J. Math. Soc. Japan
Vol. 32, No. 1, 1980

Weakly closed dihedral 2-subgroups in finite groups

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(Received June 13, 1978)
(Revised Sept. 27, 1978)

1. Introduction.

Let S be a Sylow 2-subgroup of a finite group G. A subgroup T of S is said to be weakly closed in S with respect to G if T has the following property; "Whenever $T\subseteqq S$, $g\in G$, and $T^g\subseteqq S$, then $T^g=T$." Let φ be the natural homomorphism from G onto $G/O(G)$. Then we set $Z^*(G)=\varphi^{-1}(Z(G/O(G)))$. The object of this paper is to prove the following results.

THEOREM I. *Let S be a Sylow 2-subgroup of a finite group G. Suppose a dihedral subgroup T of S is weakly closed in S (with respect to G). Then one of the following holds;*

(i) $T\not\subseteq\langle Z(T)^G\rangle$,

(ii) $Z(T)\subseteqq Z^*(G)$,

(iii) $|T|=4$, *and Sylow 2-subgroups of $\langle T^G\rangle$ are T or dihedral of order 8 or $\langle T^G\rangle/O(\langle T^G\rangle)\cong U_3(4)$,*

(iv) $|T|=8$, *and Sylow 2-subgroups of $\langle T^G\rangle$ are dihedral or semi-dihedral,*

(v) $|T|\geqq 16$, *and Sylow 2-subgroups of $\langle Z(T)^G\rangle$ are dihedral or semi-dihedral or are wreath products.*

THEOREM II. *Let S be a Sylow 2-subgroup of a finite group G. Suppose a generalized quaternion subgroup Q of S is weakly closed in S. Let $\langle z\rangle=Z(Q)$. Then one of the following holds;*

(i) $Q\not\subseteq\langle z^G\rangle$,

(ii) $z\in Z^*(G)$,

(iii) $\langle z^G\rangle/O(\langle z^G\rangle)$ *is* M_{11}, M_{12}, $\hat{M}_{12}$, $L_3(q)$, $U_3(q)$, $G_2(q)$, *or* ${}^3D_4(q)$, *q odd.*

THEOREM III. *Let S be a Sylow 2-subgroup of a finite group G. Suppose a semi-dihedral subgroup D of S is weakly closed in S. Then $Z(D)\subseteqq Z^*(G)$ or Sylow 2-subgroups of $\langle Z(D)^G\rangle$ are dihedral or semi-dihedral.*

2. Preliminaries.

LEMMA 2.1. *Suppose a subgroup A of a Sylow p-subgroup P of G is conjugate to a normal subgroup B of P in G. Then there exists an element $g\in G$ such that $A^g=B$ and $N_P(A)^g\subseteqq P$.*

PROOF. See Lemma 2.1 in [**3**].

LEMMA 2.2. *Let S be a Sylow 2-subgroup of a finite group G. Suppose there exist an involution z and a subgroup S_0 of S such that $\langle x\rangle \lhd S$, $x^2=z$, $|S:S_0|\leqq 2$, and $\{z^G\}\cap S_0=\{z\}$. Then $z\in Z^*(G)$ or Sylow 2-subgroups of $\langle z^G\rangle$ are dihedral or semi-dihedral.*

PROOF. See [**3**].

LEMMA 2.3. *Let S be a Sylow 2-subgroup of a finite group G. Suppose a cyclic subgroup X of S is weakly closed in S. Then $\Omega_1(X)\subseteqq Z^*(G)$ or Sylow 2-subgroups of $\langle \Omega_1(X)^G\rangle$ are dihedral or semi-dihedral.*

PROOF. See [**3**].

LEMMA 2.4. *Let V be the transfer of G to H/K and let a be an involution in H. Then $V(a)\equiv\prod_g a^g$, modulo K where g ranges over a set of coset representatives for the cosets gH of H in G fixed by a.*

PROOF. See Lemma 14.4.1 in [**9**].

