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On (x)-complexes

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Let N be a finitely generated module over a noetherian local ring R with maximal ideal $\mathfrak{m}$. It is well known that a maximal R-sequence and a maximal N-sequence have connections with a minimal injective resolution of N. For example, the length of a maximal R-sequence, namely the depth of R, is equal to the length of a minimal injective resolution of N, namely the injective dimension of N, if it is finite, and the length of a maximal N-sequence, namely the depth of N, is equal to the minimal integer of i with $\mu^i(\mathfrak{m}, N)>0$ where $\mu^i(\mathfrak{m}, N)$ is the dimension of an $R/\mathfrak{m}$-vector space $\mathrm{Ext}_R^i(R/\mathfrak{m}, N)$. But we have thought there are more connections between them. In particular we are interested in studying possible connections between the terms of an R-sequence or of an N-sequence, and the terms of a minimal injective resolution of N.

First we shall introduce a complex associated to a minimal injective resolution of N for a sequence of elements in $\mathfrak{m}$ [see Definition 1]. This complex characterizes some N-sequence. We shall study properties of this complex. In particular we shall give, using a term of this complex, a necessary and sufficient condition for the following conjecture of Bass to hold: a noetherian local ring is Cohen-Macaulay if it possesses a finitely generated module of finite injective dimension. Moreover we shall show some property of a minimal injective resolution, applying this complex.

Throughout this note, R is a noetherian local ring with a unique maximal ideal $\mathfrak{m}$. The unlabeled Hom and Ext mean Hom_R and Ext_R, respectively.

We begin by introducing a definition. Let $x_0, x_1, \cdots, x_r$ be a sequence of elements in $\mathfrak{m}$. We denote this sequence by $(x_0, x_1, \cdots, x_r)$ or, for brevity, (x). The ideal generated by $x_0, x_1, \cdots, x_r$ is also denoted by $(x_0, x_1, \cdots, x_r)$ or (x). Let N be an R-module and $0\longrightarrow N\xrightarrow{d^{-1}}E^0\xrightarrow{d^0}E^1\xrightarrow{d^1}\cdots$ be a minimal injective resolution of N.

DEFINITION 1. Let $N_{(x)}^0=\{e\in E^0 \mid x_0e\in d^{-1}(N)\}$. For any integer i with $0<i\leqq r$, we define inductively $N_{(x)}^i$ as follows; $N_{(x)}^i=\{e\in(0:(x_0, x_1, \cdots, x_{i-1}))_{E^i} \mid x_ie\in d^{i-1}(N_{(x)}^{i-1})\}$. For $i>r$, $N_{(x)}^i=(0:(x_0, x_1, \cdots, x_r))_{E^i}$. Each $N_{(x)}^i$ is a submodule of E^i. Each d^i induces an R-homomorphism: $N_{(x)}^i\rightarrow N_{(x)}^{i+1}$, denoted again by d^i. In this case we have a complex of R-modules

$$0 \longrightarrow N \xrightarrow{d^{-1}} N^0_{(x)} \xrightarrow{d^0} N^1_{(x)} \longrightarrow \cdots.$$

This complex is unique for a sequence (x) and N up to isomorphism. This complex is also denoted by $0 \to N \to N^0 \to N^1 \to \cdots$ for simplicity, which is called (x)-complex under N.

As its obvious properties, we obtain the followings;

1) For each i N^i can be naturally identified with a submodule of $\mathrm{Hom}(R/(x_0, x_1, \cdots, x_{i-1}), E^i)$, since $(0:(x_0, x_1, \cdots, x_{i-1}))_{E^i}$ is naturally isomorphic to $\mathrm{Hom}(R/(x_0, x_1, \cdots, x_{i-1}), E^i)$. In this case, for each i with $i>r$, $N^i = \mathrm{Hom}(R/(x), E^i)$.

2) For each $i>r$, N^i is $R/(x)$-injective.

