

## The modified analytic trivialization of singularities

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We consider the classification of certain classes of singularities. They include in particular those of Kuiper, Whitney and Zeeman. The kite singularities, such as  $y^2 = t^2 x^3 + x^5$  in  $\mathbf{R}^3$ , which arise from the Ratio Test ((4)), are also considered.

While facing a classification problem, it is often very difficult, and yet most interesting, to decide which equivalence relation is the best. It should be as strong as possible, whilst keeping the number of classes to a minimum.

A typical situation is reflected in the Whitney example

$$W(x, y; t) = y(y-x)(y-2x)(y-tx), \quad 2 < t < \infty.$$

This is considered as a  $t$ -parameter family of function germs in  $\mathbf{R}^2$ . Since the contour maps of  $W$ , for fixed values of  $t$ , have the "same type", these function germs should be put in one equivalence class.

It is easy to see that there exists a  $t$ -level preserving homeomorphism  $h: \mathbf{R}^2 \times I \rightarrow \mathbf{R}^2 \times I$ ,  $h(0, t) = (0, t)$ ,  $I = [a, b] \subset (2, \infty)$ , for which  $W \circ h$  is independent of  $t$ . We then ask whether it is possible to find an  $h$  which is  $C^r$ -diffeomorphic, or even bianalytic.

Accordingly, we call:  $\text{homeo} \rightarrow C^1\text{-diffeo} \rightarrow \dots \rightarrow C^\infty\text{-diffeo} \rightarrow \text{bianalytic}$  the canonical route of advance. An equivalence relation by a homeomorphism preserves only the topology, it is too weak for analysis; that by a  $C^r$ -diffeomorphism preserves some formality of analysis, but not computability. A bianalytic equivalence, whilst desirable, rarely exists.

In 1965, Whitney pointed out that for his example, no local  $C^1$ -diffeomorphism could exist! Thus one can not edge forward at all along the canonical route.

We introduce the notion of *modified analytic trivialization* (MAT). The associated equivalence relation preserves computability, but is slightly weaker than bianalyticity; it is independent of  $C^r$ -diffeomorphism ( $1 \leq r \leq \infty$ ), and much stronger than homeomorphism. This yields an alternative route of advance.

The General Theorem in §3 establishes MAT for a class of singularities in  $\mathbf{R}^n$ . Trivializations for the Kuiper, Whitney and Zeeman singularities are special cases (Theorems 1 to 3). On the other hand, in §4, we find that the kite

singularities do not admit MAT, as one expects. It is interesting to note, however, that all kite singularities are Whitney regular ((4)). Some satisfy Verdier's  $w$ -regularity condition ((7)) (e. g.  $y^4=t^2x^5+x^7$ , (6)). Thus MAT seems to be a better notion.

One of the main features of MAT is its special character of being only over  $\mathbf{R}$ . Serious difficulties can hardly be overcome when one attempts a generalization to  $\mathbf{C}$ . These are clearly exposed in the constructive proof of Theorem 5. We like to thank J.M. Mack for discovering this proof for us. Thanks are also due M.J. Field, R. Thom, D. Trotman and J.N. Ward for many inspiring communications.

This paper is written up in response to some queries raised by E.C. Zeeman.

### §1. The fundamental notion: modified analytic trivialization (abbreviated to MAT).

We begin with the elementary case. Let  $\mathcal{M}_2$  be the Möbius strip, considered as the analytic submanifold of  $\mathbf{R}P^1 \times \mathbf{R}^2$  consisting of all points  $([\xi, \eta], (x, y))$  satisfying the equation  $\xi y = \eta x$ . The projection map restricted to  $\mathcal{M}_2$ ,

$$\pi : \mathcal{M}_2 \longrightarrow \mathbf{R}^2$$

is onto and  $\pi^{-1}(0) = \mathbf{R}P^1 \times \{0\}$ . Call  $\pi$  the blowing-up of  $\mathbf{R}^2$  at 0. Note that in the complement of  $\pi^{-1}(0)$ ,  $\pi$  is bianalytic.