LEMMA 2.5. *Let T be a 2-group acting on G with $T\cap G=1$ and let S be a Sylow 2-subgroup of TG containing T. If T is weakly closed in S with respect to TG, then $[T, G]\subseteqq O(G)$.*

PROOF. See Lemma 4.2 in [**1**].

LEMMA 2.6. *Let G be a finite simple group and let S be a Sylow 2-subgroup of G. Let z be an involution in S and let K be a normal subgroup of $C_G(z)$ such that a Sylow 2-subgroup of K is a generalized quaternion subgroup $Q\subseteqq S$. Assume Q is weakly closed in S with respect to G. Then G is M_{11}, M_{12}, $L_3(q)$, $U_3(q)$, $G_2(q)$, or ${}^3D_4(q)$, q odd.*

PROOF. See Theorem 6 in [**2**].

LEMMA 2.7. *Let T be a subgroup of a Sylow 2-subgroup S of G. Assume T is a four-group and T is weakly closed in S. Then one of the following holds;*

(i) *T is a Sylow 2-subgroup of $\langle T^G\rangle$,*

(ii) *$\langle T^G\rangle/O(\langle T^G\rangle)$ is $U_3(4)$,*

(iii) *Sylow 2-subgroups of $\langle T^G\rangle$ are dihedral of order 8.*

PROOF. See Theorem 3.2 in [**8**].

3. Proof of Theorem I.

Let G be a mininimal counterexample to Theorem I. By Lemma 2.7, we may assume $|T|\geqq 8$.

LEMMA 3.1. *G is simple.*

PROOF. We may assume $O(G)=1$. Let $\langle z\rangle=Z(T)$. Suppose $z\notin E(G)$. Then $[T, E(G)]=1$ by Lemma 2.5. Hence $z\in C_G(F^*(G))\subseteqq O_2(G)$. Since G is a counterexample, $T\subseteqq\langle z^G\rangle$, so $T\subseteqq O_2(G)$. Then $T\lhd G$ since T is weakly closed in S. Hence $z\in Z(G)$, this is a contradiction. Hence $z\in E(G)$. Since $\langle z^G\rangle\subseteqq E(G)$,

$T\subseteqq E(G)$. By a Frattini argument, $G=N_G(S\cap E(G))E(G)$. Since $N_G(S\cap E(G))\subseteqq N_G(T)\subseteqq C_G(z)$, $\langle z^G\rangle=\langle z^{E(G)}\rangle$. Hence we may assume $G=E(G)$. We set $G=E_1E_2\cdots E_n$, where E_i is a component of G for $i=1,\cdots,n$. Assume $z\notin E_i$ for $i=1,\cdots,n$, then $[T, E_i]=1$ by Lemma 2.5, hence $[T, E(G)]=1$. Then $T\subseteqq O_2(G)$, this implies $z\in Z(G)$, a contradiction. Hence we may assume $z\in E_1$. Furthermore we may assume $G=E_1$ since $\langle z^G\rangle\subseteqq E_1$, $T\subseteqq E_1$, and $\langle z^G\rangle=\langle z^{E_1}\rangle$. Suppose $Z(G)\neq 1$. We set $\bar{G}=G/Z(G)$, then $\bar{G}$ is simple. By induction, $\bar{G}$ is M_{11}, $L_3(q)$, $U_3(q)$, $L_2(q)$, q odd, or A_7. If $\bar{G}$ is M_{11}, $L_3(q)$, $U_3(q)$, q odd, then $Z(G)=1$ since Schur multiplier of $\bar{G}$ is trivial, a contradiction. If $\bar{G}$ is A_7 or $L_2(q)$, q odd, then G is $\hat{A}_7$ or $SL(2, q)$. Hence $z\in Z(G)$, a contradiction. Hence $Z(G)=1$, which proves Lemma 3.1.

