J. Math. Soc. Japan
Vol. 32, No. 2, 1980

The first Chern class and holomorphic symmetric tensor fields

To Fumio Sakai and Carmen Silva

By Shoshichi KOBAYASHI*)

(Received July 11, 1978)

§ 1. Introduction.

Let M be a complex manifold and TM (resp. T^*M) be the holomorphic tangent (resp. cotangent) bundle of M. Let S^mTM (resp. S^mT^*M) be the m-th symmetric tensor power of TM (resp. T^*M). Thus $\Gamma(S^mTM)$ (resp. $\Gamma(S^mT^*M)$) denotes the space of holomorphic contravariant (resp. covariant) symmetric tensor fields of degree m.

The purpose of this paper is to answer affirmatively the following question raised by F. Sakai[1]. If M is a Kähler K3 surface, are $\Gamma(S^mTM)$ and $\Gamma(S^mT^*M)$ trivial ? (For Kummer surfaces and quartic surfaces, the answer has been known to Sakai). We shall show that if M is a compact, simply connected Kähler manifold with vanishing first Chern class, then

$$\Gamma(S^mTM)=\Gamma(S^mT^*M)=0 \qquad \text{for} \quad m>0\,.$$

Several related results, some applicable to manifolds of general type, will be proved on the way. The main ingredients of the proof consist of (i) Bochner's method of proving vanishing theorems, (ii) Berger's classification of irreducible holonomy groups, (iii) representations of $SU(n)$ and $Sp(n)$, and (iv) Yau's solution of the Calabi conjecture. Using (i), (ii) and (iii) we obtain differential geometric results in which assumptions are stated in terms of the Ricci tensor. Applying (iv) we can restate the results in terms of the first Chern class.

§ 2. Ricci tensor and holomorphic symmetric tensor fields.

By the well known method of Bochner we prove the following

---

*) Guggenheim Fellow, partially supported by SFB in Bonn and NSF Grant MCS 76-01692.

1) The question was raised and answered aboard Motorship "Carmen Silva" on the Rhine during the 19th "Arbeitstagung", 1978.

THEOREM 1. *Let M be an n-dimensional compact Kähler manifold with positive (resp. negative) semi-definite Ricci tensor. Let $\xi\in\Gamma(S^mT^*M)$ (resp. $\Gamma(S^mTM)$). Then*

(a) $\nabla\xi=0$, *i.e., ξ is parallel,*

(b) *if $\xi_{k_1\cdots k_m}$ (resp. $\xi^{k_1\cdots k_m}$) are the components of ξ with respect a local coordinate system of M, then*

$$Q(\xi, \xi):=\sum R_j^k\xi_{kk_2\cdots k_m}\bar{\xi}^{jk_2\cdots k_m}=0$$

$$(resp. :=\sum R_j^k\xi^{jk_2\cdots k_m}\bar{\xi}_{kk_2\cdots k_m}=0)\,.$$

PROOF. We consider only the case $\xi\in\Gamma(S^mT^*M)$ since the proof for the case $\xi\in\Gamma(S^mTM)$ is similar. Let

$$\xi=\sum \xi_{k_1\cdots k_m}dz^{k_1}\otimes\cdots\otimes dz^{k_m} \tag{1}$$

and

$$f=\|\xi\|^2=\sum \xi_{k_1\cdots k_m}\bar{\xi}^{k_1\cdots k_m}\,. \tag{2}$$

We calculate the Laplacian

$$\Delta f=\sum g^{p\bar{q}}\nabla_{\bar{q}}\nabla_p f\,. \tag{3}$$

Since ∇_p (resp. $\nabla_{\bar{q}}$) annihilates anti-holomorphic (resp. holomorphic) tensor fields, we obtain

$$\nabla_p f=\sum \nabla_p\xi_{k_1\cdots k_m}\bar{\xi}^{k_1\cdots k_m} \tag{4}$$

$$\nabla_{\bar{q}}\nabla_p f=\sum (\nabla_p\xi_{k_1\cdots k_m}\nabla_{\bar{q}}\bar{\xi}^{k_1\cdots k_m}+\nabla_{\bar{q}}\nabla_p\xi_{k_1\cdots k_m}\bar{\xi}^{k_1\cdots k_m})\,. \tag{5}$$