Let  $I = [a, b]$  be an interval, and  $\tau^*$  a homeomorphism between two neighborhoods of  $\pi^{-1}(0) \times I$  in  $\mathcal{M}_2 \times I$ , with the following properties:

(1.1)  $\tau^*$  is  $t$ -level preserving,  $t \in I$ ;

(1.2) It is bianalytic;

(1.3) It leaves  $\pi^{-1}(0) \times I$  invariant.

In this case,  $\tau^*$  induces a  $t$ -level preserving homeomorphism  $\tau$  between two neighborhoods of  $\{0\} \times I$  in  $\mathbf{R}^2 \times I$ ; and  $\tau$  leaves  $\{0\} \times I$  pointwise fixed. We then call  $\tau$  a *modified analytic twisting* of  $\mathbf{R}^2 \times I$  along  $I$ . It should be noticed that although  $\tau$  is in general not analytic, it "lifts" to the bianalytic map  $\tau^*$ . Thus  $\tau$  is *essentially analytic in some sense*.

Let  $F(x, y; t)$  be a real-valued function defined and analytic in  $U \times I$ ,  $F(0, 0; t) = 0$ , where  $U$  is a neighborhood of 0. We say  $F(x, y; t)$  admits a MAT along  $I$  if there exists a modified analytic twisting  $\tau$  such that  $F \circ \tau$  is independent of  $t$ . (This says that the  $t$ -parameter family of singularities is "trivialized" by  $\tau$ .)

A straightforward generalization of this notion to  $\mathbf{R}^n$  is as follows. Let

$$\pi : \mathcal{M}_n \longrightarrow \mathbf{R}^n$$

be the blowing-up of  $\mathbf{R}^n$  at 0, where  $\pi^{-1}(0) = \mathbf{R}P^{n-1} \times \{0\}$  is a copy of the real

projective  $(n-1)$ -space. Here  $\mathcal{M}_n$  is the analytic submanifold of  $\mathbf{R}P^{n-1} \times \mathbf{R}^n$  whose points  $([\xi_1, \dots, \xi_n], (x_1, \dots, x_n))$  satisfy the equations  $\xi_i x_j = \xi_j x_i$ , for all  $i, j$ . Let  $I = [a_1, b_1] \times \dots \times [a_m, b_m]$  be a cube in  $\mathbf{R}^m$ , and  $\tau^*$  a bianalytic map between two neighborhoods of  $\pi^{-1}(0) \times I$  in  $\mathcal{M}_n \times I$ . Suppose  $\tau^*$  is  $t$ -level preserving and leaves  $\pi^{-1}(0) \times I$  invariant. The induced homeomorphism  $\tau$  between two neighborhoods of  $\{0\} \times I$  leaves  $\{0\} \times I$  pointwise fixed. We call  $\tau$  a modified analytic twisting of  $\mathbf{R}^n \times I$  along  $I$ . For a function  $F(x; t)$ ,  $F(0; t) = 0$ , defined and analytic in  $U \times I$ ,  $U$  an open neighborhood of 0, suppose there exists a  $\tau$  so that  $F \circ \tau$  is independent of  $t$ , then we say  $F(x; t)$  admits a MAT along  $I$ .

In this case, it is easy to see that  $F(x; t)$  admits a semi-analytic fibration in the sense of Whitney ([8], p. 230). We shall not consider this notion here.

A general theorem about MAT is stated in §3. We first state some special cases that might interest experts.

**§2. MAT for the singularities of Kuiper, Whitney and Zeeman.**

**THEOREM 1.** *The Kuiper jet ((2))*

$$K(x, y; t) = x^5 + y^5 + t x^3 y^3$$

*admits a MAT along any  $I = [a, b]$ .*

**THEOREM 2.** *The Whitney function*

$$W(x, y; t) = y(y-x)(y-2x)(y-tx), \quad t \in \mathbf{R}$$

*admits a MAT along any  $I$  not containing 0, 1 or 2.*

**THEOREM 3.** *The Zeeman double cusp*

$$Z(x, y; t) = x^4 + y^4 + t x^2 y^2, \quad t \in \mathbf{R}$$

*admits a MAT along any  $I$  not containing  $-2$ .*

**§3. The General Theorem.**

We can write

$$F(x; t) = H_k(x; t) + H_{k+1}(x; t) + \dots, \quad H_k \neq 0,$$

where  $H_j(x, t) = \sum a_\alpha(t) x^\alpha$ ,  $|\alpha| = j$ , represents a homogenous  $j$ -form in  $x$ , with coefficients in  $t$ . We say  $H_j(x; t)$  is non-degenerate in  $x$  for  $t \in I$  if, for each fixed  $t$ , 0 is the only point in  $\mathbf{R}^n$  at which