Next we consider the case of $|T|\geqq 16$. We set $T=\langle t, x\mid x^t=x^{-1}, |x|=2^n, |t|=2\rangle$, $\Omega_3(\langle x\rangle)=\langle y_2\rangle$, $y_2{}^2=y_1$, $y_1{}^2=z$, and $\bar{C}=C_G(z)/\langle z\rangle$.

LEMMA 3.2. $\{\bar{y}_1{}^{\bar{C}}\}\cap C_{\bar{S}}(\bar{y}_2)=\{\bar{y}_1\}$.

PROOF. Let $\bar{k}\in\{\bar{y}_1{}^{\bar{C}}\}\cap C_{\bar{S}}(\bar{y}_2)$ and let k be an element in an inverse image of $\bar{k}$ such that $k^2=z$. Since $\bar{k}\in C_{\bar{S}}(\bar{y}_2)$, $x^k=x$ or xz. By Lemma 2.1, there exists an element $h\in C_G(z)$ such that $k^h=y_1$ and $N_S(\langle k\rangle)^h\subseteqq S$. Since $[k, x]\subseteqq\langle z\rangle\subseteqq\langle k\rangle$, $x\in N_S(\langle k\rangle)$, hence $x^h\in S$. Hence x^h acts on $\langle x\rangle$. Consider the automorphisms of $\langle x\rangle$, then we have $[y_1{}^h, x]=1$. We set $y_1{}^h=b$. Since $b\in S$, b acts on T. Let $t^b=tx^i$, then $t=t^{b^2}=tx^{2i}$, hence $x^i\in\langle z\rangle$. Therefore $T\subseteqq N_S(\langle b\rangle)$ since $[b, T]\subseteqq\langle z\rangle\subseteqq\langle b\rangle$. On the other hand, there exists an element $g\in G$ such that $\langle b\rangle^g=\langle y_1\rangle$ and $N_S(\langle b\rangle)^g\subseteqq S$ by Lemma 2.1. Hence $T^g\subseteqq S$. Then $T^g=T$ since T is weakly closed in S, this implies $\langle b\rangle=\langle y_1\rangle$. Since $y_1{}^h=b$, $h\in N_G(\langle y_1\rangle)$. This yields that $\langle k\rangle=\langle y_1\rangle$. Hence we have $\bar{k}=\bar{y}_1$, which proves Lemma 3.2.

By Lemma 2.2, $\bar{y}_1\in Z^*(\bar{C})$ or Sylow 2-subgroups of $\langle\bar{y}_1{}^{\bar{C}}\rangle$ are dihedral or semi-dihedral. We consider the case of $\bar{y}_1\in Z^*(\bar{C})$. Then $\langle y_1\rangle O(C)\lhd C$.

LEMMA 3.3. $TO(C)\lhd C$.

PROOF. We set $L=N_G(\langle y_1\rangle)$, and $\bar{L}=L/\langle y_1\rangle$. As in the proof of Lemma 3.2, we can prove that $\{\bar{y}_2{}^{\bar{L}}\}\cap\bar{S}=\{\bar{y}_2\}$. By Z^*-theorem, $\langle y_2\rangle O(L)\lhd L$. Since $\langle y_1\rangle O(C)\lhd C$, $C=O(C)N_G(\langle y_1\rangle)$ by a Frattini argument. Similarly we have $L=O(L)N_G(\langle y_2\rangle)$. Hence $\langle y_2\rangle O(C)\lhd C$. If we repeat this method, we have $TO(C)\lhd C$, this proves Lemma 3.3.

