J. Math. Soc. Japan
Vol. 32, No. 2, 1980

Hölder estimates on higher derivatives of the solution for $\bar{\partial}$-equation with C^k-data in strongly pseudoconvex domain

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(Received Aug. 22, 1977)

§ 0. Introduction.

In 1971 Kerzman [4] showed there exists a solution of $\bar{\partial}$-equation with bounded data which is Hölder continuous for any exponent smaller than 1/2. Since then many results have been obtained concerning this problem. Henkin-Romanov [3] and Range-Siu [5] proved the exact 1/2-Hölder estimate. Moreover Siu [6] showed the Hölder continuity of higher derivatives of the solution assuming the data are sufficiently smooth. In this paper we shall improve Siu's result and get a new estimate which is sharper in some tangential directions. We follow the method of Siu [6]; however, various parts of his calculus are ameliorated. I thank Professor H. Tanabe, who encouraged me to write this paper and corrected my manuscript.

0.1. Notations.

Let Ω be a bounded strongly pseudoconvex domain in $\boldsymbol{C}^n$ with C^N-boundary. We assume that Ω is represented as $\{z\in\boldsymbol{C}^n;\ \rho(z)<0\}$, where ρ is a function of class C^N and in some neighborhood of $\partial\Omega$ is strictly plurisubharmonic and satisfies $d\rho\neq 0$. We use the following notations;

$$D_j=\frac{\partial}{\partial z_j}=\frac{1}{2}\Big(\frac{\partial}{\partial x_j}-i\frac{\partial}{\partial y_j}\Big),\quad \bar{D}_j=\frac{\partial}{\partial\bar{z}_j}=\frac{1}{2}\Big(\frac{\partial}{\partial x_j}+i\frac{\partial}{\partial y_j}\Big),$$

$$\|u\|_0=\sup\{|u(z)|;\ z\in\Omega\},$$

$$\|u\|_\varepsilon=\sup\{|u(z)-u(\zeta)|/|z-\zeta|^\varepsilon;\ \zeta,\ z\in\Omega,\ \zeta\neq z\}+\|u\|_0$$

$$\|u\|_k=\max\{\|D^\alpha\bar{D}^\beta u\|_0;\ |\alpha|+|\beta|\leqq k\},$$

$$\|u\|_{k+\varepsilon}=\max\{\|D^\alpha\bar{D}^\beta u\|_\varepsilon;\ |\alpha|+|\beta|\leqq k\}$$

where $k\in\boldsymbol{N}$ and $0<\varepsilon<1$. For a form $f=\sum f_i d\bar{z}_i$,

$$\|f\|_k=\max\{\|f_i\|_k;\ 1\leqq i\leqq n\}.$$

DEFINITION 0.1. A vector field Y on $\bar{\Omega}$ is called holomorphic tangential when $\partial\rho(Y)=0$ on $\partial\Omega$, i.e. if $Y=\sum a_i(z)\partial/\partial z_i+\sum b_i(z)\partial/\partial\bar{z}_i$, then $\sum a_i(z)\partial\rho/\partial z_i=0$ on $\partial\Omega$.

When we differentiate forms, we use the Euclid connection, that is, we differentiate them componentwise. In the next section, following Henkin [2] we shall construct an inverse operator for $\bar{\partial}$ which we denote by T. Our main results are the followings.

THEOREM 1. *Let $N\geqq k+2$ and f be a $\bar{\partial}$-closed C^k-(0.1) form on $\bar{\Omega}$. Then*

$$\|T(f)\|_{k+1/2}\leqq C_k\|f\|_k.$$

REMARK. Our improvement consists in the condition $N\geqq k+2$. Y. T. Siu required $N\geqq k+4$ in [6].

THEOREM 2. *Let $N\geqq k+2$, f be a $\bar{\partial}$-closed C^k-(0.1) form on $\bar{\Omega}$ and Y be a C^2-holomorphic tangential vector field. Then for $|\alpha|=k-1$,*

$$\|YD^\alpha T(f)\|_\beta\leqq C_\beta\|f\|_k, \qquad \text{for any } \beta \text{ smaller than } 1.$$

§1. Henkin's kernel.

In this section we construct an inverse operator for $\bar{\partial}$ following Henkin [2].

1.1. Let

$$F^*(\zeta, z)=\sum_{i=1}^n \frac{\partial\rho}{\partial\zeta_i}(\zeta)(z_i-\zeta_i)+\frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2\rho}{\partial\zeta_i\partial\zeta_j}(\zeta)(z_i-\zeta_i)(z_j-\zeta_j).$$

Then

$$-\mathrm{Re}\,F^*(\zeta, z)\geqq\rho(\zeta)-\rho(z)+C|\zeta-z|^2,$$

for $|\zeta-z|$ and $|\rho(\zeta)|$ small. Moreover if we set

$$F(\zeta, z)=\sum\frac{\partial\rho}{\partial\zeta_i}(\zeta)(z_i-\zeta_i)+\sum\phi_{i,j}(\zeta)(z_i-\zeta_i)(z_j-\zeta_j)$$

where $\phi_{i,j}(\zeta)=\dfrac{1}{2}\dfrac{\partial^2\rho}{\partial\zeta_i\partial\zeta_j}*\chi_\varepsilon(\zeta)$ (χ_ε; a mollifier), then

$$-\mathrm{Re}\,F(\zeta, z)\geqq C\{\rho(\zeta)-\rho(z)+|\zeta-z|^2\}$$

for $|\zeta-z|$, $|\rho(\zeta)|$ and ε small. (c.f. [1], [4] and [5].)

1.2. As in Henkin [1] we can prove the existence of a function $\Phi(\zeta, z)$ defined in $V\times\tilde{\Omega}=\{\zeta;|\rho(\zeta)|<\nu\}\times\{z;\rho(z)<\nu\}$, $\nu>0$, with the following properties:

(1) $\Phi(\zeta, z)$ belongs to $C^{N-1}(V\times\tilde{\Omega})$ and is holomorphic in z.

(2) $\Phi(\zeta, z)\neq 0$ for $z\in\tilde{\Omega}$, $\zeta\in V$, $\zeta\neq z$ and $\rho(\zeta)\geqq\rho(z)$.

(3) For $\zeta^0 \in \partial\Omega$, there exist a neighborhood U of ζ^0 in V and a C^{N-1} non-vanishing function $H(\zeta, z)$ on $U\times U$ holomorphic in z such that $\Phi(\zeta, z)=F(\zeta, z)H(\zeta, z)$ on $U\times U$.

(4) There exist n C^{N-1} functions $P_i(\zeta, z)$, $1\leqq i\leqq n$, on $V\times\tilde{\Omega}$ holomorphic in z such that

$$\Phi(\zeta, z)=\sum_{i=1}^{n}(z_i-\zeta_i)P_i(\zeta, z) \qquad \text{on } V\times\tilde{\Omega}.$$

1.3. DEFINITION 1.1.

1) $C(\zeta, z)=C_0\Phi^{-n}(\zeta, z)\sum_{j=1}^{n}(-1)^j P_j\bar{\partial}_\zeta P_1\wedge\cdots\hat{j}\cdots\wedge\bar{\partial}_\zeta P_n\wedge\omega(\zeta)$,

where the notation $\hat{j}$ means $\bar{\partial}_\zeta P_j$ is omitted, $\omega(\zeta)=d\zeta_1\wedge\cdots\wedge d\zeta_n$ and $C_0=(n-1)!/(2\pi i)^n$.

2) Letting

$$\eta_j=\lambda(\bar{z}_j-\bar{\zeta}_j)|z-\zeta|^{-2}+(1-\lambda)P_j(\zeta, z)\Phi^{-1}(\zeta, z),$$

define

$$K(\zeta, z, \lambda)=C_0\sum_{j=1}^{n}(-1)^j\eta_j\bar{\partial}_{\zeta,\lambda}\eta_1\wedge\cdots\hat{j}\cdots\wedge\bar{\partial}_{\zeta,\lambda}\eta_n\wedge\omega(\zeta).$$

3) Letting $\tau_j(\bar{\zeta})=(-1)^{j+1}d\bar{\zeta}_1\wedge\cdots\hat{j}\cdots\wedge d\bar{\zeta}_n$, we set

$$L(\zeta, z)=-C_0|\zeta-z|^{-2n}\sum_{j=1}^{n}(\bar{z}_j-\bar{\zeta}_j)\tau_j(\bar{\zeta})\wedge\omega(\zeta).$$

4) $K(\zeta, z)$ is given by integrating $K(\zeta, z, \lambda)$ from 0 to 1 in λ.

5) Moreover we write

$$C(\zeta, z)=\sum C_j(\zeta, z)\tau_j(\bar{\zeta})\wedge\omega(\zeta), \qquad L(\zeta, z)=\sum L_j(\zeta, z)\tau_j(\bar{\zeta})\wedge\omega(\zeta),$$

where C_j and L_j are defined by these equalities.