$$\frac{\partial H_j}{\partial x_1} = \dots = \frac{\partial H_j}{\partial x_n} = 0.$$

**THE GENERAL THEOREM.** *Suppose the initial form  $H_k(x; t)$  of  $F(x; t)$  is non-degenerate in  $x$  for  $t \in I$ . Then  $F(x; t)$  admits a MAT along  $I$ .*

§ 4. **Non-existence of MAT for the kite singularities.**

For integers  $k, l, m, 1 \leq k < l < m$ , put

$$K_{k,l,m}(x, y; t) = y^{2k} - t^2 x^{2l+1} - x^{2m+1}.$$

The variety  $K_{k,l,m} = 0$  looks like a kite, for a picture of  $K_{1,2,3}$ , see (5), p. 123.

THEOREM 4. *There is no MAT for the kite singularity  $K_{k,l,m}$  along any  $I = [-\varepsilon, \varepsilon]$ .*

§ 5. **An illustration via  $Z(x, y; t)$ .**

We give a complete proof of Theorem 3. The proof of the General Theorem is more or less a straightforward generalization of the argument given here.

We only consider the case  $I = [a, b] \subset (-2, \infty)$ . The argument for  $I \subset (-\infty, -2)$  is similar. For  $t > -2$ , note that

$$(5.1) \quad Z_x^2 + Z_y^2 \neq 0, \quad \text{for } (x, y) \neq (0, 0), \quad (x, y) \text{ near } 0.$$

Let us consider the vector field

$$(5.2) \quad v(x, y; t) = -\frac{Z_t Z_x}{Z_x^2 + Z_y^2} \frac{\partial}{\partial x} - \frac{Z_t Z_y}{Z_x^2 + Z_y^2} \frac{\partial}{\partial y} + \frac{\partial}{\partial t}, \quad (x, y) \neq (0, 0),$$

$$(5.3) \quad v(0, 0; t) = \frac{\partial}{\partial t}.$$

The construction of  $v$ , first appeared in (3), is as follows. At an arbitrary point  $P$  off the  $t$ -axis, let the level surface  $Z(x, y; t) = \text{const.}$  that passes through  $P$  be denoted by  $\mathcal{L}_P$ , and the tangent plane by  $T(\mathcal{L}_P, P)$ . By (5.1),  $T(\mathcal{L}_P, P)$  is not perpendicular to  $\partial/\partial t$ . Hence  $\partial/\partial t$  has a non-zero projection there. Normalizing the  $t$ -component of the projection to 1, we find (5.2).

We shall see later that the flow of  $v$  yields the desired trivialization  $\tau$  for  $Z(x, y; t)$ . We first make some preliminary observations. Let  $\phi(t; P)$  denote the trajectory of  $v$  satisfying the initial condition  $\phi(0; P) = P$ . Then  $\phi(t; P)$  lies entirely on  $\mathcal{L}_P$ , and hence  $Z$  is constant along  $\phi$ . Moreover, since  $v$  has  $t$ -component 1,  $\tau$  is  $t$ -level preserving.

However,  $v$  is not in general analytic on  $\{0\} \times I$ . (We do not yet know it is continuous!) The *essential observation* is that in a neighborhood of  $\{0\} \times I$ ,  $v$  can be lifted to a vector field  $v^*$  which is defined and analytic *throughout* a neighborhood of  $\pi^{-1}(0) \times I$ . (This implies, in particular, that  $v$  is continuous.)