3) If $(x)=(x_0, x_1, \cdots, x_r)$ is an R-sequence, the sequence $N^{r+1} \to N^{r+2} \to \cdots$ is exact, because $\mathrm{Ext}^i(R/(x), N)=0$ for $i>r+1$ follow from proj. $\dim_R R/(x)=r+1$.

PROPOSITION 2. *If $(x)=(x_0, x_1, \cdots, x_r)$ is an R-sequence, then an (x)-complex under any R-module N is always acyclic.*

PROOF. Let $0 \longrightarrow N \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots$ be a minimal injective resolution of N. From an exact sequence $0 \to E^0/d^{-1}(N) \to E^1 \to E^2$, we obtain an exact sequence $0 \to \mathrm{Hom}(R/(x_0), E^0/d^{-1}(N)) \to \mathrm{Hom}(R/(x_0), E^1) \to \mathrm{Hom}(R/(x_0), E^2)$. Since $\mathrm{Hom}(R/(x_0), E^0/d^{-1}(N)) \cong N^0/d^{-1}(N) \cong d^0(N^0)$, $0 \to d^0(N^0) \to \mathrm{Hom}(R/(x_0), E^1) \to \mathrm{Hom}(R/(x_0), E^2)$ is exact. So $N^0 \to N^1 \to N^2$ is exact. Since $\mathrm{Ext}^i(R/(x_0), N)=0$ for $i>1$, $\mathrm{Hom}(R/(x_0), \ )$ is a functor from R-modules to $R/(x_0)$-modules which preserves minimal injective resolutions [see **1**, Lemma 2.1]. Hence we have a minimal injective resolution of the $R/(x_0)$-module $d^0(N^0)$: $0 \to d^0(N^0) \to \mathrm{Hom}(R/(x_0), E^1) \to \cdots \to \mathrm{Hom}(R/(x_0), E^i) \to \cdots$. Using the same argument as above, $N^1 \to N^2 \to N^3$ is exact, and so on.

COROLLARY 3. *Let $0 \to N \to E^0 \to E^1 \to \cdots \to E^r \to 0$ be a minimal injective resolution of a finitely generated R-module N where r is finite. Then, for any R-sequence $x_0, x_1, \cdots, x_s$, there is an element y_i of E^i $(i=1, 2, \cdots, s)$ such that $(x_0, x_1, \cdots, x_{i-1})y_i=0$ and $x_i y_i \neq 0$ for $i=1, 2, \cdots, s$.*

PROOF. Let $0 \to N \to N^0 \to \cdots \to N^r \to 0$ be the $(x_0, x_1, \cdots, x_i)$-complex under N. This complex is acyclic and $N^{i+k}=(0:(x_0, x_1, \cdots, x_i))_{E^{i+k}}$ for $1 \leqq k \leqq r-i$. Since depth $R/(x_0, x_1, \cdots, x_i) + \mathrm{Sup}\,\{j \mid \mathrm{Ext}^j(R/(x_0, x_1, \cdots, x_i), N) \neq 0\} = \mathrm{depth}\, R$ $(=r)$ [see **5**, Théorème (4.15)], we have $\mathrm{Sup}\,\{j \mid \mathrm{Ext}^j(R/x_0, x_1, \cdots, x_i), N) \neq 0\} = i+1$. Hence the sequence $\mathrm{Hom}(R/(x_0, x_1, \cdots, x_i), E^i) \to \mathrm{Hom}(R/(x_0, x_1, \cdots, x_i), E^{i+1}) \to \mathrm{Hom}(R/(x_0, x_1, \cdots, x_i), E^{i+2})$ is not exact and so $N^i - (0:(x_0, x_1, \cdots, x_i))_{E^i}$ is not empty. Choose any element y_i in $N^i - (0:(x_0, x_1, \cdots, x_i))_{E^i}$. Then we obtain $(x_0, x_1, \cdots, x_{i-1})y_i=0$ and $x_i y_i \neq 0$.

