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Non existence of irreducible birecurrent Riemannian manifold of dimension $\geqq 3$

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Introduction.

Formerly, A. Lichnerowicz [1] defined a *birecurrent* (recurrent of the 2nd order) Riemannian manifold by $\nabla^2 R = R \otimes a$, where R is the Riemannian curvature tensor field, a is a covariant tensor field of order 2 and ∇ is the covariant differential. He proved that if a birecurrent M is compact and the scalar curvature does nowhere vanish it is recurrent in the ordinary sense: $\nabla R = R \otimes \alpha$, where α is a 1-form on M. W. Roter [2] treated this problem, but it contains some errors.

It is known (Kobayashi-Nomizu [3], p. 305) that an irreducible recurrent Riemannian manifold of dimension n is locally symmetric if $n \geqq 3$ and whether it is irreducible or not, the universal covering manifold $\tilde{M}$ of a connected complete recurrent Riemannian M is either a globally symmetric space or $M = R^{n-2} \times V^2$, where R^{n-2} is an $(n-2)$-dimensional flat manifold and V^2 is a 2-dimensional Riemannian manifold. The main purpose of this paper is to prove the following theorem.

THEOREM. *If an irreducible Riemannian manifold M of dimension n $(\geqq 3)$ is birecurrent, then M is recurrent in the ordinary sense.*

The case where $n=2$ or M is reducible will be also considered in § 3.

§ 1. Preliminary lemmas.

Although the following discussions are available for Riemannian manifolds of class C^4, we suppose the manifolds to be of class C^∞ for simplicity. 'Differentiable' always means 'C^∞-differentiable'. We use the local expression of each tensor field with respect to a local coordinate system $(x^1, \cdots, x^n)$. The indices run from 1 to n and the summation convention is adopted. The Riemannian metric of M is denoted by g whose components are (g_{ij}) or (g^{ij}). The components of curvature tensor field R are given by

$$R^i{}_{jkh}=\partial\left\{{i \atop hj}\right\}\Big/\partial x^k-\partial\left\{{i \atop kj}\right\}\Big/\partial x^h+\left\{{m \atop hj}\right\}\left\{{i \atop km}\right\}-\left\{{m \atop kj}\right\}\left\{{i \atop hm}\right\},$$

where $\left\{{i \atop jk}\right\}$ are Christoffel's symbols obtained from g. The Ricci tensor field is denoted by S whose components are $R_{ij}=R^k{}_{ikj}$ and $K=g^{ij}R_{ij}$ is the scalar curvature field. The components of ∇S, for example, are denoted by $\nabla_i R_{jk}$ or $\nabla^i R_{jk}$. For a contra- or covariant tensor field T of degree p (components $T^{i_1\cdots i_p}$ or $T_{i_1\cdots i_p}$), we use the notation $|T|^2=g(T, T)=T^{i_1\cdots i_p}T_{i_1\cdots i_p}$, in particular $|R|^2=R^{ijkh}R_{ijkh}$ and $|S|^2=R^{ij}R_{ij}$. The value of R at $p\in M$, for example, is denoted by R_p.

As indicated in the Introduction, M is said to be birecurrent or recurrent of the 2nd order, if

(1.1) $$\nabla^2 R=R\otimes a \quad \text{or} \quad \nabla_m\nabla_l R_{ijkh}=a_{lm}R_{ijkh}$$

where a is a covariant tensor field of order 2 with components (a_{ij}). From (1.1), we have immediately

(1.2) $$\nabla^2 S=S\otimes a$$

(1.3) $$\nabla^2 K=Ka\,.$$

Hereafter, we consider such a birecurrent Riemannian M. We call a the *birecurrence tensor field* and a point $p\in M$ such that $a_p\neq 0$ is said to be *regular.* If a vanishes identically on M, then $\nabla^2 R=0$. It is known by Nomizu-Ozeki [4] that in a complete Riemannian manifold, $\nabla^m R=0$ $(m\geqq 2)$ implies $\nabla R=0$ and it is remarked later that the assumption of completeness is not necessary. Namely, if $a=0$ on M, then M is locally symmetric.

In a general birecurrent M, the following equation holds which is easily verified.

(1.4) $$\nabla_j\nabla_i(|R|^2)=2a_{ij}|R|^2+2(\nabla_i R^{khlm})(\nabla_j R_{khlm})\,.$$

Then, without loss of generality, we can assume that the birecurrence tensor field a of (1.1) *is symmetric.* In fact, suppose the open submanifold $M'=\{p\in M\,|\,R_p\neq 0\}$ of M, then a is symmetric on M' by (1.4). Let $a'=(a'_{ij})$ be a symmetric covariant tensor field defined by $a'_{ij}=\dfrac{1}{2}(a_{ij}+a_{ji})$, then clearly $\nabla^2 R=R\otimes a'$ holds on M'. Let $p\in M-M'$ be an arbitrary point, then $R_p=0$ and $(\nabla^2 R)_p=0$ by (1.1). Hence the above expression of $\nabla^2 R$ holds also at p.

