x_1, \cdots, x_{j-1}, \cdot, x_{j+1}, \cdots, x_s))$$

$$=\max_{x_j} |f_j(x_1, \cdots, x_s)|$$

(see [8, Chapter 8]). Thus, we have (1.3). Similarly, we have (1.4). (q. e. d.)

Let w and ω be two modulus functions of continuity. Then we denote by $E_W(0)(W_w^{(0)})$ the degree of approximation of the class $W_w^{(0)}$ by the class $W_\omega^{(0)}$. We know the exact value of $E_{W_\omega^{(0)}}(W_w^{(0)})$, that is,

$$E_{W_\omega^{(0)}}(W_w^{(0)})=(1/2)\max_h[w(h)-\omega(h)]$$

(see [8, Chapter 8, Theorem 8]). We can extend this to the case of many variables (Theorem 2.2). To prove it we need the following lemma. We omit its proof because it is not so difficult.

LEMMA 6. *Let ω be a modulus function of continuity, and let $W(x)$ be defined such that it is even and 2π-periodic, and*

$$W(x)=\omega(x), \qquad 0\leqq x\leqq\pi .$$

Then, for $|x|, |y|\leqq\pi$ we have

$$W(x+y)\leqq W(x)+W(y).$$

We use the notation $\|(x_1, \cdots, x_s)\|=\max_i|x_i|$. Theorem 2.2 is applied to estimate the exact order of decrease of $E_{n-1;s}(W_w^{(0,\cdots,0)})$, where w is concave (Theorem 2.3). The following lemmas are used to estimate the exact order of $E_{n-1;s}(W_w^{(p_1\cdots p_s)})$.

LEMMA 7. *Let w be a concave modulus function of continuity. Then, for each h, $0<h\leqq\pi$, there is an $M\geqq 0$ for which*

$$\max_{0\leqq x\leqq\pi}[w(x)-Mx]=w(h)-Mh \tag{1.5}$$

(see [8, Chapter 8]).

LEMMA 8. *Let f satisfy the Lipschitz conditions*

$$|f(x_1, \cdots, x_i+t, \cdots, x_s)-f(x_1, \cdots, x_i, \cdots, x_s)|\leqq|t|, \quad i=1, \cdots, s. \tag{1.6}$$

Then there is a constant M such that

$$E_{n-1;s}(f)\leqq Mn^{-1}. \tag{1.7}$$

Also, for each $i=1, \cdots, s$ the function

$$f_i(x_1, \cdots, x_s)=\int_K D^{[1]}(t_i)\operatorname{sign} U_{n_i;1}(x_i-t_i)dt_i$$

satisfies the condition (1.6), *and*

$$E_{n-1;s}(f_i)\geqq(\pi/2)n^{-1}. \tag{1.8}$$

PROOF. (1.7) follows from Theorem E, and (1.8) has been obtained in Lemma 5. (q. e. d.)

§ 2. Theorems.

THEOREM 2.1. *Let $p_1, \cdots, p_s$ be the positive integers, then we have an estimation*

$$\max_{i\in I(\{p_j\};s)} K_{p_i} n_i^{-p_i} \leqq E_{n_1-1,\cdots,n_s-1}(W^{(p_1\cdots p_s)})$$

$$\leqq \sum_{(r_1,\cdots,r_s)\in P(\{p_j\};s)} \sum_{\emptyset\neq Q\subset I(\{r_j\};s)} \prod_{i\in I(\{r_j\};s)\setminus Q} \|D^{[r_i]}\|_1 \prod_{i\in Q} K_{r_i} n_i^{-r_i} .$$