LEMMA 3.4. *Let D be a non-abelian subgroup of T. Whenever $D^g\subseteqq S$, $g\in G$, then $D^g\subseteqq T$.*

PROOF. Suppose false. So there exists a non-abelian subgroup D of T such that D is conjugate to a subgroup D_1 of S not contained in T. Choose D maximal with respect to inclusion, subject to the condition that D does not satisfy Lemma 3.4. Then there exist a subgroup H of S and an element $k\in N_G(H)$ such that $D^k=D_1$ and $D\subseteqq H$. By the choice of D, $D=T\cap H$, hence D and D_1 are normal in H. If $D\cap D_1\neq 1$, then $Z(D)=Z(D_1)$, hence $k\in C_G(z)$.

Since $TO(C_G(z)) \lhd C_G(z)$, $D_1 = D^k \subseteqq TO(C_G(z)) \cap S = T$, this is a contradiction. Therefore $D_1 \cap D = 1$, this implies $[D, D_1] = D \cap D_1 = 1$. We set $\langle z_1 \rangle = Z(D_1)$. Since $[D_1', x] = 1$, $[z_1, x] = 1$. Since z_1 is conjugate to z and T is weakly closed in S, we have $z_1 = z$ by Lemma 2.1, this is a contradiction. Hence Lemma 3.4 is proved.

LEMMA 3.5. *There is a contradiction.*

PROOF. By Lemma 2.3, we may assume that $\langle x \rangle$ is not weakly closed in S. Therefore there exist a subgroup H of S and an element $g \in N_G(H)$ such that $x \in H$ and $\langle x \rangle \neq \langle x^g \rangle$. If $\langle x \rangle \cap \langle x^g \rangle \neq 1$, then $g \in C_G(z)$, hence $x^g \in TO(C_G(z)) \cap S = T$, this implies $\langle x \rangle = \langle x^g \rangle$, a contradiction. Then $[x, x^g] \subseteqq \langle x \rangle \cap \langle x^g \rangle = 1$. Let $y = x^g$. If $[y^{2^{n-1}}, T] = 1$, then $[z^g, T] = 1$. Since T is weakly closed in S, $z^g = z$ by Lemma 2.1, this is a contradiction. Hence $[y^{2^{n-1}}, T] \neq 1$. So we may assume that $t^y = tx^{-1}$, $x^y = x$. Then $[T, xy^2] = 1$. Now we consider the transfer of G to $S/C_S(x)$. Suppose $Sx_1, \cdots, Sx_m$ are the distinct left cosets of S in G. By Lemma 2.4, $V(t) \equiv \prod x_i t x_i^{-1}$, modulo $C_S(x)$ where Sx_i is fixed by t. Furthermore $V(t) \equiv \prod x_i t x_i^{-1}$, modulo $C_S(x)$, where Sx_i is fixed by t and xy^2, since $[xy^2, t] = 1$. Let $\Omega_2(\langle x \rangle) = \langle y_1 \rangle$. Since $Sx_i y_1 t = Sx_i t z y_1 = Sx_i y_1$, y_1 induces permutation on the set of left cosets of S which is fixed by $\langle t, xy^2 \rangle$. Then $V(t) \equiv \prod x_i t x_i^{-1} \cdot \prod x_j t x_j^{-1} (x_j y_1) t (x_j y_1)^{-1}$, modulo $C_S(x)$, where Sx_i is fixed by $\langle t, xy^2, y_1 \rangle$, and Sx_j is fixed by $\langle t, xy^2 \rangle$, and Sx_j is not fixed by y_1. Since t is an involution and $y_1 t y_1^{-1} = tz$, $x_j t x_j^{-1} (x_j y_1) t (x_j y_1)^{-1} = x_j t z x_j^{-1}$. Since $x_j (xy^2) x_j^{-1} \in S$ and $(x_j (xy^2) x_j^{-1})^{2^{n-1}} = x_j z x_j^{-1}$, $x_j z x_j^{-1} \equiv 1$, modulo $C_S(x)$ by cosidering the automorphisms of $\langle x \rangle$. Hence $V(t) \equiv \prod x_i t x_i^{-1}$, modulo $C_S(x)$, where Sx_i is fixed by $\langle t, xy^2, y_1 \rangle$. By Lemma 3.4, $\langle t, y_1 \rangle^{x_i^{-1}} \subseteqq S$ implies $\langle t y_1 \rangle^{x_i^{-1}} \subseteqq T$, hence $t^{x_i^{-1}} \equiv t$, modulo $C_S(x)$. Since the number of the cosets of S which is fixed by $\langle t, xy^2, y_1 \rangle$ is odd, $V(t) \equiv t$, modulo $C_S(x)$. Hence $V(t) \neq 1$. This contradicts Lemma 3.1.

