J. Math. Soc. Japan
Vol. 32, No. 3, 1980

On a criterion of Picard principle for rotation free densities

By Toshimasa TADA

(Received Nov. 16, 1978)

A nonnegative locally Hölder continuous function $P(z)$ on $0<|z|\leqq 1$ will be referred to as a *density* on $0<|z|\leqq 1$. After Bouligand we say that the *Picard principle* is valid for P if the dimension of the half module of nonnegative solutions of the equation $\Delta u(z)=P(z)u(z)$ on $0<|z|\leqq 1$ with boundary values 0 on $|z|=1$ is 1. For tests of Picard principle for rotation free densities, i.e. densities $P(z)$ satisfying $P(z)=P(|z|)$ on $0<|z|\leqq 1$, it was shown in [2] that *the Picard principle is valid for a rotation free density* $P(z)$ *if and only if* $\alpha(P)=0$, where $\alpha(P)$, referred to as the *singularity index* of P, is the quantity in $[0, 1)$ associated with P. Moreover for complete characterization of $\alpha(P)=0$ in terms of e_P, it was shown in [1] that $\alpha(P)=0$ *if and only if*

(1)
$$\int_0^1 \frac{1}{r\left(r\frac{d}{dr}\log e_P(r)+1\right)}dr=\infty ,$$

where e_P, referred to as the *P-unit*, is a unique bounded solution of

$$\frac{d^2}{dr^2}e(r)+\frac{1}{r}\frac{d}{dr}e(r)=P(r)e(r)$$

on $0<r<1$ with $e(1)=1$. The principal part for the proof of the condition (1) was in analysis of solutions of a Riccati type equation

$$-\frac{d}{dt}a(t)+a(t)^2=e^{-2t}P(e^{-t})$$

on $0<t<\infty$. The purpose of this paper is to prove the condition (1) only by using a comparison principle given in the following section.

§ 1. Comparison principle.

Let $P(z)$ be a rotation free density on $0<|z|\leqq 1$. Consider the ordinary differential operators

$$L_n e(r)=\frac{d^2}{dr^2}e(r)+\frac{1}{r}\frac{d}{dr}e(r)-\Big(P(r)+\frac{n^2}{r^2}\Big)e(r) \qquad (n=0, 1, 2)$$

for C^2 function $e(r)$ on $0<r<\rho$ with $0<\rho\leqq 1$. Then we have the following comparison principle for the proof of which we refer to p. 417 in [3]:

LEMMA. *If a bounded C^2 function $e(r)$ satisfies $L_n e(r)\leqq 0$ on $0<r<\rho$ for some n and $e(\rho)\geqq 0$, then $e(r)\geqq 0$ on $0<r\leqq\rho$.*

§ 2. Fundamental inequalities.

There exists a unique bounded solution $e_n(r, \rho)$ $(n=0, 1, 2)$ of $L_n e(r)=0$ on $0<r<\rho$ with $e(\rho)=1$ and hence $0<e_n(r, \rho)\leqq 1$ on $0<r\leqq\rho$. As for the unique existence of $e_n(r, \rho)$, we refer to [2]. By the unique existence of $e_n(r, \rho)$, we infer that

(2) $$e_n(r, \rho)=\frac{e_n(r, \sigma)}{e_n(\rho, \sigma)} \qquad (n=0, 1, 2)$$

on $0<r\leqq\rho$ for any σ with $\rho\leqq\sigma\leqq 1$. In particular we set $e_n(r)=e_n(r, 1)$. We will show that the limit

(3) $$\alpha_n(P)=\lim_{r\to 0}\frac{e_{n+1}(r)}{e_n(r)} \qquad (n=0, 1)$$

exists and satisfies the inequality $0\leqq\alpha_n(P)<1$, and that the following relation is valid:

(4) $$\alpha_0(P)\geqq\alpha_1(P)\geqq\alpha_0(P)^3 .$$

Although the existence of (3) is shown in § 1.2 of [2] and the relation (4) can be derived off hand from Theorem 1 in [2], we give here direct and simpler proofs of these facts based on the comparison principle. By the fact that

$$L_n\frac{r^n}{\rho^n}=-\frac{r^n}{\rho^n}P(r)\leqq 0 \qquad (n=0, 1, 2)$$

and

$$L_{n+1}e_n(r, \rho)=-\frac{2n+1}{r^2}e_n(r, \rho)\leqq 0 \qquad (n=0, 1),$$

we may apply Lemma to $r^n/\rho^n-e_n(r, \rho)$ and $e_n(r, \rho)-e_{n+1}(r, \rho)$. Then we have that