But from the definition of the curvature we obtain

$$\nabla_{\bar{q}}\nabla_p\xi_{k_1\cdots k_m}=\nabla_p\nabla_{\bar{q}}\xi_{k_1\cdots k_m}+\sum R^j_{k_1p\bar{q}}\xi_{jk_2\cdots k_m}+\cdots \tag{6}$$
$$+\sum R^j_{k_mp\bar{q}}\xi_{k_1\cdots k_{m-1}j}\,.$$

Since $\nabla_{\bar{q}}\xi_{k_1\cdots k_m}=0$ and since ξ is symmetric, we obtain from (5) and (6) the following identity:

$$\Delta f=\|\nabla\xi\|^2+m\cdot Q(\xi, \xi)\,. \tag{7}$$

Since both terms on the right are non-negative, we obtain $\Delta f\geqq 0$. On a compact manifold, this is possible only when $\Delta f=0$, that is,

$$\nabla\xi=0 \quad\text{and}\quad Q(\xi, \xi)=0\,. \tag{8}$$

COROLLARY 2. *Let M be as in Theorem 1. If the Ricci tensor is positive (resp. negative) definite, then*

$$\Gamma(S^m T^* M)=0 \quad (resp.\ \Gamma(S^m TM)=0).$$

But we are more interested in the case where the Ricci tensor is semi-definite. To obtain sharper results in this case, we study the holonomy group of M.

§ 3. Holonomy groups.

Let M be an n-dimensional riemannian manifold. We denote the linear holonomy group of M by Ψ and the restricted linear holonomy group of M by Ψ^0. Then Ψ^0 is the identity component of Ψ and is a closed subgroup of $SO(n)$. The holonomy classification theorem of Berger [2] states:

THEOREM 3. *If M is irreducible, i.e., if Ψ^0 is irreducible, then either* (a) *M is locally symmetric (i.e., the curvature is parallel) or* (b) *Ψ^0 is a closed subgroup of $SO(n)$ which acts transitively on S^{n-1}. Explicitly, Ψ^0 is one of the following* 9 *groups:*

(i) $SO(n)$, (ii) $U(k)$, $n=2k$, (iii) $SU(k)$, $n=2k$, (iv) $Sp(k)\times Sp(1)$, $n=4k$, (v) $Sp(k)\times U(1)$, $n=4k$, (vi) $Sp(k)$, $n=4k$, (vii) G_2, $n=7$, (viii) Spin (7), $n=8$, (ix) Spin (9), $n=16$.

COROLLARY 4. *If M is a Kähler manifold of complex dimension n and if Ψ^0 is irreducible, then either* (a) *M is locally symmetric or* (b) *Ψ^0 is one of the following* 4 *groups:*

(i) $U(n)$, (ii) $SU(n)$, (iii) $Sp(k)\times U(1)$, (iv) $Sp(k)$, $(n=2k)$.

This corollary follows from the fact that, among the groups listed in Theorem 3, only the groups listed in Corollary 4 leave a non-trivial 2-form (the Kähler form) invariant, see Berger [2].

We quote the following classical result (see Weyl [4]).

THEOREM 5. *Let G be one of the four groups listed in Corollary* 4. *It is a linear group acting on $V=C^n$. For each positive integer m, the natural representation of G on the symmetric tensor product space $S^m V$ is irreducible.*

We only need the fact that there is no nonzero element in $S^m V$ which remains invariant under the action of G.

§ 4. Sharpening of Theorem 1.

Making use of the results in § 3, we shall now strengthen the result of § 2.

Let M be a simply connected, complete Kähler manifold. Let

(9) $$M=M_0\times M_1\times \cdots \times M_r, \quad \text{(biholomorphic and isometric)}$$

be the de Rham decomposition of M, where M_0 is biholomorphically isometric to the complex Euclidean space and $M_1, \cdots, M_r$ are Kähler manifolds whose holonomy groups are irreducible. (The Euclidean factor M_0 may reduce to a single point). By Corollary 4, each M_i, $(i=1, \cdots, r)$, is either an irreducible hermitian symmetric space or has one of the four groups listed in Corollary 4 as its holonomy group.