Recall that  $\mathbf{R}P^1$  can be covered by two compact coordinate neighborhoods,  $\{\theta_1, \theta_2\}$ , and then  $D_i = \theta_i \times \mathbf{R}$ ,  $i=1, 2$ , cover  $\mathcal{M}_2$ . The projections  $\pi(D_1)$  and  $\pi(D_2)$  are sectors around the  $x$ -axis and  $y$ -axis respectively.

Let us consider  $\pi(D_1)$  first. A coordinate system  $\{X, Y\}$  can be chosen in  $D_1$ , expressing  $\pi$  as

$$\pi(X, Y) = (X, XY),$$

or

$$x = X, \quad y = XY.$$

In the interior of  $\pi(D_1)$ , the inverse map  $\pi^{-1}$  then induces

$$d\pi^{-1}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial X} - \frac{Y}{X} \frac{\partial}{\partial Y}$$

$$d\pi^{-1}\left(\frac{\partial}{\partial y}\right) = \frac{1}{X} \frac{\partial}{\partial Y},$$

and so

$$(5.4) \quad d(\pi^{-1} \times id_I)(v) = -\frac{Z_t Z_x}{Z_x^2 + Z_y^2} \frac{\partial}{\partial X} + \frac{Y Z_t Z_x - Z_t Z_y}{X(Z_x^2 + Z_y^2)} \frac{\partial}{\partial Y} + \frac{\partial}{\partial t}.$$

Let  $v^*$  denote this last vector field. So far it is defined only for  $X \neq 0$ . Now observe:

$$Z_x^2 + Z_y^2 = X^6 U(X, Y; t)$$

where  $U$  is defined and analytic in a neighborhood of  $X=0$ ; and  $U$  is strictly positive. Moreover, the numerators in (5.4) are divisible by  $X^7$ . Hence  $v^*$  can be extended analytically across  $\theta_1 \times I$  and expressed in the form

$$(5.5) \quad v^* = X V_1(X, Y; t) \frac{\partial}{\partial X} + V_2(X, Y; t) \frac{\partial}{\partial Y} + \frac{\partial}{\partial t}$$

where  $V_1, V_2$  are analytic throughout a neighborhood of  $\theta_1 \times I$  in  $D_1 \times I$ .

This extension is obviously the unique continuous extension of  $v^*$  of (5.4). Since the coefficient of  $\partial/\partial X$  in (5.5) vanishes for  $X=0$ ,

$$(5.6) \quad d(\pi \times id_I)(v^*) = \frac{\partial}{\partial t} = v(0, 0; t) \quad \text{on } \theta_1 \times I.$$

Now consider  $\pi(D_2)$ . In the same way  $v$  is lifted into an analytic vector field in a neighborhood of  $\theta_2 \times I$ .

Recall that in the complement of  $\pi^{-1}(0) \times I$ ,  $\pi \times id_I$  is bianalytic, hence the lifting is unique and independent of the coordinate systems. Thus the above two liftings, together with their extensions, must patch up automatically. The properties of the resulting field  $v^*$  can be summarized as follows:

(5.7)  $v^*$  is defined and analytic throughout a neighborhood of  $\pi^{-1}(0) \times I$ .

(5.8)  $d(\pi \times id_I)(v^*(\mu, t)) = v(\pi(\mu), t)$  for all  $\mu$  in a neighborhood of  $\pi^{-1}(0)$ .

The rest of the argument is standard. Let  $\phi^*(t; \mu_0, t_0)$  denote the trajectory of  $v^*$  with  $\phi^*(0; \mu_0, t_0) = (\mu_0, t_0)$ , and define

$$\tau^*(\mu, t) = \phi^*(t-a; \mu, a).$$

Since  $v^*$  is analytic,  $\tau^*$  is analytic in  $(\mu, t)$  ((1) p. 119). Hence  $\tau^*$  yields the desired modified analytic twisting  $\tau$  for  $Z(x, y; t)$ .

The functions  $W(x, y; t)$  and  $K(x, y; t)$  can be considered in exactly the same way.