(5) $$e_n(r, \rho)\leqq\frac{r^n}{\rho^n} \qquad (n=0, 1, 2)$$

and

(6) $$e_{n+1}(r, \rho)\leqq e_n(r, \rho) \qquad (n=0, 1)$$

on $0<r\leqq\rho$. From (2), (5) and (6) it follows that the functions $e_n(r, \rho)/r^n$ and $e_{n+1}(r, \rho)/e_n(r, \rho)$ are decreasing as $r\to 0$. Then we have that

$$\frac{e_n'(r, \rho)}{e_n(r, \rho)}\geqq\frac{n}{r} \qquad (n=0, 1, 2) \tag{7}$$

and

$$\frac{e_{n+1}'(r, \rho)}{e_{n+1}(r, \rho)}\geqq\frac{e_n'(r, \rho)}{e_n(r, \rho)} \qquad (n=0, 1) \tag{8}$$

on $0<r\leqq\rho$. By the monotonousness of $e_{n+1}(r, \rho)/e_n(r, \rho)$ the limit

$$\alpha_n(P)=\lim_{r\to 0}\frac{e_{n+1}(r)}{e_n(r)} \qquad (n=0, 1)$$

exists and $0\leqq\alpha_n(P)<1$. In particular $e_P(r)=e_0(r)$ is referred to as the *P-unit* and $\alpha(P)=\alpha_0(P)$ is referred to as the *singularity index* of P at $z=0$.

To show a relation between $\alpha(P)$ and $\alpha_1(P)$ we observe that

$$L_{n+1}\Big(\frac{r}{\rho}e_n(r, \rho)\Big)=\frac{2e_n(r, \rho)}{\rho}\Big(\frac{e_n'(r, \rho)}{e_n(r, \rho)}-\frac{n}{r}\Big) \qquad (n=0, 1), \tag{9}$$

$$L_1\Big(\frac{e_0(r)e_2(r)}{e_1(r)}\Big)=\frac{2e_0(r)e_2(r)}{e_1(r)}\Big(\frac{1}{r^2}-\Big(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\Big)\Big(\frac{e_1'(r)}{e_1(r)}-\frac{e_0'(r)}{e_0(r)}\Big)\Big) \tag{10}$$

and

$$L_2\Big(\frac{e_1(r)^4}{e_0(r)^3}\Big)=\frac{12e_1(r)^4}{e_0(r)^3}\Big(\frac{e_1'(r)}{e_1(r)}-\frac{e_0'(r)}{e_0(r)}\Big)^2. \tag{11}$$

In view of (7) and (9) we may apply Lemma to $e_{n+1}(r, \rho)-(r/\rho)e_n(r, \rho)$. Then we have that

$$e_{n+1}(r, \rho)\geqq\frac{r}{\rho}e_n(r, \rho) \qquad (n=0, 1)$$

on $0<r\leqq\rho$. From this and (2) it follows that $re_n(r, \rho)/e_{n+1}(r, \rho)$ is decreasing as $r\to 0$ and hence

$$\frac{e_{n+1}'(r)}{e_{n+1}(r)}-\frac{e_n'(r)}{e_n(r)}\leqq\frac{1}{r} \qquad (n=0, 1) \tag{12}$$

on $0<r\leqq 1$. By virtue of (10), (11) and (12) we may also apply Lemma to $e_1(r)-e_0(r)e_2(r)/e_1(r)$ and $e_2(r)-e_1(r)^4/e_0(r)^3$. Thus we have that

$$\frac{e_1(r)}{e_0(r)}\geqq\frac{e_2(r)}{e_1(r)}\geqq\Big(\frac{e_1(r)}{e_0(r)}\Big)^3$$

on $0<r\leqq 1$ and hence

$$\alpha(P)\geqq\alpha_1(P)\geqq\alpha(P)^3. \tag{13}$$

§ 3. *P*-unit criterion.