PROOF. The first inequality follows from Lemma 5. We must show the second inequality. Let $P=P(\{p_j\};s)$ and $I=I(\{r_j\};s)$. Put

$$E_{n_1-1,\cdots,n_s-1}(\prod_{i\in I} D^{[r_i]})_1 = \prod_{i\in I} \|D^{[r_i]} - S_{n_1-1,\cdots,n_s-1}\|_1 ,$$

where $S_{n_1-1,\cdots,n_s-1}$ is a trigonometric polynomial of degree $n_1-1, \cdots, n_s-1$. For each $f\in W^{(p_1\cdots p_s)}$ we put

$$T_{n_1-1,\cdots,n_s-1}(x_1, \cdots, x_s) = (2\pi)^{-s}\int_K \cdots \int_K f(x_1, \cdots, x_s)dx_1 \cdots dx_s$$

$$+ \sum_{(r_1,\cdots,r_s)\in P} \int_K \cdots \int_K S_{n_1-1,\cdots,n_s-1}(x_1-t_1, \cdots, x_s-t_s)$$

$$\times f^{[r_1\cdots r_s]}(t_1, \cdots, t_s)dt_1 \cdots dt_s .$$

Then we have

$$\|f-T_{n_1-1,\cdots,n_s-1}\| \leqq \sum_{(r_1,\cdots,r_s)\in P} E_{n_1-1,\cdots,n_s-1}(\prod_{i\in I} D^{[r_i]})_1 .$$

By Lemma 4 we have the second inequality. (q. e. d.)

If we put $E_{W^{(0\cdots 0)}_\omega}(W^{(0\cdots 0)}_w) = \sup_{f\in W^{(0\cdots 0)}_w} \inf_{g\in W^{(0\cdots 0)}_\omega} \|f-g\|$ we have

THEOREM 2.2. *Let w and ω be two modulus functions of continuity. Then we have*

$$E_{W^{(0\cdots 0)}}(W^{(0\cdots 0)}_w) = (1/2) \max_h (w(h)-\omega(h)) .$$

PROOF. Let W stand for the function defined by ω as in Lemma 6. Then we put

$$g(x_1, \cdots, x_s) = \min_{\|(t_1,\cdots,t_s)\|\leqq\pi} \{f(x_1+t_1, \cdots, x_s+t_s) + W(\|(t_1, \cdots, t_s)\|)\} .$$

Let $\|(x_1-y_1, \cdots, x_s-y_s)\| \leqq \pi$. For each t_i we can find an integer $k(t_i)$ such that

$$u_i = x_i+t_i-y_i-2k(t_i)\pi , \qquad |u_i| \leqq \pi .$$

Thus we have

$$\begin{aligned} & g(x_1, \cdots, x_s) \\ &= \min_{\|(t_1,\cdots,t_s)\|\leqq\pi} \{f(y_1+x_1+t_1-y_1, \cdots, y_s+x_s+t_s-y_s)+W(\|t_1, \cdots, t_s)\|)\} \\ &= \min_{\|(t_1,\cdots,t_s)\|\leqq\pi} \{f(y_1+u_1, \cdots, y_s+u_s) \\ &\qquad +W(\|(u_1-x_1+y_1+2k(t_1)\pi, \cdots, u_s-x_s+y_s+2k(t_s)\pi)\|)\}. \end{aligned}$$

Here, by Lemma 6 we have

$$\begin{aligned} & W(\|(u_1-x_1+y_1+2k(t_1)\pi, \cdots, u_s-x_s+y_s+2k(t_s)\pi)\|) \\ &\quad =W(|u_j-x_j+y_j+2k(t_j)\pi|) \qquad (\text{for some } j=1, \cdots, s) \\ &\quad =W(|u_j-x_j+y_j|) \\ &\quad \leqq W(|u_j|)+W(|x_j-y_j|) \\ &\quad \leqq W(\|(u_1, \cdots, u_s)\|)+W(\|(x_1-y_1, \cdots, x_s-y_s)\|). \end{aligned}$$

Therefore, we see

$$g(x_1, \cdots, x_s)\leqq g(y_1, \cdots, y_s)+W(\|(x_1-y_1, \cdots, x_s-y_s)\|).$$

Consequently, we have

$$|g(x_1, \cdots, x_s)-g(y_1, \cdots, y_s)|\leqq W(\|(x_1-y_1, \cdots, x_s-y_s)\|).$$