REMARK. The relations $\sum(z_j-\zeta_j)\eta_j=1$ and $\sum(z_j-\zeta_j)P_j=\Phi$ imply $d_\zeta C(\zeta, z)=0$ and $d_{\zeta,\lambda}K(\zeta, z, \lambda)=0$.

1.4. LEMMA 1.2. *Let*

$$K(\zeta, z)=\sum_{i<j}K_{i,j}(\zeta, z)d\bar{\zeta}_1\wedge\cdots\hat{i}\cdots\hat{j}\cdots\wedge d\bar{\zeta}_n\wedge\omega(\zeta).$$

Then $K_{i,j}$ has the following form:

$$K_{i,j}(\zeta, z)=\sum_{k=1}^{n-1}\Phi^{-k}(\zeta, z)|\zeta-z|^{2k-2n}\sum_{m=1}^{n}(\bar{\zeta}_m-\bar{z}_m)C_{i,j}^{m,k}(\zeta, z)$$

where $C_{i,j}^{m,k}$ is C^{N-2} and holomorphic in z.

PROOF. From the definition, the coefficient of $d\lambda\wedge d\bar{\zeta}_1\wedge\cdots\hat{i}\cdots\hat{j}\cdots\wedge d\bar{\zeta}_n\wedge\omega(\zeta)$ is as follows.

$$\sum_{k=1}^{n}(-1)^k\eta_k\begin{vmatrix}\frac{\partial\eta_1}{\partial\lambda}\frac{\partial\eta_1}{\partial\bar\zeta_1}\cdots\overset{\vdots}{}\cdots\overset{\vdots}{}\cdots\frac{\partial\eta_1}{\partial\bar\zeta_n}\\ \cdots\cdots\cdots\cdots\cdots\\ \frac{\partial\eta_n}{\partial\lambda}\frac{\partial\eta_n}{\partial\bar\zeta_1}\cdots\underset{\hat i}{\vdots}\cdots\underset{\hat j}{\vdots}\cdots\frac{\partial\eta_n}{\partial\bar\zeta_n}\end{vmatrix}\hat{k}=-\begin{vmatrix}\eta_1\frac{\partial\eta_1}{\partial\lambda}\frac{\partial\eta_1}{\partial\bar\zeta_1}\cdots\overset{\vdots}{}\cdots\overset{\vdots}{}\cdots\frac{\partial\eta_1}{\partial\bar\zeta_n}\\ \eta_n\frac{\partial\eta_n}{\partial\lambda}\frac{\partial\eta_n}{\partial\bar\zeta_1}\cdots\underset{\hat i}{\vdots}\cdots\underset{\hat j}{\vdots}\cdots\frac{\partial\eta_n}{\partial\bar\zeta_n}\end{vmatrix}.$$

The terms which occur by derivating denominators are represented as some linear combinations of the first two column vectors. So they can be omitted from the above determinant. Hence computing the resulting determinant proves the lemma.

1.5. The following formula is proved in Henkin [2].

$$u(z)=\int_{\partial\Omega}u(\zeta)C(\zeta,z)+\int_{\partial\Omega}\bar\partial u(\zeta)\wedge K(\zeta,z)-\int_{\Omega}\bar\partial u(\zeta)\wedge L(\zeta,z)$$

for $u\in C^1(\bar\Omega)$.

DEFINITION 1.2. We define the operator T from the space of continuous (0, 1) forms into the space of continuous functions by

$$T(f)=\int_{\partial\Omega}f(\zeta)\wedge K(\zeta,z)-\int_{\Omega}f(\zeta)\wedge L(\zeta,z).$$

One can see $\bar\partial T(f)=f$, if $\bar\partial f=0$.

§ 2. Estimates of kernels.

2.1. DEFINITION 2.1.

1) $\sigma(\zeta)=\sum_{j=1}^{n}\partial\rho/\partial\zeta_j\,\tau_j(\bar\zeta)\wedge\omega(\zeta)$.

2) $S_j=\dfrac{\partial\rho}{\partial\zeta_j}|\partial\rho|^{-2}\Big(\sum_{i=1}^{n}\dfrac{\partial\rho}{\partial\zeta_i}\dfrac{\partial}{\partial\bar\zeta_i}+\Delta\rho\Big)$

where Δ is the laplacian.

S_j is a first order differential operator in the neighborhood of $\partial\Omega$ where $d\rho\neq 0$.

LEMMA 2.1. *The following two equalities hold on* $\partial\Omega$.

1) $|\partial\rho|^2\omega(\bar\zeta)\wedge\tau_k(\zeta)=(-1)^n\partial\rho/\partial\zeta_k\sigma(\zeta)$,

2) $|\partial\rho|^2\tau_k(\bar\zeta)\wedge\omega(\zeta)=\partial\rho/\partial\bar\zeta_k\sigma(\zeta)$.

PROOF. Since $\sum\partial\rho/\partial\zeta_j d\zeta_j+\sum\partial\rho/\partial\bar\zeta_j d\bar\zeta_j=0$ on $\partial\Omega$,

(1) $$\begin{aligned}\partial\rho/\partial\bar\zeta_j\omega(\bar\zeta)\wedge\tau_k(\zeta)&=-\partial\rho/\partial\zeta_k d\bar\zeta_1\wedge\cdots d\bar\zeta_{j-1}\wedge d\zeta_k\wedge d\bar\zeta_{j+1}\cdots\wedge d\bar\zeta_n\wedge\tau_k(\zeta)\\&=(-1)^{n+j+1}\partial\rho/\partial\zeta_k d\bar\zeta_1\wedge\cdots\hat{j}\cdots\wedge d\bar\zeta_n\wedge d\zeta_k\wedge\tau_k(\zeta)\\&=(-1)^n\partial\rho/\partial\zeta_k\tau_j(\bar\zeta)\wedge\omega(\zeta);\end{aligned}$$

(2) similarly $\partial\rho/\partial\bar{\zeta}_j\tau_k(\bar{\zeta})\wedge\omega(\zeta)=\partial\rho/\partial\bar{\zeta}_k\tau_j(\bar{\zeta})\wedge\omega(\zeta)$.

Multiplying both sides of (1) and (2) by $\partial\rho/\partial\zeta_j$ and summing up in j, we get 1) and 2).

2.2. Cartan's formula.

Let ω be a form and X be a vector field. Then the following formula (Cartan's formula) holds. (c.f. Sternberg [7].)

$$L_X\omega=X \lrcorner\, d\omega+d(X \lrcorner\, \omega)\,,$$

where L_X is the Lie derivative and $\lrcorner$ is the interier product. In the above formula we take $X=\partial/\partial\zeta_j$. Then the Lie derivative L_X agrees with the Euclid connection because X is parallel. So Stoke's theorem implies the following lemma.

LEMMA 2.2. *If ω be a $(2n-1)$ form, then*

$$\int_{\partial\Omega}\frac{\partial}{\partial\zeta_j}\omega=\int_{\partial\Omega}\frac{\partial}{\partial\zeta_j}\lrcorner\, d\omega\,.$$

EXAMPLE 1. Let $K(\zeta)$ be a $\bar{\partial}$-closed $(n, n-1)$ form and $u(\zeta)$ be a C^1 function. Then

$$\int_{\partial\Omega}\frac{\partial}{\partial\zeta_j}\{u(\zeta)K(\zeta)\}=\int_{\partial\Omega}\bar{\partial}u(\zeta)\wedge\left\{\frac{\partial}{\partial\zeta_j}\lrcorner\, K(\zeta)\right\}.$$

We shall use this formula in section 3 for $u=(\zeta-z)^\alpha(\bar{\zeta}-\bar{z})^\beta$ and $K=X^\gamma C(\zeta, z)$, where X is coming later.

EXAMPLE 2. Let f be a $\bar{\partial}$-closed (0, 1) form and $K(\zeta)$ be a C^1-$(n, n-2)$ form. Then

$$\int_{\partial\Omega}\frac{\partial}{\partial\zeta_j}\{f(\zeta)\wedge K(\zeta)\}=-\int_{\partial\Omega}f(\zeta)\wedge\left\{\frac{\partial}{\partial\zeta_j}\lrcorner\, dK(\zeta)\right\}.$$

We shall use this in the proof of Proposition 6.