LEMMA 1.1.

(1.5) $$|R|^2 a=2|S|^2 a=K^2 a\,.$$

PROOF. First, making use of (1.2) we have

$$R_{ai}R^{a}{}_{kjh}+R_{ka}R^{a}{}_{ijh}=0\,,$$

by the Ricci identity and the symmetricity of a. By a contraction, we get

$$R_{ai}R^{a}{}_{j}+R^{ab}R_{iabj}=0\,. \tag{1.6}$$

On the other hand, operate ∇_m to the 2nd Bianchi's identity:

$$\nabla_l R_{ijkh}+\nabla_i R_{jlkh}+\nabla_j R_{likh}=0\,,$$

then we get

$$a_{lm}R_{ijkh}+a_{im}R_{jlkh}+a_{jm}R_{likh}=0\,, \tag{1.7}$$

by (1.1). By a contraction, we have

$$a_{lm}R_{ik}-a_{im}R_{lk}+a_{am}R_{lik}{}^{a}=0\,, \tag{1.8}$$

where $R_{lik}{}^{a}=g^{ab}R_{likb}$. By a contraction, we get

$$a_{lm}|S|^2-a_{im}R_{lk}R^{ik}+a_{am}R_{lik}{}^{a}R^{ik}=0\,,$$

which induces

$$a_{lm}|S|^2=2a_{im}R_{lk}R^{ik}, \tag{1.9}$$

by (1.6). Next, multiplying R^{ijkh} to (1.7) and contracting with respect to the indices, we get easily

$$a_{lm}|R|^2=4a_{im}R_{lk}R^{ik}, \tag{1.10}$$

where we have used (1.8) and (1.6). Lastly, by a contraction of (1.8), we have

$$a_{lm}K=2a_{im}R^{i}{}_{l}\,.$$

Multiplying $R^{l}{}_{k}$ and contraction with respect to l, we get

$$a_{lm}K^2=4a_{im}R_{lk}R^{ik} \tag{1.11}$$

where we have used $a_{lm}R^{l}{}_{k}=\nabla_m\nabla_l R^{l}{}_{k}=\dfrac{1}{2}\nabla_m(\nabla_k K)=\dfrac{1}{2}a_{km}K$. Summing up (1.9), (1.10) and (1.11), we get the required equations. q. e. d.

Suppose that $K\not\equiv 0$ on M. Then in the open submanifold $M'=\{p\in M\,|\,K_p\neq 0\}$, a_{ij} is of the form

$$a_{ij}=(\nabla_j\nabla_i K)/K\,, \tag{1.12}$$

by (1.3).

LEMMA 1.2. *Suppose that the scalar curvature field K does not vanish identically on M of dimension $\geqq 2$. Then, in the open submanifold $M'=\{p\in M\,|\,K_p\neq 0\}$, the following identity holds:*

$$(1.13)\qquad \nabla^i\nabla_i\Big(\Big(\frac{1}{2}|R|^2+|S|^2-K^2\Big)/K\Big)$$
$$=(|\nabla R-R\otimes\nabla K/K|^2+2|\nabla S-S\otimes\nabla K/K|^2)/K.$$

PROOF. By straightforward calculations, we get

$$\frac{1}{2}\nabla^i\nabla_i(|R|^2/K)=|\nabla R-R\otimes\nabla K/K|^2/K+\frac{1}{2}(\mathrm{Tr}\ a)|R|^2/K,$$
$$\nabla^i\nabla_i(|S|^2/K)=2|\nabla S-S\otimes\nabla K/K|^2/K+(\mathrm{Tr}\ a)|S|^2/K,$$
$$\nabla^i\nabla_i K=(\mathrm{Tr}\ a)K,$$

where $\mathrm{Tr}\ a=g^{ij}a_{ij}$. Now the Lemma immediately follows from Lemma 1.1.

q. e. d.