### §6. Proof of the General Theorem.

For each  $j$ ,  $1 \leq j \leq m$ , we shall define a continuous vector field  $v_j$  with the following properties.

$$(6.1) \quad \text{On } \{0\} \times I, \quad v_j = \frac{\partial}{\partial t_j}.$$

$$(6.2) \quad \text{At any } P \in \{0\} \times I, \quad v_j \in T(\mathcal{L}_P, P).$$

$$(6.3) \quad \text{The } I\text{-component of } v_j \text{ is } \frac{\partial}{\partial t_j}.$$

$$(6.4) \quad v_j \text{ lifts to a vector field } v_j^* \text{ which is analytic throughout a neighborhood of } \pi^{-1}(0) \times I \text{ in } \mathcal{M}_n \times I.$$

Assuming  $v_j$  and  $v_j^*$ ,  $1 \leq j \leq m$ , have been found, the rest of the proof proceeds as follows. Let  $\phi_1^*(t_1; \mu, c)$  denote the trajectory of  $v_1^*$  with  $\phi_1^*(0; \mu, c) = (\mu, c)$ ;  $\phi_1^*$  is analytic in its variables. Let  $\phi_2^*$  denote the trajectory of  $v_2^*$ , then  $\phi_2^*(t_2; \phi_1^*(t_1; \mu, c))$  is analytic in  $(t_1, t_2, \mu, c)$ . Intuitively, we let  $(\mu, c)$  "flow" along  $v_1^*$ , obtaining an analytic "stream line"  $\phi_1^*$ ; we then let the stream line  $\phi_1^*$  flow along  $v_2^*$ , getting an analytic surface  $\phi_2^*(t_2; \phi_1^*)$ , etc. Continue in this way until  $\phi_m^*$ , we then define

$$\tau^*(\mu, t) = \phi_m^*(t_m - a_m; \phi_{m-1}^*(\cdots; \phi_1^*(t_1 - a_1; \mu, a))),$$

where  $t = (t_1, \dots, t_m) \in I$ ,  $a = (a_1, \dots, a_m)$ .

Thus  $\tau^*$  is a bianalytic map between two neighborhoods of  $\pi^{-1}(0) \times I$ . By the properties (6.1) to (6.4),  $\tau$  is the desired trivialization for  $F(x; t)$ . (Note that the Frobenius integrability conditions may not be satisfied in our case; but we are not concerned.)

It remains to construct  $v_j$  and  $v_j^*$ . For simplicity of notation, we only construct  $v_1$  and  $v_1^*$ . By assumption,  $H_k(x; t)$  is non-degenerate in  $x$ , hence the gradient vector with respect to  $x$ ,

$$\nabla_x F(x; t) = \frac{\partial F}{\partial x_1} \frac{\partial}{\partial x_1} + \cdots + \frac{\partial F}{\partial x_n} \frac{\partial}{\partial x_n}$$

is non-vanishing for  $x \neq 0$ ; in fact, we have

$$|\nabla_x F| = u(x, t) |x|^{k-1}$$

where  $u(x, t)$  is a unit. Now we define

$$v_1(x, t) = -|\nabla_x F|^{-2} \frac{\partial F}{\partial t_1} \nabla_x F + \frac{\partial}{\partial t_1}, \quad x \neq 0,$$

$$v_1(0, t) = \frac{\partial}{\partial t_1}.$$

Let us construct  $v_1^*$ . The projective space  $\pi^{-1}(0) = \mathbf{R}P^{n-1}$  can be covered by  $n$  compact coordinate neighborhoods  $\{\theta_1, \dots, \theta_n\}$ . Let  $D_i = \theta_i \times \mathbf{R}$ . Each  $\pi(D_s)$  is a solid cone in  $\mathbf{R}^n$  around the  $x_s$ -axis, having its vertex at 0,  $1 \leq s \leq n$ .