Next we consider the case of $\bar{y}_1 \notin Z^*(\bar{C})$.

LEMMA 3.6. *There exists a normal subgroup L of $C_G(z)$ such that Sylow 2-subgroups of L are generalized quaternion. Moreover, $x^2 \in L$.*

PROOF. By Lemma 2.2, Sylow 2-subgroups of $\langle \bar{y}_1^{\bar{C}} \rangle$ are dihedral or semidihedral. Since $\bar{y}_1 \notin Z^*(\bar{C})$, there exists an element k such that $\langle k \rangle$ is conjugate to $\langle y_1 \rangle$ in $C_G(z)$ and $\langle k \rangle \neq \langle y_1 \rangle$. By Lemma 3.2, $x^k = x^{-1}$ or $x^{-1}z$. Hence $k^x = x^{-2}k$ or $x^{-2}zk$. Since $\bar{k} \in \langle \bar{y}_1^{\bar{C}} \rangle$, $\bar{x}^2 \in \langle \bar{y}_1^{\bar{C}} \rangle$. Let $\langle \bar{j} \rangle$ be a maximal cyclic subgroup of $\bar{S} \cap \langle \bar{y}_1^{\bar{C}} \rangle$, and let j be an element of inverse image of $\bar{j}$. Since $\langle \bar{x}^2 \rangle \lhd \bar{S} \cap \langle \bar{y}_1^{\bar{C}} \rangle$, $\bar{x}^2 \in \langle \bar{j} \rangle$. Hence $x^2 \in \langle j \rangle$. Since $(x^2)^k = x^{-2}$, $j^k = j^{-1}$ or $j^{-1}z$. Hence Sylow 2-subgroups of $\langle \bar{y}_1^{\bar{C}} \rangle$ are dihedral. Let $\bar{L} = \langle \bar{y}_1^{\bar{C}} \rangle$, and let L be an inverse image of $\bar{L}$. If Sylow 2-subgroups of $\bar{L}$ are four-group, then $\langle k, x^2 \rangle$ is a Sylow 2-subgroup of L, so Sylow 2-subgroups of L are quaternion. Thus we may assume that Sylow 2-subgroups of $\bar{L}$ are not four-group. Since Sylow 2-subgroups of $\bar{L}$ are dihedral, one of the following holds;

(i) $\bar{L}/O(\bar{L})$is a dihedral 2-group,

(ii) $PSL(2, q)\subseteqq\bar{L}/O(\bar{L})\subseteqq PGL(2, q)$, q odd,

(iii) $\bar{L}/O(\bar{L})\cong A_7$.

If (i) occurs, then $\bar{y}_1\in Z^*(\bar{C})$, since Sylow 2-subgroups of $\bar{L}$ are non-abelian dihedral 2-subgroups, this is a contradiction. If (ii) or (iii) occurs, then we have that Sylow 2-subgroups of L are generalized quaternion. This completes the proof of Lemma 3.6.