Now, if we consider the function

$$g_0=g+d, \qquad d=(1/2)\max_h \{w(h)-\omega(h)\}$$

we see $g_0\in W_\omega^{(0\cdots0)}$, and since

$$\begin{aligned} & 0\leqq f(x_1, \cdots, x_s)-g(x_1, \cdots, x_s) \\ &\quad \leqq |f(x_1, \cdots, x_s)-f(x_1+t_1, \cdots, x_s+t_s)|-\omega(\|(t_1, \cdots, t_s)\|) \\ &\qquad\qquad (\text{for some } t_1, \cdots, t_s) \\ &\quad \leqq w(\|(t_1, \cdots, t_s)\|)-\omega(\|(t_1, \cdots, t_s)\|), \end{aligned}$$

we have

$$\|f-g_0\|\leqq d.$$

To complete the proof, we have to construct a function $f_0\in W_w^{(0\cdots0)}$, which cannot be approximated by any function $g\in W_\omega^{(0\cdots0)}$ with an error $<d$. Define a 2π-periodic function

$$f_0(x_1, \cdots, x_s)=w(\|(x_1, \cdots, x_s)\|) \qquad \text{for } \|(x_1, \cdots, x_s)\|\leqq\pi.$$

If $\|(x_1-y_1, \cdots, x_s-y_s)\|\leqq\pi$ we can find two points $(x_1', \cdots, x_s')$ and $(y_1', \cdots, y_s')$

for which $x_i-2k_i\pi=x_i'$, $y_i-2j_i\pi=y_i'$ for some integers k_i and j_i ($i=1, \cdots, s$), and $\|(x_1', \cdots, x_s')\|\leqq\pi$, $\|(y_1', \cdots, y_s')\|\leqq\pi$. Then we have $|x_i-y_i|=\min\{|x_i'-y_i'|, |x_i'+y_i'|\}$. Let $\|(x_1', \cdots, x_s')\|=|x_i'|\leqq\|(y_1', \cdots, y_s')\|=|y_j'|$, then we see

$$\begin{aligned}|f_0(x_1, \cdots, x_s)-f_0(y_1, \cdots, y_s)|&=|f_0(x_1', \cdots, x_s')-f_0(y_1', \cdots, y_s')|\\&\leqq w(||x_i'|-|y_j'||)\\&\leqq w(|y_j'|-|x_j'|)\\&\leqq w(\min\{|x_j'-y_j'|, |x_j'+y_j'|\})\\&=w(|x_j-y_j|)\\&\leqq w(\|(x_1-y_1, \cdots, x_s-y_s)\|).\end{aligned}$$

Thus $f_0\in W_w^{(0\cdots0)}$. Let $g\in W_\omega^{(0\cdots0)}$ and $\|(x_1, \cdots, x_s)\|\leqq\pi$, then we have

$$\begin{aligned}2\|f_0-g\|&\geqq|f_0(x_1, \cdots, x_s)|-|g(x_1, \cdots, x_s)-g(0, \cdots, 0)|\\&\geqq w(\|(x_1, \cdots, x_s)\|)-\omega(\|(x_1, \cdots, x_s)\|).\end{aligned}$$

Therefore we see

$$\|f_0-g\|\geqq d. \qquad \text{(q. e. d.)}$$

THEOREM 2.3. *If w is a concave modulus function of continuity we have an estimation*

$$(1/2)w(\pi/n)\leqq E_{n-1;s}(W_w^{(0\cdots0)})\leqq w(Cn^{-1}),$$

where C is a constant.