EXAMPLE 3. If $a(\zeta)$ be a C^1 function, then

$$\int_{\partial\Omega}\frac{\partial}{\partial\zeta_j}\{a(\zeta)\sigma(\zeta)\}=\int_{\partial\Omega}\{S_ja(\zeta)\}\,\sigma(\zeta)\,.$$

Indeed on $\partial\Omega$,

$$\frac{\partial}{\partial\zeta_j}\lrcorner\, d\{a(\zeta)\sigma(\zeta)\}=(-1)^n\sum\partial/\partial\bar{\zeta}_i\{a(\zeta)\partial\rho/\partial\zeta_i\}\,\omega(\bar{\zeta})\wedge\tau_j(\zeta)$$

$$=\partial\rho/\partial\zeta_j|\partial\rho|^{-2}\{\sum\partial\rho/\partial\zeta_i\partial a/\partial\bar{\zeta}_i+a(\zeta)\Delta\rho\}\,\sigma(\zeta)\,.$$

2.3. DEFINITION 2.2.

1) For $f\in C^k(\Omega)$,

$$f^{(k)}(\zeta, z)=f(\zeta)-\sum_{|\alpha|+|\beta|\leq k}\frac{(\zeta-z)^{\alpha}(\bar{\zeta}-\bar{z})^{\beta}}{\alpha!\quad\beta!}D^{\alpha}\bar{D}^{\beta}f(z).$$

2) For $f=\sum_{i=1}^{n}f_i(\zeta)d\bar{\zeta}_i$, $\bar{\partial}$-closed ($f_i\in C^k$),

$$f^{(k)}(\zeta, z)=\sum_{i=1}^{n}f_i^{(k)}(\zeta, z)d\bar{\zeta}_i,$$

and for $|\gamma|\leqq k+1$, $f_{\gamma}(\zeta)=\bar{D}^{\beta}f_i(\zeta)$ where $D^{\beta}D_i=D^{\gamma}$.

LEMMA 2.3. *If $f\in C^k$, then*

1) $|f^{(k-1)}(\zeta, z)|\leqq C\|f\|_k|\zeta-z|^k,$

2) $|W[f^{(k-1)}(\zeta, z)]|\leqq C\|f\|_k|\zeta-z|^{k-1},$

where W is $\partial/\partial z_j$, $\partial/\partial\bar{z}_j$, $\partial/\partial\zeta_j$ or $\partial/\partial\bar{\zeta}_j$.

3) *If $f=\sum f_i d\bar{\zeta}_i$ is $\bar{\partial}$-closed ($f_i\in C^k$), then*

$$f^{(k)}(\zeta, z)=f(\zeta)-\sum_{|\alpha|+|\gamma|\leq k+1}\frac{(\zeta-z)^{\alpha}}{\alpha!\ \gamma!}D^{\alpha}f_{\gamma}(z)\bar{\partial}_{\zeta}(\bar{\zeta}-\bar{z})^{\gamma}.$$

PROOF. 1) is a simple consequence of

$$f^{(k-1)}(\zeta, z)=k\sum_{|\alpha|+|\beta|=k}\frac{(\zeta-z)^{\alpha}(\bar{\zeta}-\bar{z})^{\beta}}{\alpha!\quad\beta!}\int_0^1(1-\theta)^{k-1}D^{\alpha}\bar{D}^{\beta}f(z+\theta(\zeta-z))d\theta.$$

2) is obvious from 1) and some simple calculations. 3) See Siu [**6**] p. 174.

2.4. LEMMA 2.4. *Let c be a positive constant. Then*

$$\int_{|x|=1,\,x_1>0}\frac{d\sigma}{x_1+c}<\frac{\pi}{2}\log\left(1+\frac{1}{c}\right)$$

where $x=(x_1, \cdots, x_N)\in\boldsymbol{R}^N$, $N\geqq 3$ and $d\sigma$ is the canonical measure of the unit sphere in $\boldsymbol{R}^N$. (See Range-Siu [**5**] p. 342.)

LEMMA 2.5.

1) $\int_{x_1>0,\,|x|<R}(x_1+r^2)^{-1}r^{2-N}dx_1\cdots dx_N<C,$

2) $\int_{x_1>0,\,|x|<R}(x_1+\delta+r^2)^{-1}(r^2+\delta^2)^{-1}r^{3-N}dx_1\cdots dx_N<C(1+|\log\delta|^2),$

3) $\int_{x_1>0,\,|x|<R}(x_1+\delta+r^2)^{-2}r^{2-N}dx_1\cdots dx_N<C\delta^{-1/2},$

where $r^2=x_1^2+\cdots+x_N^2$ and $\delta>0$.

PROOF. 1) and 3) See Range-Siu [**5**].

2) $$\int_{|x|<R,\, x_1>0}(x_1+\delta+r^2)^{-2}(r^2+\delta^2)^{-1}r^{3-N}dx_1\cdots dx_N$$

$$=C\int_0^R r(r^2+\delta^2)dr\int_{|x|=1,\, x_1>0}(x_1+r+\delta/r)d\sigma$$

$$<C\int_0^R r(r^2+\delta^2)^{-1}\log(1+r/(r^2+\delta))dr<C(1+|\log\delta|^2).$$

2.5. Later we shall prove that a function u on Ω is α-Hölder continuous by showing $|\mathrm{grad}\, u|<C|\rho(z)|^{-\alpha}$, so the following proposition is important.

PROPOSITION 3.

1) $$\int_{\partial\Omega}|\zeta-z|^{1-2n}d\mu<C(1+|\log(-\rho(z))|),$$

2) $$\int_{\partial\Omega}|\Phi|^{-1}|\zeta-z|^{2-2n}d\mu<C[1+(\log|\rho(z)|)^2],$$

3) $$\int_{\partial\Omega}|\Phi|^{-2}|\zeta-z|^{3-2n}d\mu<C|\rho(z)|^{-1/2},$$

4) $$\int_{\partial\Omega}|\Phi|^{-1}|\zeta-z|^{3-2n}d\mu<C,$$

where $d\mu$ is the Lebesgue measure induced on $\partial\Omega$.

PROOF. It suffices to show the above inequalities when $|\rho(z)|$ is small. Let ζ^0 be the orthogonal projection of z to $\partial\Omega$. We calculate the integral near ζ^0 using some new coordinate system $(x_1, \cdots, x_{2n-1})$ with $x_1=\mathrm{Im}\, F(\zeta, z)$ (c.f. Henkin [**1**]).

Let $S_1=\{\zeta\in\partial\Omega;\ |\zeta^0-\zeta|<\varepsilon\}$. We divide the integral $\int_{\partial\Omega}$ into $\int_{S_1}+\int_{S_1^c}$. Then $\left|\int_{S_1^c}\right|<C$ where C depends on ε but not on $|\rho(z)|$. Thus we have only to estimate the integrals on S_1. Let $\delta=|\rho(z)|$.

1) $$\int_{S_1}|\zeta-z|^{1-2n}d\mu<C\int_{|x|<R}(r^2+\delta^2)^{-1}r^{3-2n}dx_1\cdots dx_{2n-1}$$

$$<C\log(1+R^2/\delta^2).$$

2) $$\int_{S_1}|\Phi|^{-1}|\zeta-z|^{2-2n}d\mu$$

$$<\int_{|x|<R,\, x_1>0}r^{4-2n}/[(x_1+\delta+r^2)(r^2+\delta^2)]dx_1\cdots dx_{2n-1}$$

$$<C(1+|\log\delta|^2).$$

We can show 3) and 4) analogously to the above using Lemma 2.5.

2.6. DEFINITION 2.4. We introduce the notations,

1) $X_j=\partial/\partial z_j+\partial/\partial\zeta_j$,

2) $K(z)=1+[\log|\rho(z)|]^2$.

LEMMA 2.6. 1) $|X_j\Phi(\zeta, z)|\leqq C|\zeta-z|$.

2) *Let u be a function defined in Ω such that $|\text{grad}\, u|<CK(z)$. Then u is Hölder continuous for any exponent smaller than* 1.

PROOF. 1) is an easy consequence of (4) of the properties of $\Phi(\zeta, z)$.

2) One can show $|u(z)-u(z')|\leqq A_\alpha|z-z'|^\alpha$ for $0<\alpha<1$ by integrating du along an appropriate path connecting z and z'.

§ 3. Estimates in the polynomial case.