LEMMA 3.7. *A contradiction.*

PROOF. Let Q be a Sylow 2-subgroup of L which is contained in S. Then we shall prove that Q is weakly closed in S with respect to G. Suppose false. Then there exist a subgroup H of S and an element $g\in N_G(H)$ such that $Q\subseteqq H$ and $Q^g\neq Q$. If $Q\cap Q^g\neq 1$, then $g\in C_G(z)$, hence $Q^g\subseteqq S\cap L=Q$, a contradiction. Therefore $[Q, Q^g]\subseteqq Q\cap Q^g=1$. Let $Q_0=Q^g$, and let $Q=\langle a, b\,|\,a^b=a^{-1}, a^{2^{h-2}}=b^2, |b|=4\rangle$, $Q_0=\langle c, d\,|\,c^d=c^{-1}, c^{2^{h-2}}=d^2, |d|=4\rangle$. Let $\Omega_2(\langle a\rangle)=\langle v\rangle$, then $\langle v\rangle=\Omega_2(\langle x\rangle)$ since $x^2\in Q$ and $|x^2|\geqq 4$, in particular $v^2=z$. Let $\Omega_2(\langle c\rangle)=\langle w\rangle$ and $w^2=z_0$. Since $[Q_0', x]=1$, $[z_0, x]=1$. Let $t^d=tx^i$, then $t^{d^2}=tx^{2i}$ or $tx^{2i}z$ since $[d, v]=1$ and $\langle v\rangle=\Omega_2(\langle x\rangle)$. On the other hand $t^{d^2}=t^{z_0}=tz$. Hence $x^{2i}\in\langle z\rangle$, so $x^i\in\Omega_2(\langle x\rangle)=\langle v\rangle$. Therefore we may assume $t^d=tv$. Similarly we have $t^w=tv^{-1}$. Then $t^{dw}=(tv)^w=t$. Since $(dw)^2=z_0$, $[z_0, T]=1$. Since z_0 is conjugate to z and T is weakly closed in S, we have $z_0=z$ by Lemma 2.1, a contradiction. Hence Q is weakly closed in S with respect to G.

By Lemma 2.6, G is M_{12}, $G_2(q)$, or ${}^3D_4(q)$, q odd, since G is simple and counterexample to Theorem I. Then $C_G(z)$ has a subgroup K_1*K_2 of index 2 such that K_i is $SL(2, q_i)$, q_i odd, for $i=1, 2$, where we shall write K_1*K_2 for a central product of K_1 and K_2. Let $Q_i=S\cap K_i$. Then we may assume $x^2\in Q_1$.

Suppose first $|Q_1|=8$. If $x\in Q_1*Q_2$, then $|x|\leqq 4$, so $|T|\leqq 8$, which is a contradiction by the choice of T. Hence $x\notin Q_1*Q_2$. Let $\tilde{C}=C_G(z)/K_1$, then $\tilde{T}$ is weakly closed in $\tilde{S}$ with respect to $\tilde{C}$ and $|\tilde{T}|\leqq 4$. By Lemma 2.7, one of the following holds;

(i) $\tilde{T}$ is a Sylow 2-subgroup of $\langle\tilde{T}^{\tilde{C}}\rangle$,

(ii) Sylow 2-subgroups of $\langle\tilde{T}^{\tilde{C}}\rangle$ are dihedral of order 8,

(iii) $\langle\tilde{T}^{\tilde{C}}\rangle/O(\langle\tilde{T}^{\tilde{C}}\rangle)\cong U_3(4)$.

If (ii) or (iii) occurs, then $\tilde{x}\in\Phi(\tilde{S})$, which is a contradiction. Hence we may assume that (i) holds. Suppose $\langle\tilde{T}^{\tilde{C}}\rangle$ is solvable, then $[x, K_2]\subseteqq TK_1\cap K_2$, which contradicts to the structure of $C_G(z)$ since $TK_1\cap K_2$ is a 2-group. Hence $\langle\tilde{T}^{\tilde{C}}\rangle$ is non-solvable, then $\langle\tilde{T}^{\tilde{C}}\rangle\subseteqq\tilde{K}_2$, so $x\in Q_1*Q_2$, this contradicts to the choice of x. Next we suppose $|Q_1|\geqq 16$. Then K_i is non-solvable for $i=1, 2$. By a Frattini argument, $C_G(z)=K_1N_G(Q_1)$. Since $|Q_1|\geqq 16$ and $|x^2|\geqq 4$, $N_G(Q_1)\subseteqq N_G(\langle x^2\rangle)$, hence $C_G(z)=K_1N_G(\langle x^2\rangle)$. We now repeat the argument of Lemma 3.3 to conclude that $TO(N_G(\langle x^2\rangle)\lhd N_G(\langle x^2\rangle)$. Since $O(C)=1$, we have $\tilde{T}\lhd\tilde{C}$. Hence