LEMMA 1.3. *Suppose M to be connected and of dimension $\geqq 2$. If there exists a point $p\in M$ such that $R_p=(\nabla R)_p=0$, then M is flat.*

PROOF. At first, we prove that if a point $q\in M$ can be joined with p by a geodesic, then $R_q=0$. In fact, let $c:[t_0, t_1]\to M$ be such a geodesic: $c(t_0)=p$, $c(t_1)=q$, where $[t_0, t_1]$ is a closed interval of real number field. Let X, Y, Z, W be vector fields along c obtained by parallel displacements from arbitrary tangent vectors X_p, Y_p, Z_p, W_p at p. By (1.1), we have

$$\dot{c}(R(X, Y, Z, W))=\nabla_{\dot{c}}R(X, Y, Z, W)=(\nabla R)(X, Y, Z, W;\ \dot{c})$$
$$\dot{c}((\nabla R)(X, Y, Z, W;\ \dot{c}))=\nabla_{\dot{c}}((\nabla R)(X, Y, Z, W;\ \dot{c}))$$
$$=a(\dot{c}, \dot{c})R(X, Y, Z, W)$$

where $\dot{c}$ denotes the tangent vector of c and $\nabla_{\dot{c}}$ means the covariant differentiation in the direction $\dot{c}$. By the assumption and by the uniqueness of solutions of linear differential equations, we have

$$R(X, Y, Z, W)=(\nabla R)(X, Y, Z, W;\ \dot{c})=0,\quad \text{along } c.$$

Since the initial tangent vectors X_p, Y_p, Z_p, W_p can be arbitrarily chosen, we get $R=0$ along c, in particular $R_q=0$.

Now let q be an arbitrary point, then we can easily see that q is joined with p by a finite number of geodesic arcs, so that we also have $R_q=0$.

LEMMA 1.4. *Suppose a connected M of dimension $\geqq 2$ whose scalar curvature K does not vanish identically and has some zero point. Then the subset $M_0=\{p\in M|K_p=0\}$ is a closed $(n-1)$-dimensional submanifold with respect to the induced topology. Moreover, M_0 is totally geodesic as a Riemannian submanifold with respect to the induced metric.*

PROOF. At first we prove that $\nabla K=dK\neq 0$ on M_0, where d denotes the differential. In fact, assume that $(\nabla K)_p=0$ at $p\in M_0$. Then $K_p=(\nabla K)_p=0$ hold, so that $K=0$ all over M, which is verified quite analogously to the proof of Lemma 1.3. This contradicts to the assumption for K. Then as is well known, M_0 becomes a closed $(n-1)$-dimensional submanifold of M with respect to the induced topology and it is a Riemannian submanifold with respect to the induced metric.

Now, the normal vector field ∇K to M_0 satisfies $\nabla(\nabla K)=Ka=0$ along M_0 so that M_0 is auto-parallel, namely totally geodesic. q. e. d

§ 2. Proof of Theorem.

Without loss of generality, we can assume that M is connected. If otherwise, we may apply the following proof to each connected component. The proof is divided into the following three cases 1)$\sim$3).

1) *Case where the scalar curvature* $K\neq 0$ *all over* M.

We denote the set of all regular points of M by M'': $M''=\{p\in M \mid a_p\neq 0\}$, which is an open submanifold. By Lemma 1.1, $\frac{1}{2}R^2+S^2-K^2=0$ holds on M'', so that by Lemma 1.2 we have

$$\nabla R=R\otimes\nabla K/K, \tag{2.1}$$

on M''. Now consider an arbitrary point $q\in M-M''$. If q is a limiting point of a sequence of regular points, then (2.1) also holds at q by the continuity. If q is not such a limiting point, there exists a neighborhood N of q such that $a=0$ on N. By (1.1), we have $\nabla^2 R=0$ hence $\nabla R=0$ on N, as is remarked in § 1. Then $K=$const. and $\nabla K/K=0$, so that (2.1) holds also on N. Namely, M is recurrent.

2) *Case where the scalar curvature* $K=0$ *all over* M.

If $a=0$ all over M, then $\nabla R=0$ on M as is remarked in § 1. Suppose now that a does not vanish identically and consider the open submanifold M'' in 1). Since $K=0$ on M'', we have $R=0$ on M'' by Lemma 1.1. Then $\nabla R=0$ on M'', because M'' is open. Namely, at each $p\in M''$, $R_p=(\nabla R)_p=0$ hold. By Lemma 1.3, M is flat.

3) *Case where the scalar curvature* $K\not\equiv 0$ *and has some zero point.*

This case can not occur by the following reason. Let M be such a manifold. Then $M_0=\{p\in M \mid K_p=0\}$ is a closed $(n-1)$-dimensional submanifold by Lemma 1.4, so that $M-M_0$ is an open Riemannian submanifold which is birecurrent. As in the case 1), (2.1) holds on $M-M_0$, in particular

$$\nabla S=S\otimes\nabla K/K, \tag{2.2}$$

on $M-M_0$. Now the covariant tensor field S/K of order 2 is well defined on $M-M_0$. Let p be an arbitrary point of M_0, then $(\nabla K)_p\neq 0$ as in the proof of

Lemma 1.4. Namely there exists a differentiable vector field X on M such that $(\nabla_X K)_p \neq 0$, where ∇_X denotes the covariant differentiation in the direction X. Hence $\nabla_X K \neq 0$ on a neighborhood N of p. Taking account of (2.2), $S/K = (\nabla_X S)/(\nabla_X K)$ is defined outside of $M_0 \cap N$ in N. Since the right hand side is defined on N, S/K is also defined on N as a differentiable tensor field. Since $p \in M_0$ is an arbitrary point, we see that S/K is well defined as a differentiable tensor field all over M. We can easily see that $\nabla(S/K)=0$ on $M-M_0$, making use of (2.2). By the continuity, this holds also on M_0 hence all over M. Since M is irreducible, we have $S/K=\lambda g$, λ=const. Namely, $S=\lambda K g$ on $M-M_0$ and this holds on M by the continuity. Hence M is an Einstein manifold because $n \geqq 3$, so that K=const. on M. This contradicts to the assumption for K.