Here we shall establish the Hölder estimates of

$$\int_{\partial\Omega}(\zeta-z)^\alpha(\bar\zeta-\bar z)^\beta D_z^\gamma C(\zeta, z). \qquad (\text{Recall } D_j=\partial/\partial z_j.)$$

3.1. LEMMA 3.1. *For a function $a(\zeta, z)$ which is smooth in z*

$$(\zeta-z)^\alpha D_z^\beta a(\zeta, z)=\sum_{\beta_1+\beta_2=\beta,\ \beta_2\leq\alpha}\binom{\beta}{\beta_1}\binom{\alpha}{\beta_2}D_z^{\beta_1}[(\zeta-z)^{\alpha-\beta_2}a(\zeta, z)].$$

PROOF. By induction on β, we shall prove

$$(*) \qquad (\zeta-z)^\alpha D_z^\beta a=\sum_{\beta_1+\beta_2=\beta}\binom{\beta}{\beta_1}D_z^{\beta_1}[\{(\partial/\partial\zeta)^{\beta_2}(\zeta-z)^\alpha\}a],$$

from which the conclusion follows easily. If $\beta=0$, it is trivial. Suppose $(*)$ is true for β. We replace a by $D_j a$. Combining the equality thus obtained and the following relation

$$\{(\partial/\partial\zeta)^{\beta_2}(\zeta-z)^\alpha\}D_j a=D_j[\{(\partial/\partial\zeta)^{\beta_2}(\zeta-z)^\alpha\}a]+\{(\partial/\partial\zeta)^{\beta_2}\partial/\partial\zeta_j(\zeta-z)^\alpha\}a,$$

we get $(*)$ with $D^{\beta'}=D_jD^\beta$ in the place of D^β. This proves the lemma.

PROPOSITION 4. *Let $|\gamma|=M\leqq N-2$. Then*

$$\int_{\partial\Omega}(\zeta-z)^\alpha(\bar\zeta-\bar z)^\beta D_z^\gamma C(\zeta, z)=\sum_{\gamma_1+\gamma_2=\gamma,\ \gamma_2\leq\alpha}\binom{\gamma}{\gamma_1}\binom{\alpha}{\gamma_2}D_z^{\gamma_1}\int_{\partial\Omega}(\zeta-z)^{\alpha-\gamma_2}(\bar\zeta-\bar z)^\beta C(\zeta, z).$$

PROOF. This is an easy conclusion from Lemma 3.1.

3.2. LEMMA 3.2. *Let $|\gamma|=M\leqq N-2$. Then*

$$(°)\qquad D_z^{\alpha}\int_{\partial\Omega}(\zeta-z)^{\beta}(\bar{\zeta}-\bar{z})^{\gamma}C(\zeta, z)$$

$$=\int_{\partial\Omega}(\zeta-z)^{\beta}(\bar{\zeta}-\bar{z})^{\gamma}X^{\alpha}C(\zeta, z)+\sum_{k=1}^{M-1}\sum_{|\varepsilon|=k}\int_{\partial\Omega}(\zeta-z)^{\beta+\varepsilon}a_{\varepsilon}(\zeta, z)/\Phi^{n+k}\sigma(\zeta)$$

where a_{ε} *is* $C^{N+k-M-1}$ *and* C^{∞} *in* z.

PROOF. We shall prove this by induction on α. If $\alpha=0$, there is nothing to be proved. We assume (°) is true for α. Then we apply $\partial/\partial z_j$ to (°), regard $\partial/\partial z_j$ as $X_j-\partial/\partial\zeta_j$ under the integral sign and convert $\partial/\partial\zeta_j$ to S_j using the formula in Example 3 after Lemma 2.2. Then we get

$$D_jD^{\alpha}\int_{\partial\Omega}(\zeta-z)^{\beta}(\bar{\zeta}-\bar{z})^{\gamma}C(\zeta, z)=\int_{\partial\Omega}(\zeta-z)^{\beta}(\bar{\zeta}-\bar{z})^{\gamma}X_jX^{\alpha}C(\zeta, z)$$

$$-\int_{\partial\Omega}\partial/\partial\zeta_j[(\zeta-z)^{\beta}(\bar{\zeta}-\bar{z})^{\gamma}X^{\alpha}C(\zeta, z)]$$

$$+\sum_{k=0}^{M-1}\sum_{|\varepsilon|=k}\int_{\partial\Omega}X_j\{(\zeta-z)^{\beta+\varepsilon}a_{\varepsilon}(\zeta, z)\Phi^{-n-k}(\zeta, z)\sigma(\zeta)\}$$

$$-\int(\zeta-z)^{\beta+\varepsilon}\{S_j(a_{\varepsilon}(\zeta, z)/\Phi^{n+k}(\zeta, z)\}\,\sigma(\zeta)\,.$$

Now,

$$\int_{\partial\Omega}\frac{\partial}{\partial\zeta_j}[(\zeta-z)^{\beta}(\bar{\zeta}-\bar{z})^{\gamma}X^{\alpha}C(\zeta, z)]=\int_{\partial\Omega}(\zeta-z)^{\beta}\bar{\partial}\rho(\bar{\zeta}-\bar{z})^{\gamma}\wedge\left[\frac{\partial}{\partial\zeta_j}\lrcorner X^{\alpha}C(\zeta, z)\right].$$

(See Example 1 after Lemma 2.2.) But

$$X^{\alpha}C(\zeta, z)=\sum_{k=0}^{M}\sum_{l=1}^{n}\sum_{|\varepsilon|=k}(\zeta-z)^{\varepsilon}a_{\varepsilon,l}(\zeta, z)/\Phi^{n+k}(\zeta, z)\tau_l(\bar{\zeta})\wedge\omega(\zeta)\,,$$

where $a_{\varepsilon,l}$ is in the class $C^{N+k-M-2}$, so these terms are allowed to appear in the formula for α'. ($D^{\alpha'}=D_jD^{\alpha}$.)

$$(\zeta-z)^{\beta+\varepsilon}S_j[a_{\varepsilon}/\Phi^{n+k}(\zeta, z)]$$

$$=(\zeta-z)^{\beta+\varepsilon}B_j(\zeta)\left[\frac{a_{\varepsilon}\Delta\rho+S^*a_{\varepsilon}}{\Phi^{n+k}(\zeta, z)}-\frac{(n+k)a_{\varepsilon}(\zeta, z)\sum(z_i-\zeta_i)S^*P_i(\zeta, z)}{\Phi^{n+k+1}(\zeta, z)}\right],$$

where $B_j(\zeta)=\dfrac{\partial\rho}{\partial\zeta_j}/|\partial\rho|^2$ and $S^*=\sum\partial\rho/\partial\zeta_i\,\partial/\partial\bar{\zeta}_i$.

From the hypothesis $a_{\varepsilon}\in C^{N+k-M-1}$, so $B_j\Delta\rho a_{\varepsilon}$, $B_ja_{\varepsilon}\in C^{N+k-M-2}$ and $B_ja_{\varepsilon}S^*P_i\in C^{N+k-M-1}$. Hence the above terms are allowed to appear in the formula for α'. In a similar way we can decompose the term $(\zeta-z)^{\beta+\varepsilon}X_j[a_{\varepsilon}(\zeta, z)/\Phi^{n+k}(\zeta, z)\sigma(\zeta)]$ into sums of the desired form.

3.3. The following lemma is proved in Siu [6].

LEMMA 3.3. *Let k and m be integers such that $m \leqq k \leqq N$ and G be an open set of $\boldsymbol{C}^n$. Suppose $\partial\rho/\partial\bar{\zeta}_j \neq 0$ on U where U is the open set appeared in* 1.2.

Then for any $h(\zeta, z) \in C^k((U \cap \partial\Omega) \times G)$, there exists an $h^\circ(\zeta, z) \in C^{k-m+1}(U \times G)$ such that $h^\circ(\zeta, z) = h(\zeta, z)$ for $\zeta \in \partial\Omega \cap U$ and $\partial h^\circ/\partial\bar{\zeta}_j = \gamma(\zeta, z)\rho^{m-1}(\zeta)$ for some C^{k-m} function $\gamma(\zeta, z)$ on $U \times G$.

EXAMPLE. Suppose $\partial\rho/\partial\bar{\zeta}_j \neq 0$ on U. Then applying the lemma to $\partial\rho/\partial\zeta_i$ and $\phi_{i,k}$ we get a C^1 function $F^\circ(\zeta, z)$ such that

$$F^\circ(\zeta, z) = F(\zeta, z) \qquad \text{on} \quad (U \cap \partial\Omega) \times \bar{\Omega}\,,$$

$$|\partial F^\circ/\partial\bar{\zeta}_j(\zeta, z)| \leqq C|\zeta - z|\,|\rho(\zeta)|^{N-2}\,,$$

$$-\operatorname{Re} F^\circ(\zeta, z) \geqq C\{\rho(\zeta) - \rho(z) + |\zeta - z|^2\} \qquad \text{for } |\rho(\zeta)| \text{ and } |\zeta - z|$$

small. $F^\circ(\zeta, z)$ will play an important role in the proofs of the following lemmas.