$[\tilde{T}, \tilde{K}_2] \subseteqq \tilde{K}_2 \cap \tilde{T} = 1$, this contradicts to the structure of $C_G(z)$. Therefore Lemma 3.7 is proved.

Next we consider the case of $|T| = 8$.

LEMMA 3.8. *$TO(C_G(z)) \triangleleft C_G(z)$.*

PROOF. Let $\bar{C} = C_G(z)/\langle z \rangle$. Then $\bar{T}$ is weakly closed in $\bar{S}$ with respect to $\bar{C}$. By Lemma 2.7, one of the following holds;

(i) $\bar{T}O(\bar{C}) \triangleleft \bar{C}$,

(ii) elements of $\bar{T}^{\#}$ are conjugate in $\bar{C}$,

(iii) Sylow 2-subgroups of $\langle \bar{T}^{\bar{C}} \rangle$ are dihedral of order 8.

If (i) holds, then $TO(C_G(z)) \triangleleft C_G(z)$. If (ii) holds, then there exists an element $g \in C$ such that $\bar{t}^{\bar{g}} = \bar{x}$, this is a contradiction. Assume (iii) holds. Let $\bar{W} = \bar{S} \cap \bar{C}$, then $\bar{W}$ is dihedral of order 8. Let $\langle \bar{h} \rangle$ be a normal subgroup of $\bar{W}$ of order 4. Then $\bar{h}^2 = \bar{x}$, since $\bar{x} \in Z(\bar{W})$. Let h be an element of inverse image of $\bar{h}$ such that $h^2 = x$ and $\langle h \rangle \triangleleft S$. Let $u \in \{z^G\} \cap C_S(x)$. Then there exists an element $k \in G$ such that $u^k = z$ and $C_S(u)^k \subseteqq S$. Since $[x, u] = 1$, $x^k \in S$. Hence $[z^k, h] = 1$. Let $z_1 = z^k$, then there exists an element $r \in G$ such that $z_1{}^r = z$ and $C_S(z_1)^r \subseteqq S$. Since $[z_1, h] = 1$, $h^r \in S$. Since $|h| = 8$, $[z^r, T] = 1$. Since T is weakly closed in S, $z^r = z$ by Lemma 2.1. Hence $z_1 = z$, so $u = z$. Thus we have $\{z^G\} \cap C_S(x) = \{z\}$. By Lemma 2.2, we have a contradiction. Hence Lemma 3.8 is proved.

LEMMA 3.9. *$O^2(G) \neq G$.*

PROOF. If we repeat the argument of Lemma 3.5, we have that $O^2(G) \neq G$.

Since G is simple, we have a final contradiction. This completes the proof of Theorem I.

4. Proof of Theorem II.

Let G be a minimal counterexample to the Theorem II. Let $Z(Q) = \langle z \rangle$.

LEMMA 4.1. *G is simple.*

PROOF. Suppose false. If we repeat the argument of Lemma 3.1, we have G is quasisimple and $O(G) = 1$. Let $\bar{G} = G/Z(G)$. Then $\bar{G}$ is M_{11}, M_{12}, $L_3(q)$, $U_3(q)$, $G_2(q)$, or ${}^3D_4(q)$, q odd, by induction. If $\bar{G} \neq M_{12}$, then Schur multiplier of $\bar{G}$ is odd, this is a contradiction. Hence $\bar{G} = M_{12}$. Then $G = \hat{M}_{12}$. This contradicts to the choice of G.