Summing up the cases 1)$\sim$3), the proof is complete.

§ 3. Case where $n=2$ or M is reducible.

Suppose that $n=2$. If $K \neq 0$ on M, M is recurrent, and if $K \equiv 0$ on M, M is flat. Hence, if M is not recurrent, then $K \not\equiv 0$ and K has some zero point. By Lemma 1.4, the orbit defined by $K=0$ is a geodesic c (or a set of geodesics). Let p_0 be an arbitrary point of c. By (1.3), $\lim_{p \to p_0} (\nabla^2 K/K)(p)$ exists. We denote such a non recurrent M by V^2.

Next, suppose that M is reducible and $n \geqq 3$. Since the discussions in the cases 1) and 2) of § 2 are valid without the irreducibility of M, M is recurrent if $K \neq 0$ or $K=0$ all over M. Only the case 3) of § 2 remains. Each point of M has a neighborhood U admitting an orthogonal decomposition $U=U_1 \times U_2$, where U_1 and U_2 are Riemannian manifolds. Let R_1 (resp. R_2) be the curvature tensor field of U_1 (resp. U_2). As in the recurrent case (Kobayashi-Nomizu, [3], p. 306), there are only the following possibilities: (1) $\nabla^2 R_1=0$ and $\nabla^2 R_2=0$ (2) $R_1=0$ and $\nabla^2 R_2=R_2 \otimes a \neq 0$ (3) $\nabla^2 R_1=R_1 \otimes a \neq 0$ and $R_2=0$, where R_1 (resp. R_2) is supposed as a tensor field on U in a natural way. In the case (1), $\nabla^2 R=0$ and hence $\nabla R=0$ by the remark in § 1. Hence U is locally symmetric. In other cases, since we can assume that, for example, U_2 is irreducible, only the case (2) remains. Since U is the same situation as M in the case 3) of § 2, $(\nabla K)_p \neq 0$ at any point $p \in U$ if $K_p=0$ and $U_0=\{p \in U \mid K_p=0\}$ is an $(n-1)$-dimensional Riemannian submanifold of U. The scalar curvature field K_2 of U_2 does not vanish identically and the natural projection $U \to U_2$ maps U_0 onto a subset $U_0'=\{p \in U_2 \mid K_2=0\}$. In this case, $\nabla_2 K_2 \neq 0$ along U_2. Hence U_0' is a Riemannian submanifold of codimension 1 of U_2. Now, a induces a non zero birecurrent tensor field on U_2-U_0', and also on U_2. By the former consideration of this section, the only possible case is $U_2=V^2$. Namely, $U_1=R^{n-2}$ and $U_2=V^2$, where R^{n-2} is an $(n-2)$-dimensional flat manifold and V^2 is a 2-dimensional birecurrent manifold explained in the first part of this section. Now we have the following proposition.

PROPOSITION. *Let M be a reducible birecurrent Riemannian manifold of dimension n ($\geqq 3$). If either $R \neq 0$ or $R=0$ all over M, then M is recurrent. In other cases, let U be a neighborhood of M admitting an orthogonal decomposition into two Riemannian manifolds. Then, either U is locally symmetric or $U=R^{n-2}\times V^2$, where R^{n-2} is an $(n-2)$-dimensional flat manifold and V^2 is a 2-dimensional birecurrent Riemannian manifold explained in the first part of this section.*

If M is connected and complete, the universal covering manifold $\tilde{M}$ of M is either globally symmetric or $\tilde{M}=R^{n-2}\times V^2$, where R^{n-2} and V^2 are of the same meaning in the above Proposition.

References

[1] A. Lichnerowicz, Courbure, nombres de Betti et espaces symétriques, Proc. Intern. Congr. Math. Cambridge, Mass., 1950, Vol. 2, 216-223.

[2] W. Roter, Some remarks on second order recurrent spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **12** (1964), 661-670.

[3] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Interscience, New York, Vol. 1, 1963.

[4] K. Nomizu and H. Ozeki, A theorem on curvature tensor fields, Proc. Nat. Acad. Sci. U.S.A., **48** (1962), 206-207.

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