3.4. In what follows in this section all functions are to be C^∞ in z.

LEMMA 3.4. *Let $a(\zeta, z)$ be a C^k function and $|\beta| = k \leqq N-2$. Then*

$$\left|\int_{\partial\Omega} (\zeta - z)^\beta a(\zeta, z)/\Phi^{n+k}(\zeta, z)\sigma(\zeta)\right| < C\,K(z).$$

PROOF. If $k=0$, it is already shown in Proposition 3. Let $k \geqq 1$. It suffices to show the above inequality when $|\rho(z)|$ is small. As in the proof of Proposition 3, let ζ^0 be the orthogonal projection of z to $\partial\Omega$ and U be the neighborhood of ζ^0 introduced in 1.2. Similarly to Proposition 3 we divide the integal $\int_{\partial\Omega}$ into the sum of $\int_{S_1}$ and $\int_{S_1^c}$ where $S_1 = \{\zeta \in \partial\Omega\,;\ |\zeta - \zeta^0| < \varepsilon\}$. By the same reason as in Proposition 3 we have only to estimate the integral on S_1. Since $d\rho \neq 0$ near $\partial\Omega$, we can assume $\partial\rho/\partial\bar{\zeta}_1 \neq 0$ near S_1. By Lemma 2.1 there exists a C^k function $b(\zeta, z)$ such that on S_1

$$(\zeta - z)^\beta a(\zeta, z)/\Phi^{n+k}(\zeta, z)\sigma(\zeta) = (\zeta - z)^\beta b(\zeta, z)/F^{n+k}(\zeta, z)\tau_1(\bar{\zeta}) \wedge \omega(\zeta)\,.$$

If we apply Lemma 3.3 to $b(\zeta, z)$, then we get a C^1 function $b^\circ(\zeta, z)$ such that $b(\zeta, z) = b^\circ(\zeta, z)$ for $\zeta \in S_1$ and

$$|\partial b^\circ/\partial\bar{\zeta}_1(\zeta, z)| \leqq C|\rho(\zeta)|^{k-1}\,.$$

Let $F^\circ(\zeta, z)$ be the function in the example after Lemma 3.3 and

$$B = \{\zeta \in C^n\,;\ |\zeta - \zeta^0| < \varepsilon\} \cap \Omega^c \quad \text{and} \quad S_2 = \partial B \cap \Omega^c\,.$$

Then we get the following by Stokes' theorem.

$$\int_{S_1}(\zeta-z)^\beta b(\zeta, z)F^{-n-k}(\zeta, z)\tau_1(\bar\zeta)\wedge\omega(\zeta)$$

$$=\int_B d\{(\zeta-z)^\beta b^\circ(\zeta, z)F^{\circ-n-k}(\zeta, z)\tau_1(\bar\zeta)\wedge\omega(\zeta)\}$$

$$-\int_{S_2}(\zeta-z)^\beta b^\circ(\zeta, z)F^{\circ-n-k}(\zeta, z)\tau_1(\bar\zeta)\wedge\omega(\zeta).$$

Since the distance between z and S_2 is larger than ε, the integral on S_2 is bounded.

$$|d[(\zeta-z)^\beta b^\circ(\zeta, z)F^{\circ-n-k}(\zeta, z)\tau_1(\bar\zeta)\wedge\omega(\zeta)]|$$

$$<C|\zeta-z|^k|\rho(\zeta)|^{k-1}|F^{\circ-n-k}(\zeta, z)|+C|\zeta-z|^{k+1}|\rho(\zeta)|^{N-2}|F^{\circ-n-k-1}(\zeta, z)|$$

$$<C|\zeta-z||F^{\circ-n-1}(\zeta, z)|.$$

We compute the integral on B in an appropriate coordinate system $(x_1, \cdots, x_{2n})$ with $x_1=\rho(\zeta)$. Then

$$C|\zeta-z||F^\circ(\zeta, z)|^{-n-1}<C(x_1+\delta+r^2)^{-1}r^{3-2n}(r^2+\delta^2)^{-1}$$

where $r^2=x_1^2+\cdots+x_{2n}^2$ and $\delta=|\rho(z)|$. Hence by Lemma 2.5

$$\left|\int_B d[(\zeta-z)^\beta b^\circ(\zeta, z)F^{\circ-n-k}(\zeta, z)\tau_1(\bar\zeta)\wedge\omega(\zeta)]\right|<C\,K(z).$$

REMARK. In the above lemma if $|\beta|\geqq k$, then the bound $C\,K(z)$ can be replaced by a constant C.

LEMMA 3.5. *Let* $k\leqq N-3$, $|\beta|=k$ *and* $a(\zeta, z)\in C^{k+1}$. *Then*

$$\left|\operatorname{grad}\int_{\partial\Omega}(\zeta-z)^\beta a(\zeta, z)/\Phi^{n+k}(\zeta, z)\sigma(\zeta)\right|<C\,K(z).$$

PROOF. We differentiate the above integral under the integral sign. For $\partial/\partial\bar z_j$ compute directly and for $\partial/\partial z_j$ regard it as $X_j-\partial/\partial\zeta_j$ and convert $\partial/\partial\zeta_j$ to S_j. Then the conclusion follows from Lemma 3.4.

LEMMA 3.6. *Let* $1\leqq k\leqq N-1$, $|\beta|=k$, $|\gamma|\geqq 1$ *and* $a(\zeta, z)\in C^{k-1}$. *Then*

$$\left|\int_{\partial\Omega}(\zeta-z)^\beta(\bar\zeta-\bar z)^\gamma a(\zeta, z)/\Phi^{n+k}(\zeta, z)\sigma(\zeta)\right|<C\,K(z).$$

PROOF. If $k=1$, then

$$|(\zeta-z)^\beta(\bar\zeta-\bar z)^\gamma a(\zeta, z)/\Phi^{n+1}(\zeta, z)|<C|\zeta-z|^{2-2n}|\Phi|^{-1},$$

thus the conclusion follows from Proposition 3. Let $k\geqq 2$. As in the proof of Lemma 3.4 we get

$$\frac{(\zeta-z)^{\beta}(\bar{\zeta}-\bar{z})^{\gamma}a(\zeta, z)}{\Phi^{n+k}(\zeta, z)}\sigma(\zeta)=\frac{(\zeta-z)^{\beta}(\bar{\zeta}-\bar{z})^{\gamma}b(\zeta, z)}{F^{n+k}(\zeta, z)}\tau_1(\bar{\zeta})\wedge\omega(\zeta)$$

on S_1 for some $b(\zeta, z)\in C^{k-1}$. Next applying Lemma 3.3 we get the extensions $b^{\circ}(\zeta, z)$ and $c(\zeta, z)$ of $b(\zeta, z)$ and $(\bar{\zeta}-\bar{z})^{\gamma}$ such that

$$|\partial b^{\circ}/\partial\bar{\zeta}_1|\leqq C|\rho(\zeta)|^{k-2} \quad\text{and}\quad |\partial c/\partial\bar{\zeta}_1|\leqq C|\rho(\zeta)|^{N-2}.$$

Moreover the fact $c(\zeta, z)=(\bar{\zeta}-\bar{z})^{\gamma}$ on S_1 implies

$$|c(\zeta, z)|\leqq|\zeta-z|+|(\bar{\zeta}-\bar{z})^{\gamma}-c(\zeta, z)|\leqq C\{|\zeta-z|+|\rho(\zeta)|\}.$$

Hence as in the calculation of Lemma 3.4 we obtain

$$\begin{aligned}&|d\{(\zeta-z)^{\beta}b^{\circ}(\zeta, z)c(\zeta, z)/F^{\circ}(\zeta, z)^{n+k}\tau_1(\bar{\zeta})\wedge\omega(\zeta)\}|\\ &\quad<C|\zeta-z|^{k+1}|\rho(\zeta)|^{N-2}\{|\zeta-z|+|\rho(\zeta)|\}/|F^{\circ}(\zeta, z)|^{n+k+1}\\ &\qquad+C|\zeta-z|^{k}\{(|\zeta-z|+|\rho(\zeta)|)|\rho(\zeta)|^{k-2}+|\rho(\zeta)|^{k-1}\}/|F^{\circ}(\zeta, z)|^{n+k}\\ &\quad<C|\zeta-z|/|F^{\circ}(\zeta, z)|^{n+1}.\end{aligned}$$

From this the lemma is proved by applying the argument of Lemma 3.4.

LEMMA 3.7. *Let $N-2\geqq k\geqq 1$, $|\beta|=k$, $|\gamma|\geqq 1$ and $a(\zeta, z)\in C^{k}$. Then*

$$\left|\text{grad}\int_{\partial\Omega}(\zeta-z)^{\beta}(\bar{\zeta}-\bar{z})^{\gamma}a(\zeta, z)/\Phi^{n+k}\sigma(\zeta)\right|<C\,K(z).$$

PROOF. This is easily shown with the aid of Lemma 3.6 analogously to Lemma 3.5.

REMARK. If $|\beta|>k$, the above bound can also be replaced by C.