PROOF. If f belongs to $W_{w_0}^{(0\cdots0)}$, where $w_0(t)=t$, it satisfies the Lipschitz conditions (1.6). Thus, by Lemma 8 there is a constant C such that

$$E_{n-1;s}(W_{w_0}^{(0\cdots0)})=Cn^{-1}. \tag{2.1}$$

Let $M\geqq0$ denote a value of M for which (1.5) holds, with $h=2Cn^{-1}$. By Theorem 2.2, for each $f\in W_w^{(0\cdots0)}$, there exists $g\in W_{w_1}^{(0\cdots0)}$ with $w_1(t)=Mt$ such that

$$\|f-g\|\leqq(1/2)[w(2Cn^{-1})-M(2Cn^{-1})].$$

By (2.1), there is a trigonometric polynomial $T_{n-1;s}$ of degree $n-1$ or less such that

$$\|g-T_{n-1;s}\|\leqq MC\,n^{-1}.$$

Therefore we have

$$\|f-T_{n-1;s}\|\leqq(1/2)w(2Cn^{-1})\leqq w(Cn^{-1}).$$

Let $f_n(x_1, \cdots, x_s)$ be an odd and $2\pi/n$-periodic function such that

$$f_n(x_1, \cdots, x_s)=\begin{cases}(1/2)w(2x_1), & 0\leqq x_1\leqq\pi/2n,\\(1/2)w(2\pi/n-2x_1), & \pi/2n\leqq x_1\leqq\pi/n,\end{cases}$$

then we see $f_n \in W_w^{(0\cdots 0)}$. For each fixed point $(x_2, \cdots, x_s)$, the polynomial of best approximation for the function f_n with one variable x_1 is $T_{n-1}(x_1)\equiv 0$ (see [8, Chapter 2]). Thus, for each fixed point $(x_2, \cdots, x_s)$

(2.2) $$E_{n-1;s}(f_n)=\|f_n-T_{n-1;s}\|\geqq E_{n-1}(f_n)=\max_{x_1}|f_n(x_1, \cdots, x_s)|=\|f_n\|.$$

Consequently, we see that the polynomial of best approximation for f_n with s variables is $T_{n-1;s}=0$. Thus, we have

$$E_{n-1;s}(W_w^{(0\cdots 0)})\geqq E_{n-1;s}(f_n)=\|f_n\|=(1/2)w(\pi/n). \qquad \text{(q. e. d.)}$$

THEOREM 2.4. *Let $p_1, \cdots, p_s$ be the positive integers, and let w be a concave modulus function of continuity. Then we have*

(2.3) $$\begin{aligned}&\max_i (1/\pi)K_{p_i+1}n^{-p_i}w(\pi/n)\leqq E_{n-1;s}(W_w^{(p_1\cdots p_s)})\\ &\leqq\Big[\sum_{(r_1,\cdots,r_s)\in P}\ \sum_{\emptyset\neq Q\subset I}\ \prod_{i\in I\setminus Q}\|D^{[r_i]}\|_1\prod_{i\in Q}K_{r_i}n^{-r_i}\Big]w(C/n),\end{aligned}$$

where $P=P(\{p_j\}; s)$, $I=I(\{r_j\}; s)$ and C is a constant.

PROOF. For each $f\in W_w^{(p_1\cdots p_s)}$ we have a representation

$$\begin{aligned}f(x_1, \cdots, x_s)=&(2\pi)^{-s}\int_K\cdots\int_K f(t_1, \cdots, t_s)dt_1\cdots dt_s\\ &+\sum_{(r_1,\cdots,r_s)\in P}\int_K\cdots\int_K\prod_{i=1}^s D^{[r_i]}(x_i-t_i)f^{[r_1\cdots r_s]}(t_1, \cdots, t_s)dt_1\cdots dt_s.\end{aligned}$$

Let $E_{n-1;s}(\prod_{i=1}^s D^{[r_i]})_1=\|\prod_{i=1}^s D^{[r_i]}-S_{n-1;s}\|_1$, where $S_{n-1;s}$ is a polynomial of degree $n-1$ or less, and let

$$\begin{aligned}T_{n-1;s}(x_1, \cdots, x_s)=&(2\pi)^{-s}\int_K\cdots\int_K f(t_1, \cdots, t_s)dt_1\cdots dt_s\\ &+\sum_{(r_1,\cdots,r_s)\in P}\int_K\cdots\int_K S_{n-1;s}(x_1-t_1, \cdots, x_s-t_s)f^{[r_1\cdots r_s]}(t_1, \cdots, t_s)dt_1\cdots dt_s.\end{aligned}$$