Fixing an  $s$  and consider  $D_s$ . A coordinate system  $\{X_1, \dots, X_n\}$  can be chosen in  $D_s$  so that  $\pi$  is expressed as

$$x_s = X_s, \quad x_j = X_s X_j \quad j \neq s.$$

Then for  $x_s \neq 0$ ,

$$\frac{\partial}{\partial x_s} = \frac{\partial}{\partial X_s} - \sum_{j \neq s} \frac{X_j}{X_s} \frac{\partial}{\partial X_j}; \quad \frac{\partial}{\partial x_j} = \frac{1}{X_s} \frac{\partial}{\partial X_j} \quad j \neq s.$$

Note that  $\partial F / \partial t_1$  and  $\partial F / \partial x_i$  ( $1 \leq i \leq n$ ) are of orders  $k$  and  $k-1$  in  $x$  respectively; they are therefore divisible by  $X_s^k$  and  $X_s^{k-1}$  respectively. Define

$$v_1^* = d(\pi^{-1} \times id_I)(v_1), \quad 0 < |X_s| \leq \varepsilon$$

which can then be written as

$$v_1^* = X_s U_s(X, t) \frac{\partial}{\partial X_s} + \sum_{j \neq s} U_j \frac{\partial}{\partial X_j} + \frac{\partial}{\partial t_1}$$

where  $U_i$ ,  $1 \leq i \leq n$ , are defined and analytic for  $|X_s| \leq \varepsilon$ . Hence  $v_1^*$  can be extended uniquely across  $\theta_s \times I$ . The coefficient of  $\partial / \partial X_s$  vanishes when  $X_s = 0$ .

Now the liftings for  $s=1, \dots, n$  patch up automatically, yielding the desired vector field  $v_1^*$  in a neighborhood of  $\pi^{-1}(0) \times I$ . This completes the proof.

### §7. The kite singularities.

We prove Theorem 4.

Let  $\mathcal{E}$  be a real analytic space,  $a \in \mathcal{E}$ . By an analytic arc at  $a$ , we mean (the germ of) an analytic map

$$\lambda: [0, \varepsilon) \longrightarrow \mathcal{E}$$

with  $\lambda(0) = a$ ,  $\lambda(s) \neq a$ ,  $s > 0$ .

The set of all such arcs is denoted by  $\mathcal{A}(\mathcal{E}, a)$ .

More generally, if  $\Sigma$  is a closed subset of  $\mathcal{E}$ , call  $\lambda$  an analytic arc at  $\Sigma$  if  $\lambda(0) \in \Sigma$ ,  $\lambda(s) \notin \Sigma$ ,  $s > 0$ .

Let  $\mathcal{A}(\mathcal{E}, \Sigma)$  denote all such arcs.

In the following, we take  $\Sigma = \pi^{-1}(0)$ , where  $\pi: \mathcal{M}_n \rightarrow \mathbf{R}^n$  is the blowing-up of  $\mathbf{R}^n$  at 0.

PROPOSITION 1. *The map*

$$\mathcal{A}(\pi): \mathcal{A}(\mathcal{M}_n, \Sigma) \longrightarrow \mathcal{A}(\mathbf{R}^n, 0)$$

defined by  $\mathcal{A}(\pi)(\lambda^*) = \pi \circ \lambda^*$  is a bijection.

For if  $\lambda_1^*(0) \neq \lambda_2^*(0)$ , then  $\pi \circ \lambda_1^*$  and  $\pi \circ \lambda_2^*$  have different tangents at 0. Suppose  $\lambda_1^*(0) = \lambda_2^*(0)$ ,  $\lambda_1^*(s) \neq \lambda_2^*(s)$ , then  $\pi(\lambda_1^*(s)) \neq \pi(\lambda_2^*(s))$ , since  $\pi$  is bianalytic off  $\Sigma$ . Hence  $\mathcal{A}(\pi)$  is an injection.

Now let  $\lambda \in \mathcal{A}(\mathbf{R}^n, 0)$ . The tangent of  $\lambda$  at 0 is represented by a unique point  $a^* \in \Sigma$ . Without loss of generality, we can assume that this tangent is the  $x_1$ -axis. Then in the parametrization of  $\lambda: x_j = x_j(s)$ ,  $1 \leq j \leq n$ , we must have

$$O(x_1(s)) < O(x_i(s)), \quad 2 \leq i \leq n.$$

There is a coordinate system near  $a^*$ ,  $\{X_1, \dots, X_n\}$ , with respect to which  $\pi$  is expressed as:

$$x_1 = X_1, \quad x_i = X_1 X_i \quad 2 \leq i \leq n.$$

Now the arc

$$\lambda^*: X_1(s) = x_1(s), \quad X_i(s) = \frac{x_i(s)}{x_1(s)}, \quad i \geq 2$$

is in  $\mathcal{A}(\mathcal{M}_n, \Sigma)$  and  $\pi \circ \lambda^* = \lambda$ . Hence  $\mathcal{A}(\pi)$  is also surjective.