3.5. LEMMA 3.8.

$$\int_{\partial\Omega}(\zeta-z)^{\beta}D_z^{\alpha}C(\zeta, z)=\begin{cases}0 & \alpha\neq\beta\\ \alpha! & \alpha=\beta\end{cases}.$$

PROOF. By Henkin's representation (see 1.5) $(z-z')^{\beta}=\int_{\partial\Omega}(\zeta-z')^{\beta}C(\zeta, z)$ holds.

Applying D_z^{α} to both sides and taking $z'=z$ we obtain the desired result.

3.6. LEMMA 3.9. *Let Y be a C^2-holomorphic tangential vector field. Then $|Y_z\Phi(\zeta, z)|\leqq C|\zeta-z|$ for $\zeta\in\partial\Omega$.*

PROOF. This follows easily from the properties of $\Phi(\zeta, z)$.

3.7. The following proposition is the main result of this section.

PROPOSITION 5. *Let α, β be any multi-indices. Then*

1) *If $|\gamma|\leqq N-3$, $\int_{\partial\Omega}(\zeta-z)^{\alpha}(\bar{\zeta}-\bar{z})^{\beta}D_z^{\gamma}C(\zeta, z)$ is of class C^1.*

2) *For $|\gamma|=N-2$, $\int_{\partial\Omega}(\zeta-z)^{\alpha}(\bar{\zeta}-\bar{z})^{\beta}D_z^{\gamma}C(\zeta, z)$ is $\frac{1}{2}$ Hölder continuous.*

3) *For* $|\gamma|=N-3$ *and* C^2*-holomorphic tangential vector field* Y, $\int_{\partial\Omega}(\zeta-z)^{\alpha}(\bar{\zeta}-\bar{z})^{\beta}Y_z D_z^{\gamma}C(\zeta, z)$ *is Hölder continuous for any exponent smaller than* 1.

PROOF. It suffices to show that for any multi-indices α, β

i) for $|\gamma|\leqq N-2$, $|D^{\gamma}B(\alpha, \beta, z)|$ is bounded,

ii) for $|\gamma|=N-1$, $|D^{\gamma}B(\alpha, \beta, z)|<C|\rho(z)|^{-1/2}$,

iii) for $|\gamma|=N-3$, $|D_j Y D^{\gamma}B(\alpha, \beta, z)|<C K(z)$

where $B(\alpha, \beta, z)=\int_{\partial\Omega}(\zeta-z)^{\alpha}(\bar{\zeta}-\bar{z})^{\beta}C(\zeta, z)$.

If $\beta=0$, $B(\alpha, 0, z)$ is a constant by Lemma 3.8. From Lemma 3.2 for $|\gamma|=M\leqq N-2$,

$$D^{\gamma}B(\alpha, \beta, z)=\int_{\partial\Omega}(\zeta-z)^{\alpha}(\bar{\zeta}-\bar{z})^{\beta}X^{\gamma}C(\zeta, z)$$

$$+\sum_{k=0}^{M-1}\sum_{|\varepsilon|=k}\int_{\partial\Omega}(\zeta-z)^{\alpha+\varepsilon}a_{\varepsilon}(\zeta, z)/\Phi^{n+k}\sigma(\zeta).$$

By Lemma 3.4, 3.5, 3.6 and 3.7 we have already shown i).

To prove ii), let $|\gamma|=N-2$. If we write

$$X^{\gamma}C(\zeta, z)=\sum_{k=0}^{N-2}\sum_{|\varepsilon|=k}\sum_{j=1}^{n}(\zeta-z)^{\varepsilon}a_{\varepsilon,j}(\zeta, z)/\Phi^{n+k}\tau_j(\bar{\zeta})\wedge\omega(\zeta),$$

then Lemma 3.7 implies

$$\left|\text{grad}\int_{\partial\Omega}(\zeta-z)^{\alpha+\varepsilon}(\bar{\zeta}-\bar{z})^{\beta}a_{\varepsilon,j}(\zeta, z)/\Phi^{n+k}\tau_j(\bar{\zeta})\wedge\omega(\zeta)\right|<C K(z)$$

except for $k=0$. On the other hand, Lemma 3.5 implies

$$\left|\text{grad}\int_{\partial\Omega}(\zeta-z)^{\alpha+\varepsilon}a_{\varepsilon}(\zeta, z)/\Phi^{n+k}\sigma(\zeta)\right|<C K(z).$$

In case $k=0$, we apply D_j directly under the integral sign to

$$\int_{\partial\Omega}(\zeta-z)^{\alpha}(\bar{\zeta}-\bar{z})^{\beta}a_{0,j}(\zeta, z)/\Phi^{n}\tau_j(\bar{\zeta})\wedge\omega(\zeta).$$

Then the derivation of this term is bounded by $C|\rho(z)|^{-1/2}$ in view of Proposition 3.

REMARK. If $\alpha\neq 0$ or $|\beta|\geqq 2$, the above derivation is bounded by $C K(z)$.

To prove iii), we show $|YD_jD^{\gamma}B(\alpha, \beta, z)|<CK(z)$.

From the proof of ii) it remains to show

$$\left|Y\int_{\partial\Omega}(\zeta-z)^{\alpha}(\bar{\zeta}-\bar{z})^{\beta}a_{0,j}(\zeta, z)/\Phi^{n}\tau_j(\bar{\zeta})\wedge\omega(\zeta)\right|<C K(z).$$

But this follows easily from Lemma 3.9.

§ 4. Proofs of theorems.

4.1. LEMMA 4.1. *The following relations hold:*

1) $d_\zeta\{X^\alpha K(\zeta, z)\}=X^\alpha d_\zeta K(\zeta, z)=X^\alpha\{C(\zeta, z)-L(\zeta, z)\}$.

2) $\partial/\partial\zeta_j[f^{(k)}(\zeta, z)]=(D_j f)^{(k-1)}(\zeta, z)$,

$\partial/\partial\bar{\zeta}_j[f^{(k)}(\zeta, z)]=(\bar{D}_j f)^{(k-1)}(\zeta, z)$,

$X_j[f^{(k)}(\zeta, z)]=(D_j f)^{(k)}(\zeta, z)$

for $k\geqq 0$ *(we set* $f^{(-1)}(\zeta, z)=f(\zeta)$*).*

PROOF. 2) is proved only by simple calculations.

1) $$d_\zeta K(\zeta, z)=\int_0^1 d_\zeta K(\zeta, z, \lambda)=\int_0^1 -d_\lambda K(\zeta, z, \lambda)$$

$$=K(\zeta, z, 0)-K(\zeta, z, 1)=C(\zeta, z)-L(\zeta, z).$$

(Recall $d_{\zeta,\lambda}K(\zeta, z, \lambda)=0$.)

4.2. The following proposition is the main tool for the proofs of the theorems.

PROPOSITION 6. *Let* $|\alpha|=k\leqq N-2$ *and* f *be a* C^k $\bar{\partial}$*-closed* (0, 1) *form on* $\bar{\Omega}$. *Then*

$$(°)\quad D^\alpha T(f)=\sum_{|\beta|+|\gamma|\leq k} D^\beta f_\gamma(z)\int_{\partial\Omega}\frac{(\zeta-z)^\beta(\bar{\zeta}-\bar{z})^\gamma}{\beta!\quad\gamma!}D_z^\alpha C(\zeta, z)$$

$$+\sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}\int_{\partial\Omega}(D^{\alpha_1}f)^{(|\alpha_2|-1)}(\zeta, z)\wedge X^{\alpha_2}K(\zeta, z)$$

$$+\sum_{i=1}^n\sum_{|\beta|+|\gamma|<k}\int_{\partial\Omega}(D^\beta\bar{D}^\gamma f_i)^{(k-|\beta|-|\gamma|-1)}(\zeta, z)K_{\beta,\gamma}^{\alpha,i}(\zeta, z)\sigma(\zeta)$$

$$-\int_\Omega f^{(k-1)}(\zeta, z)\wedge D_z^\alpha L(\zeta, z)$$

$$=(\mathrm{I})_{\alpha,f}+(\mathrm{II})_{\alpha,f}+(\mathrm{III})_{\alpha,f}+(\mathrm{IV})_{\alpha,f}$$

where $K_{\beta,\gamma}^{\alpha,i}(\zeta, z)$ *are written in the form*

$$\sum_{|\varepsilon|+|\varepsilon'|<k-|\beta|-|\gamma|}a_{\varepsilon,\varepsilon'}(\zeta, z)(\partial/\partial\bar{\zeta})^\varepsilon X^{\varepsilon'}\{L_i(\zeta, z)-C_i(\zeta, z)\}\quad(a_{\varepsilon,\varepsilon'}\in C^{N-k+|\varepsilon|+|\varepsilon'|}).$$

PROOF. We shall prove (°) by induction on α. If $\alpha=0$ there is nothing to be proved. Suppose (°) is true for α. We first replace $f(\zeta)$ by $g_{z'}(\zeta)=f^{(k)}(\zeta, z')$ in (°) ($\bar{\partial}g_{z'}(\zeta)=0$ by Lemma 2.3) and then set $z'=z$.