Assume that the Ricci tensor of M is positive semi-definite. Let $\xi \in \Gamma(S^m T^* M)$. Assume $\nabla\xi=0$ and $Q(\xi, \xi)=0$. (According to Theorem 1, this assumption is automatically satisfied if M is compact). We claim that ξ comes from M_0, i.e., it belongs to $\Gamma(S^m T^* M_0)$ under a natural identification

$$S^m T^* M = \sum (S^{m_0} T^* M_0) \otimes (S^{m_1} T^* M_1) \otimes \cdots \otimes (S^{m_r} T^* M_r),$$

where the summation is taken over all partitions $m=m_0+m_1+\cdots+m_r$. Our assertion follows from the following two facts.

(i) If M_i is irreducible symmetric (with positive semi-definite Ricci tensor), then its Ricci tensor is positive definite and there is no nonzero holomorphic section $\xi \in \Gamma(S^{m_i} T^* M_i)$ satisfying $Q(\xi, \xi)=0$.

(ii) If the holonomy group of M_i is one of the four groups listed in Corollary 4, then there is no nonzero holomorphic section $\xi \in \Gamma(S^{m_i} T^* M)$ satisfying $\nabla\xi=0$, i.e., invariant by the holonomy. Hence,

THEOREM 6. *Let M be a compact Kähler manifold with positive (resp. negative) semi-definite Ricci tensor. Let n_0 be the dimension of the Euclidean factor M_0 in the de Rham decomposition $M_0 \times M_1 \times \cdots \times M_r$ of the universal covering space $\tilde{M}$ of M. Then*

$$\dim \Gamma(S^m T^* M) \leqq \binom{m+n_0-1}{m}, \quad (\textit{resp.}\ \dim \Gamma(S^m T M) \leqq \binom{m+n_0-1}{m}).$$

In particular, if M_0 is trivial, then

$$\Gamma(S^m T^* M)=0 \quad (\textit{resp.}\ \Gamma(S^m T M)=0).$$

§5. Theorem 6 in algebraic geometric terms.

We shall restate Theorem 6 using the first Chern class in place of the Ricci tensor.

Let M be a compact Kähler manifold. Yau [5] has shown that every closed (1, 1)-form representing the first Chern class $c_1(M)$ is the Ricci form of some Kähler metric. (For the case $c_1(M)<0$, consult also Aubin [1]). We say that $c_1(M)$ is >0 (resp. $\geqq 0$, <0, $\leqq 0$, $=0$) if there is a representative closed (1, 1)-form which is >0 (resp. $\geqq 0$, <0, $\leqq 0$, $=0$). This is equivalent to saying that there is a Kähler metric whose Ricci tensor is >0 (resp. $\geqq 0$, <0, $\leqq 0$, $=0$).

Now we can restate Theorem 6 as follows:

THEOREM 7. *Let M be a compact Kähler manifold with $c_1(M) \geqq 0$ (resp. $c_1(M) \leqq 0$). Let $\tilde{M}$ be the universal covering space of M and k the largest integer such that $\tilde{M}$ is biholomorphic to $\boldsymbol{C}^k \times N$, where N is a complex manifold. Then*

$$\dim \Gamma(S^m T^* M) \leqq \binom{m+k-1}{m}, \quad (\textit{resp.} \dim \Gamma(S^m TM) \leqq \binom{m+k-1}{m}).$$

COROLLARY 8. *If M is a compact, simply connected Kähler manifold with $c_1(M)=0$, then $\Gamma(S^m TM)=\Gamma(S^m T^* M)=0$.*

In particular, if M is a Kähler K3-surface, then $\Gamma(S^m TM)=\Gamma(S^m T^* M)=0$.

The following result follows from Corollary 2.

COROLLARY 9. *If M is a compact Kähler manifold with $c_1(M)>0$ (resp. $c_1(M)<0$), then $\Gamma(S^m T^* M)=0$ (resp. $\Gamma(S^m TM)=0$).*

Bibliography

[1] T. Aubin, Equations du type Monge-Ampère sur les variétés kähleriennes compactes, C. R. Acad. Sci. Paris, **283** (1976), 119-121.

[2] M. Berger, Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes, Bull. Soc. Math. France, **83** (1955), 279-330.

[3] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol. II, 1968, Wiley, New York.

[4] H. Weyl, The Classical Groups, 1949, Princeton Univ. Press.

[5] S. T. Yau, Calabi's conjecture and some new results in algebraic geometry, Proc. Nat. Acad. Sci. U. S. A., **74** (1977), 1798-1799.

Shoshichi KOBAYASHI

Department of Mathematics
University of California, Berkeley
California 94720
U. S. A.