Then we see

$$\begin{aligned}&|[f(x_1, \cdots, x_s)-T_{n-1;s}(x_1, \cdots, x_s)]-[f(y_1, \cdots, y_s)-T_{n-1;s}(y_1, \cdots, y_s)]|\\ &\leqq\sum_{(r_1,\cdots,r_s)\in P}\Big|\int_K\cdots\int_K\Big[\prod_{i=1}^s D^{[r_i]}(t_i)-S_{n-1;s}(t_1, \cdots, t_s)\Big]\\ &\qquad\times[f^{[r_1\cdots r_s]}(x_1-t_1, \cdots, x_s-t_s)-f^{[r_1\cdots r_s]}(y_1-t_1, \cdots, y_s-t_s)]dt_1\cdots dt_s\Big|\\ &\leqq\Big[\sum_{(r_1,\cdots,r_s)\in P}E_{n-1;s}\Big(\prod_{i=1}^s D^{[r_i]}\Big)_1\Big]w(\|(x_1-y_1, \cdots, x_s-y_s)\|).\end{aligned}$$

By Theorem 2.3 we have

$$E_{n-1;s}(f)\leqq[\sum_{(r_1,\cdots,r_s)\in P}E_{n-1;s}(\prod_{i=1}^{s}D^{[r_i]})_1]w(Cn^{-1}).$$

Thus, by using Lemma 4 we obtain the second inequality in (2.3).

Next, we have to get the estimation from below. We consider an odd and $2\pi/n$-periodic function such that

$$g_n(x)=\begin{cases} x, & 0\leqq x\leqq\pi/2n, \\ \pi/n-x, & \pi/2n\leqq x\leqq\pi/n, \end{cases}$$

for $n=1, 2, \cdots$. For each p_i, $i=1, \cdots, s$, we put

$$g_n^{[i]}(x)=\begin{cases} g_n(x) & \text{if } p_i \text{ is odd}, \\ g_n(x-\pi/2n) & \text{if } p_i \text{ is even}, \end{cases}$$

and then we define

$$h_{n;i}(x_1, \cdots, x_s)=(n/\pi)w(\pi/n)g_n^{[i]}(x_i).$$

It is not difficult to prove $h_{n;i}\in W_w^{(0\cdots 0)}$. If we consider the function

$$f_{n;i}(x_1, \cdots, x_s)=\int_K D^{[p_i]}(t_i)h_{n;i}(x_1-t_1, \cdots, x_s-t_s)dt_i$$

we see $f_{n;i}\in W^{(p_1\cdots p_s)}$ and

$$f_{n;i}(x_1, \cdots, x_s)=(n/\pi)w(\pi/n)\int_K D^{[p_i+1]}(t_i)\,\mathrm{sign}\,U_{n;p_i+1}(x_i-t_i)dt_i.$$

Thus, we have

$$E_{n-1;s}(f_{n;i})\geqq\|f_{n;i}\|\geqq|f_{n;i}(0, \cdots, 0)|=(1/\pi)K_{p_i+1}n^{-p_i}w(\pi/n)$$

(see (2.2)). (q. e. d.)

By the same lines of consideration we obtain the following result.

THEOREM 2.5. *Let $p_1, \cdots, p_s$ be the positive integers, and let w be a concave modulus function of continuity. Then we have*

$$\max_i (1/\pi)\tilde{K}_{p_i+1}n^{-p_i}w(\pi/n)\leqq E_{n-1;s}(\tilde{W}_w^{(p_1\cdots p_s)})$$
$$\leqq[\sum_{(r_1,\cdots,r_s)\in P}\sum_{\emptyset\neq Q\subset I}\prod_{i\in I\setminus Q}\|\tilde{D}^{[r_i]}\|_1\prod_{i\in Q}\tilde{K}_{r_i}n^{-r_i}]w(Cn^{-1}),$$

where C is a constant, and $P=P(p_i; s)$, $I=I(r_i; s)$.

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