In the preceding section we have that the singularity index $\alpha(P)$ is 0 if and only if $\alpha_1(P)=0$. In this section we estimate the function $e_2(r)/e_1(r)$ which determines the value $\alpha_1(P)$. Since every $e_n(r)$ is a solution of $L_n e(r)=0$, we have that

$$\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)'=\frac{3}{r^2}-\frac{1}{r}\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)$$
$$-\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)\left(\frac{e_2'(r)}{e_2(r)}+\frac{e_1'(r)}{e_1(r)}\right)$$
$$\leqq\frac{3}{r^2}-\frac{1}{r}\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)-\frac{2e_1'(r)}{e_1(r)}\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)$$

on $0<r\leqq 1$. Hence we have that

$$\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\leqq\frac{e_1(r)}{2r^2e_1'(r)}\left(3-r\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)\right.$$
$$\left.-r^2\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)'\right)$$

on $0<r\leqq 1$. Then the integral representation $\int_r^1(e_2'(r)/e_2(r)-e_1'(r)/e_1(r))dr$ of $\log(e_1(r)/e_2(r))$ can be estimated in the following way:

$$\int_r^1\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)dr$$
$$\leqq\int_r^1\frac{e_1(r)}{2r^2e_1'(r)}\left(3-r\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)-r^2\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)'\right)dr$$
$$=\frac{3}{2}\int_r^1\frac{e_1(r)}{r^2e_1'(r)}dr-\frac{1}{2}\int_r^1\frac{e_1(r)}{re_1'(r)}\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)dr$$
$$-\left[\frac{e_1(r)}{2e_1'(r)}\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)\right]_r^1$$
$$+\frac{1}{2}\int_r^1\left(1-\left(\frac{e_1(r)}{e_1'(r)}\right)^2\left(P(r)+\frac{1}{r^2}\right)+\frac{e_1(r)}{re_1'(r)}\right)\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)dr$$
$$\leqq\frac{3}{2}\int_r^1\frac{e_1(r)}{r^2e_1'(r)}dr-\frac{e_1(1)}{2e_1'(1)}\left(\frac{e_2'(1)}{e_2(1)}-\frac{e_1'(1)}{e_1(1)}\right)$$
$$+\frac{e_1(r)}{2e_1'(r)}\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)+\frac{1}{2}\int_r^1\left(\frac{e_2'(r)}{e_2(r)}-\frac{e_1'(r)}{e_1(r)}\right)dr$$

$$\leqq \frac{3}{2}\int_r^1 \frac{e_1(r)}{r^2 e_1'(r)} dr + \frac{1}{2} + \frac{1}{2}\int_r^1 \left(\frac{e_2'(r)}{e_2(r)} - \frac{e_1'(r)}{e_1(r)}\right) dr .$$

Thus we have that

$$\log \frac{e_1(r)}{e_2(r)} \leqq 3\int_r^1 \frac{e_1(r)}{r^2 e_1'(r)} dr + 1$$

on $0<r\leqq 1$ and hence, as a lower estimation of $e_2(r)/e_1(r)$, we obtain

$$\frac{e_2(r)}{e_1(r)} \geqq \exp\left(-3\int_r^1 \frac{e_1(r)}{r^2 e_1'(r)} dr - 1\right) \tag{14}$$

on $0<r\leqq 1$.

In order to obtain an upper estimation of $e_2(r)/e_1(r)$ we consider the function

$$E(r) = \exp\left(-\int_r^1 \frac{e_1(r)}{r^2 e_1'(r)} dr\right)$$

on $0<r\leqq 1$. Then $e_1(r)E(r)$ satisfies

$$L_2(e_1(r)E(r)) = -\left(\frac{e_1(r)}{r e_1'(r)}\right)^2 P(r) e_1(r) E(r) .$$

In view of this we may apply Lemma to $e_1(r)E(r) - e_2(r)$ and hence we obtain

$$\frac{e_2(r)}{e_1(r)} \leqq \exp\left(-\int_r^1 \frac{e_1(r)}{r^2 e_1'(r)} dr\right) \tag{15}$$

on $0<r\leqq 1$. Now we summarize (13), (14) and (15) in the following

THEOREM. *The Picard principle is valid for P if and only if*

$$\int_0^1 \frac{e_1(r)}{r^2 e_1'(r)} dr = \infty .$$

This theorem is rewritten in terms of $e_P(r)$ as follows:

COROLLARY ([**1**]). *The Picard principle is valid for P if and only if*

$$\int_0^1 \frac{1}{r\left(r \frac{d}{dr} \log e_P(r) + 1\right)} dr = \infty .$$

In fact from (7), (8) and (12) it follows that

$$\frac{1}{2}\left(\frac{e_P'(r)}{e_P(r)} + \frac{1}{r}\right) \leqq \frac{e_1'(r)}{e_1(r)} \leqq \frac{e_P'(r)}{e_P(r)} + \frac{1}{r}$$

on $0<r\leqq 1$.

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Toshimasa TADA
Department of Mathematics
Daido Institute of Technology
Daido, Minami, Nagoya 457
Japan