Noting

1) $g_{z'}^{(l)}(\zeta, z)|_{z'=z}=f^{(k)}(\zeta, z)$ for $l \leqq k$,

2) $D^{\nu_1}\bar{D}^{\nu_2}f^{(k)}(\zeta, z)|_{\zeta=z}=0$ for $|\nu_1|+|\nu_2| \leqq k$,

3) $$D^\alpha T(g_{z'})=D^\alpha T(f)-\sum_{|\beta|+|\gamma|\leq k+1} D^\beta f_\gamma(z) D^\alpha\left\{\frac{(z-z')^\beta}{\beta!}\frac{(\bar{z}-\bar{z}')^\gamma}{\gamma!}\right.$$
$$\left.-\int_{\partial\Omega}\frac{(\zeta-z')^\beta}{\beta!}\frac{(\bar{\zeta}-\bar{z}')^\gamma}{\gamma!}C(\zeta, z)\right\},$$

we get

$$(*)\qquad D^\alpha T(f)=\sum_{|\beta|+|\gamma|\leq k+1} D^\beta f_\gamma(z)\int_{\partial\Omega}\frac{(\zeta-z)^\beta}{\beta!}\frac{(\bar{\zeta}-\bar{z})^\gamma}{\gamma!}D_z^\alpha C(\zeta, z)$$
$$+\sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}\int_{\partial\Omega}(D^{\alpha_1}f)^{(|\alpha_2|)}(\zeta, z)\wedge X^{\alpha_2}K(\zeta, z)$$
$$+\sum_{i=1}^{n}\sum_{|\beta|+|\gamma|<k}\int_{\partial\Omega}(D^\beta\bar{D}^\gamma f_i)^{(k-|\beta|-|\gamma|)}(\zeta, z)K_{\beta,\gamma}^{\alpha,i}(\zeta, z)\sigma(\zeta)$$
$$-\int_\Omega f^{(k)}(\zeta, z)\wedge D_z^\alpha L(\zeta, z)$$
$$=(\mathrm{i})_{\alpha,f}+(\mathrm{ii})_{\alpha,f}+(\mathrm{iii})_{\alpha,f}+(\mathrm{iv})_{\alpha,f}.$$

Next we compute $D_jD^\alpha T(f)$ regarding it as

$$\{D_jD^\alpha T(f)-D^\alpha T(D_jf)\}+D^\alpha T(D_jf).$$

If we apply $(*)$ to the inside of the bracket and $(^\circ)$ to the last term, we get

$$D_j(\mathrm{i})_{\alpha,f}-(\mathrm{i})_{\alpha,D_jf}+(\mathrm{I})_{\alpha,D_jf}$$
$$=D_j\sum_{|\beta|+|\gamma|\leq k+1} D^\beta f_\gamma(z)\int\frac{(\zeta-z)^\beta}{\beta!}\frac{(\bar{\zeta}-\bar{z})^\gamma}{\gamma!}D_z^\alpha C(\zeta, z)$$
$$-\sum_{|\beta|+|\gamma|\leq k+1} D^\beta(D_jf)_\gamma\int\frac{(\zeta-z)^\beta}{\beta!}\frac{(\bar{\zeta}-\bar{z})^\gamma}{\gamma!}D_z^\alpha C(\zeta, z)$$
$$+\sum_{|\beta|+|\gamma|\leq k} D^\beta(D_jf)_\gamma\int\frac{(\zeta-z)^\beta}{\beta!}\frac{(\bar{\zeta}-\bar{z})^\gamma}{\gamma!}D_z^\alpha C(\zeta, z)$$
$$=\sum_{|\beta|+|\gamma|\leq k+1} D^\beta f_\gamma(z)D_j\int\frac{(\zeta-z)^\beta}{\beta!}\frac{(\bar{\zeta}-\bar{z})^\gamma}{\gamma!}D_z^\alpha C(\zeta, z)$$
$$+\sum_{|\beta|+|\gamma|\leq k} D_jD^\beta f_\gamma(z)\int\frac{(\zeta-z)^\beta}{\beta!}\frac{(\bar{\zeta}-\bar{z})^\gamma}{\gamma!}D_z^\alpha C(\zeta, z)$$
$$=\sum_{|\beta|+|\gamma|\leq k+1} D^\beta f_\gamma(z)\int\frac{(\zeta-z)^\beta}{\beta!}\frac{(\bar{\zeta}-\bar{z})^\gamma}{\gamma!}D_z^{\tilde{\alpha}}C(\zeta, z).\quad (\tilde{D}^\alpha=D_jD^\alpha.)$$

$$D_j(\mathrm{ii})_{\alpha,f}-(\mathrm{ii})_{\alpha,D_jf}+(\mathrm{II})_{\alpha,D_jf}$$

$$=\sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}\int[(X_j-\partial/\partial\zeta_j)\{(D^{\alpha_1}f)^{(|\alpha_2|)}\wedge X^{\alpha_2}K(\zeta, z)\}$$

$$-(D_jD^{\alpha_1}f)^{(|\alpha_2|)}\wedge X^{\alpha_2}K(\zeta, z)+(D_jD^{\alpha_1}f)^{(|\alpha_2|-1)}\wedge X^{\alpha_2}K(\zeta, z)]$$

$$=\sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}\int[(D^{\alpha_1}f)^{(|\alpha_2|)}\wedge X_jX^{\alpha_2}K+(D_jD^{\alpha_1}f)^{(|\alpha_2|-1)}\wedge X^{\alpha_2}K$$

$$+(D^{\alpha_1}f)^{(|\alpha_2|)}\wedge(\partial/\partial\zeta_j \lrcorner X^{\alpha_2}\{C(\zeta, z)-L(\zeta, z)\})]$$

$$=\sum_{\alpha_1+\alpha_2=\tilde{\alpha}}\binom{\tilde{\alpha}}{\alpha_1}\int(D^{\alpha_1}f)^{(|\alpha_2|-1)}\wedge X^{\alpha_2}K(\zeta, z)$$

$$+(-1)^n\sum_{i=1}^{n}\sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}\int(D^{\alpha_1}f_i)^{(|\alpha_2|)}X^{\alpha_2}\{C_i(\zeta, z)-L_i(\zeta, z)\}\omega(\bar{\zeta})\wedge\tau_j(\zeta)$$

where we used Example 2 after Lemma 2.2. The terms in the last sum are to go into $(\mathrm{III})_{\tilde{\alpha},f}$; in fact we have

$$(-1)^n\int(D^{\alpha_1}f_i)^{(|\alpha_2|)}X^{\alpha_2}\{C_i(\zeta, z)-L_i(\zeta, z)\}\omega(\bar{\zeta})\wedge\tau_j(\zeta)$$

$$=\int(D^{\alpha_1}f_i)^{(|\alpha_2|)}\partial\rho/\partial\zeta_j|\partial\rho|^{-2}X^{\alpha_2}\{C_i(\zeta, z)-L_i(\zeta, z)\}\sigma(\zeta),$$

and $a_{\tilde{\alpha},i}=\partial\rho/\partial\zeta_j|\partial\rho|^{-2}$ actually belongs to $C^{N-(k+1)+|\alpha_2|}$. Now we compute $D_j(\mathrm{iii})_{\alpha,f}-(\mathrm{iii})_{\alpha,D_jf}+(\mathrm{III})_{\alpha,D_jf}$ termwisely.

$$(X_j-\partial/\partial\zeta_j)\{g^{(l)}K^{\alpha,i}_{\beta,\gamma}\sigma(\zeta)\}-(D_jg)^{(l)}K^{\alpha,i}_{\beta,\gamma}\sigma(\zeta)+g^{(l-1)}K^{\alpha,i}_{\beta,\gamma}\sigma(\zeta)$$

$$=(D_jg)^{(l-1)}K^{\alpha,i}_{\beta,\gamma}\sigma(\zeta)+g^{(l)}X_jK^{\alpha,i}_{\beta,\gamma}\sigma(\zeta)-[S_j\{g^{(l)}K^{\alpha,i}_{\beta,\gamma}\}]\sigma(\zeta),$$

where $g=D^{\beta}\bar{D}^{\gamma}f_i$ and $l=k-|\beta|-|\gamma|$. One can see that the hypothesis for $K^{\tilde{\alpha},i}_{\beta,\gamma}$ are satisfied for these terms.

$$D_j(\mathrm{iv})_{\alpha,f}-(\mathrm{iv})_{\alpha,D_jf}+(\mathrm{IV})_{\alpha,D_jf}$$

$$=-\int_{\Omega}(X_j-\partial/\partial\zeta_j)[f^{(k)}(\zeta, z)\wedge D^{\alpha}_zL(\zeta, z)]$$

$$+\int_{\Omega}(D_jf)^{(k)}(\zeta, z)\wedge D^{\alpha}_zL(\zeta, z)-\int(D_jf)^{(k-1)}(\zeta, z)\wedge D^{\alpha}_zL(\zeta, z)$$

$$=-\int f^{(k)}(\zeta, z)\wedge\{-\partial/\partial\zeta_jD^{\alpha}_zL(\zeta, z)\}=-\int f^{(k)}(\zeta, z)\wedge D^{\tilde{\alpha}}_zL(\zeta, z).$$

Thus the proof of the proposition is complete.