For  $\lambda, \mu \in \mathcal{A}(\mathcal{E}, a)$ , define their order of contact,  $O(\lambda, \mu)$ , as in Calculus. Let  $\tau$  be a modified analytic twisting of  $\mathbf{R}^n \times I$  along  $I$ . For  $t \in I$ ,  $\lambda \in \mathcal{A}(\mathbf{R}^n, 0)$ , define  $\tau_t(\lambda) \in \mathcal{A}(\mathbf{R}^n, 0)$  by

$$\tau_t(\lambda)(s) = \tau(\lambda(s), t).$$

PROPOSITION 2. *For  $\lambda, \mu \in \mathcal{A}(\mathbf{R}^n, 0)$ ,*

$$O(\tau_t(\lambda), \tau_t(\mu)) = O(\lambda, \mu), \quad t \in I.$$

Let  $\tau^*$  be the bianalytic map which induces  $\tau$ . Then

$$\begin{aligned} O(\lambda, \mu) &= O(\lambda^*, \mu^*) + 1 \\ &= O(\tau_t^*(\lambda^*), \tau_t^*(\mu^*)) + 1 \quad \text{since } \tau^* \text{ is bianalytic} \\ &= O(\tau_t(\lambda), \tau_t(\mu)). \end{aligned}$$

We are now ready to prove Theorem 4. The variety

$$K_{k,l,m}(x, y; 0) = 0$$

gives rise to two arcs:

$$\lambda_1: x = s^{2k}, \quad y = s^{2m+1} \quad (s \geq 0),$$

and

$$\lambda_2: x = s^{2k}, \quad y = -s^{2m+1} \quad (s \geq 0).$$



Yet for  $t \neq 0$ ,  $K_{k,l,m}(x, y; t) = 0$  gives rise to

$$\mu_1: x = s^{2k}, \quad y = |t|s^{2l+1} + \dots,$$

and

$$\mu_2: x = s^{2k}, \quad y = -|t|s^{2l+1} + \dots.$$

Since  $m > l$ ,  $O(\lambda_1, \lambda_2) \neq O(\mu_1, \mu_2)$ .

Thus, by the above Proposition,  $K_{k,l,m}$  can not admit MAT in  $[-\varepsilon, \varepsilon]$ .

**§ 8. Obstructions for generalizing MAT to complex spaces.**

It is well-known that if a biholomorphic map of the Riemann sphere with itself leaves three points fixed, then it must be the identity map.

Let us consider the Whitney function

$$W(z_1, z_2; t) = z_2(z_2 - z_1)(z_2 - 2z_1)(z_2 - tz_1),$$

and blow-up  $C^2$  at 0,

$$\pi: CM_2 \longrightarrow C^2,$$

where  $\pi^{-1}(0) = CP^1$ , the Riemann sphere.

Consequently, any biholomorphism  $\tau^*$  on  $CM_2 \times I$  that could yield a trivialization  $\tau$  for  $W(z_1, z_2; t)$  would have to leave 3 points fixed on  $CP^1$  while deforming a fourth with  $t$ . This is impossible.

The following theorem exposes one of the essential features in the case over  $R$ .

**THEOREM 5.** *For any given finite set  $S = \{\theta_1, \dots, \theta_p\}$  on the circle  $S^1$ , there is a (real) bianalytic map of  $S^1$  with itself, whose fixed point set is  $S$ .*

The following constructive proof is due to J. M. Mack. Let

$$\phi(\theta) = \theta + \frac{1}{p} \sum_{i=1}^p \sin^2 \frac{\theta - \theta_i}{2}.$$

Then  $\phi'(\theta) \geq 1/2$  and  $\phi$  is bianalytic.

Note that if  $\theta$  is considered as a complex variable, then  $\phi'(\theta)$  is no more non-vanishing, and  $\phi$  is not biholomorphic.

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