Suppose first $|Q| \geqq 16$.

LEMMA 4.2. *There exists a normal subgroup L of $C_G(z)$ such that Sylow 2-subgroups of L are generalized quaternion.*

PROOF. If we repeat the argument of Lemma 3.2, 3.3, 3.6, we may prove the Lemma 4.2.

Let Q^* be a Sylow 2-subgroup of L which is contained in S. Then likewise in Lemma 3.7, we may prove that Q^* is weakly closed in S with respect to G.

By Lemma 2.6, we have a contradiction.

Next we consider the case of $|Q|=8$.

LEMMA 4.3. *There exists a normal subgroup K of $C_G(z)$ such that Q is a Sylow 2-subgroup of K.*

PROOF. Let $\bar{C}=C_G(z)/\langle z\rangle$. By Lemma 2.7, one of the following holds;

(i) $\bar{Q}$ is a Sylow 2-subgroup of $\langle\bar{Q}^{\bar{C}}\rangle$,

(ii) Sylow 2-subgroups of $\langle\bar{Q}^{\bar{C}}\rangle$ are dihedral of order 8,

(iii) $\langle\bar{Q}^{\bar{C}}\rangle/O(\langle\bar{Q}^{\bar{C}}\rangle)\cong U_3(4)$.

Suppose (i) holds. Let K be an inverse image of $\langle\bar{Q}^{\bar{C}}\rangle$, then $K\lhd C_G(z)$ and Q is a Sylow 2-subgroup of K. Assume (ii) holds, then likewise in Lemma 3.8, we have a contradiction. Hence (iii) holds. Then we have a contradiction since Schur multiplier of $U_3(4)$ is trivial. This proves Lemma 4.3.

Since Q is weakly closed in S, we have a contradiction by Lemma 2.6. This completes the proof of Theorem II.

5. Proof of Theorem III.

Let G be a minimal counterexample to the Theorem III. Let $D=\langle s, y \mid y^s=y^{-1+2^{m-1}}, |s|=2, |y|=2^m, m\geqq 3\rangle$ and let $Z(D)=\langle z\rangle$, $\Omega_2(\langle y\rangle)=\langle y_1\rangle$. Let $u\in\{z^G\}\cap C_S(y_1)$. Then by Lemma 2.1 there exists an element $g\in G$ such that $u^g=z$ and $C_S(u)^g\subseteqq S$. Since $[u, y_1]=1$, $y^u=y$ or yz, this implies $[u, y^2]=1$. Hence $(y^2)^g\in C_S(u)^g\subseteqq S$. Since $\langle y\rangle\lhd S$ and $(y^2)^g\in S$, we have $[z^g, y]=1$. Let $v=z^g$. Then by Lemma 2.1 there exists an element $k\in G$ such that $v^k=z$ and $C_S(v)^k\subseteqq S$. Since $[v, y]=1$, $y^k\in C_S(v)^k\subseteqq S$. As $y^k\in S$ and $D\lhd S$, y^k acts on D. Then $[z^k, D]=1$ by considering the automorphisms of D. Let $z^k=w$. Then by Lemma 2.1 there exists an element $h\in G$ such that $w^h=z$ and $C_S(w)^h\subseteqq S$. Since $[w, D]=1$, $D^h\subseteqq C_S(w)^h\subseteqq S$. Since D is weakly closed in S, $D^h=D$, hence we have $z^h=z$. Then $w=z$, so $v=z$, hence $u=z$. This implies $\{z^G\}\cap C_S(y_1)=\{z\}$. Since $|S:C_S(y_1)|\leqq 2$, we have a conclusion by Lemma 2.2. This completes the proof of the Theorem III.

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