4.3. Proof of Theorem 1.

Let f and k be as in Theorem 1. We prove that, for each term in the decomposition (°) of $D^{\alpha}T(f)$, the 1/2 Hölder norm is estimated by the right side of the desired inequality. We have already established the estimates for $(\mathrm{I})_{\alpha,f}$ in section 3. It is well-known that $(\mathrm{IV})_{\alpha,f}$ is Hölder continuous for any exponent smaller than 1. Proposition 6 assures that each term of $(\mathrm{III})_{\alpha,f}$ is once more differentiable in ζ. So we compute its gradient.

$$D_j\int g^{(l)}(\zeta, z)\tilde{K}(\zeta, z)\sigma(\zeta)$$

$$=\int X_j\{g^{(l)}(\zeta, z)\tilde{K}(\zeta, z)\sigma(\zeta)\}-S_j[g^{(l)}(\zeta, z)\tilde{K}(\zeta, z)]\sigma(\zeta),$$

where $l=k-|\beta|-|\gamma|-1$, $g^{(l)}(\zeta, z)=(D^{\beta}\bar{D}^{\gamma}f_i)^{(k-|\beta|-|\gamma|-1)}$ and $\tilde{K}(\zeta, z)=K_{\beta,\gamma}^{\alpha,i}(\zeta, z)$.

By Proposition 6, $|W\tilde{K}|<C(|\zeta-z|^{1-l-2n}+|\zeta-z|^{l}|\Phi|^{-n-l})$ where $W=X_j$ or S_j. Hence

$$\left|\mathrm{grad}\int(D^{\beta}\bar{D}^{\gamma}f_i)^{(k-|\beta|-|\gamma|-1)}K_{\beta,\gamma}^{\alpha,i}\sigma(\zeta)\right|<C\,K(z).$$

Next we observe each term of $(\mathrm{II})_{\alpha,f}$. If α_1 is neither 0 nor α, by Lemma 1.2 $(D^{\alpha_1}f)^{(|\alpha_2|-1)}\wedge X^{\alpha_2}K(\zeta, z)$ is once more differentiable in ζ. Thus by the previous method they are shown to be Hölder continuous for any exponent smaller than 1.

Hence the essential parts for the exact 1/2 Hölder estimate are

(1) $\int f^{(k-1)}(\zeta, z)\wedge X^{\alpha}K(\zeta, z)$ and

(2) $\int D^{\alpha}f(\zeta)\wedge K(\zeta, z)$.

We apply $\partial/\partial z_j$ to (1) and (2) directly under the integral sign. Then Proposition 3 implies that their gradients are dominated by $|\rho(z)|^{-1/2}$. The proof is complete.

4.4. Proof of Theorem 2.

Let k, α, β, f and Y be as in Theorem 2. If we write $Y=\sum a_i(z)\partial/\partial z_i+\sum b_i(z)\partial/\partial\bar{z}_i$, then

$$YD^{\alpha}T(f)=Y_1D^{\alpha}T(f)+\sum b_i(z)D^{\alpha}f_i(z),$$

where $Y_1=\sum a_i(z)\,\partial/\partial z_i$. Hence we can consider $Y=Y_1$. By the argument in the proof of Theorem 1 and Proposition 5 it remains to show that

(I) $\sum_{i=1}^{n} a_i(z)\int D_iD^{\alpha}f(\zeta)\wedge K(\zeta, z)$ and

(II) $\sum_{i=1}^{n} a_i(z)\int f^{(k-1)}(\zeta, z)\wedge X_i X^\alpha K(\zeta, z)$

are both β-Hölder continuous.

(I) Set $g(\zeta)=D^\alpha f(\zeta)$. It suffices to show

$$\left|\sum a_i(z)\int D_i g(\zeta)\wedge\partial/\partial z_j K(\zeta, z)\right| < C\,K(z).$$

Then

$$D_i g(\zeta)\wedge\partial/\partial z_j K(\zeta, z)=D_i g(\zeta)\wedge X_j K(\zeta, z)-D_i g(\zeta)\wedge D_j K(\zeta, z).$$

In the right hand side the first term is bounded by $C\,K(z)$. Now

$$D_i g\wedge D_j K-D_j g\wedge D_i K=D_j\{D_i g\wedge K\}-D_i\{D_j g\wedge K\}.$$

So

$$\int D_i g(\zeta)\wedge D_j K(\zeta, z)-D_j g(\zeta)\wedge D_i K(\zeta, z)$$

$$=\int D_i g\wedge[\partial/\partial\zeta_j\lrcorner\{C(\zeta, z)-L(\zeta, z)\}]-D_j g\wedge[\partial/\partial\zeta_i\lrcorner\{C(\zeta, z)-L(\zeta, z)\}].$$

Hence the difference is dominated by $K(z)$.

REMARK. In the above calculus we need approximate $g(\zeta)$ by smooth form since it is only continuously differentiable. In Kerzman [4] he proved that for $\bar\partial$-closed (in distribution sense) C^1-(0, 1) form f on Ω, there exist C^∞ $\bar\partial$-closed forms f_ε which converge to f in C^1 topology.

Thus to estimate $\sum a_i(z)\int D_i g(\zeta)\wedge D_j K(\zeta, z)$, it suffices to estimate

$$\sum a_i(z)\int D_j g(\zeta)\wedge D_i K(\zeta, z)$$

$$=\sum a_i(z)\int D_j g(\zeta)\wedge X_i K(\zeta, z)-D_j g(\zeta)\wedge Y_z K(\zeta, z).$$

But these terms are bounded by $K(z)$ from Lemma 2.6 and 3.9.

(II) $$f^{(k-1)}(\zeta, z)\wedge X_i X^\alpha K(\zeta, z)=[f^{(k-1)}(\zeta, z)\wedge\partial/\partial z_i X^\alpha K(\zeta, z)$$
$$+\partial/\partial\zeta_i\{f^{(k-1)}(\zeta, z)\wedge X^\alpha K(\zeta, z)\}-(D_i f)^{(k-2)}(\zeta, z)\wedge X^\alpha K(\zeta, z)]$$
$$=[f^{(k-1)}(\zeta, z)\wedge\partial/\partial z_i X^\alpha K(\zeta, z)+f^{(k-1)}(\zeta, z)\wedge\{\partial/\partial\zeta_i\lrcorner X^\alpha C(\zeta, z)\}$$
$$-(D_i f)^{(k-2)}(\zeta, z)\wedge X^\alpha K(\zeta, z)].$$

The last two terms in the integrand are differentiable in ζ, so by the preceding method we can show that the last two terms are β-Hölder continuous. Hence in order to prove (II) is β-Hölder continuous, we have only to show

$$(\sharp)\qquad \left|\sum a_i(z)D_j\int f^{(k-1)}(\zeta, z)\wedge\partial/\partial z_i X^\alpha K(\zeta, z)\right| < C\, K(z)\,.$$

But

$$D_j\int f^{(k-1)}(\zeta, z)\wedge\partial/\partial z_i X^\alpha K(\zeta, z)$$

$$=\int[X_j\{f^{(k-1)}(\zeta, z)\wedge\partial/\partial z_i X^\alpha K(\zeta, z)\} - f^{(k-1)}(\zeta, z)\wedge\partial/\partial z_i X^\alpha\{\partial/\partial\zeta_j \lrcorner C(\zeta, z)\}]\,.$$

Hence

$$\sum a_i(z)D_j\int f^{(k-1)}(\zeta, z)\wedge\partial/\partial z_i X^\alpha K(\zeta, z)$$

$$=\int(D_jf)^{(k-1)}(\zeta, z)\wedge Y_z X^\alpha K(\zeta, z) + f^{(k-1)}(\zeta, z)\wedge Y_z X_j X^\alpha K(\zeta, z)$$

$$- f^{(k-1)}(\zeta, z)\wedge Y_z X^\alpha\{\partial/\partial\zeta_j \lrcorner C(\zeta, z)\}\,.$$

Hence Lemma 3.9 implies ($\sharp$).

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