THEOREM 4. *Let $x_0, x_1, \cdots, x_r$ be an R-sequence. Then $x_0, x_1, \cdots, x_r$ is an N-sequence if and only if x_i is a nonzero divisor of N^i for $i=0, 1, \cdots, r$ and*

$N/(x_0, x_1, \cdots, x_r)N \neq 0$.

PROOF. First we shall show 'if' part. Since the sequence $0 \longrightarrow R \xrightarrow{x_0} R \longrightarrow R/(x_0) \longrightarrow 0$ is exact, we have $x_0E=E$ for any injective R-module E. Let $0 \longrightarrow N \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \longrightarrow \cdots$ be a minimal injective resolution of N. It is obvious that x_0 is a nonzero divisor of N and so is a nonzero divisor of E^0. Hence the multiplication map by $x_0 : E^0 \to E^0$ is an isomorphism. This isomorphism induces $E^0/d^{-1}(N) \cong E^0/x_0d^{-1}(N)$. So we obtain $d^0(N^0) \cong N^0/d^{-1}(N) \cong \mathrm{Hom}\,(R/(x_0), E^0/d^{-1}(N)) \cong \mathrm{Hom}\,(R/(x_0), E^0/x_0d^{-1}(N)) \cong \{e \in E^0 \mid x_0e \in x_0d^{-1}(N)\} \cong d^{-1}(N)/x_0d^{-1}(N) \cong N/x_0N$. This shows x_1 is a nonzero divisor on N/x_0N. Since $0 \to d^0(N^0) \to \mathrm{Hom}\,(R/(x_0), E^1) \to \mathrm{Hom}\,(R/(x_0), E^2) \to \cdots$ is a minimal injective resolution of the $R/(x_0)$-module $d^0(N^0)$ $(\cong N/x_0N)$, repeating the same reasoning we prove inductively that $x_0, x_1, \cdots, x_r$ is an N-sequence. Conversely assume that $x_0, x_1, \cdots, x_r$ is an N-sequence. Then it is trivial that x_0 is a nonzero divisor of N^0. Moreover we have $N/(x_0, x_1, \cdots, x_i)N \cong d^i(N^i)$, proceeding in the above fashion, and $d^i(N^i)$ is an essential submodule of N^{i+1} for $i=0, 1, \cdots, r$. So x_i is a nonzero divisor of N^i for $i=1, 2, \cdots, r$. This completes the proof.

COROLLARY 5. *Let $x_0, x_1, \cdots, x_r$ be elements of $\mathfrak{m}$. Then $x_0, x_1, \cdots, x_r$ is an R-sequence if and only if the $(x_0, x_1, \cdots, x_r)$-complex under $R: 0 \to R \to R^0 \to R^1 \to \cdots$ is acyclic and x_i is a nonzero divisor of R^i for $i=0, 1, \cdots, r$.*

THEOREM 6. *Let $(x)=(x_0, x_1, \cdots, x_r)$ be an R-sequence and N a nonzero finitely generated R-module. If $x_0, x_1, \cdots, x_s$ $(s \leqq r)$ is an N-sequence, each term N^i of the (x)-complex under N is finitely generated $(0 \leqq i \leqq s)$. Conversely assume each N^i is finitely generated $(0 \leqq i \leqq s < r)$. Then $x_0, x_1, \cdots, x_s$ is an N-sequence.*

PROOF. Assume $x_0, x_1, \cdots, x_s$ is an N-sequence $(s \leqq r)$. Then we have $N^0/d^{-1}(N) \cong d^0(N^0) \cong N/x_0N$. Since both N/x_0N and $d^{-1}(N)$ are finitely generated, so is N^0. In general we have $N^i/d^{i-1}(N^{i-1}) \cong d^i(N^i) \cong N/(x_0, x_1, \cdots, x_i)N$ $(0 \leqq i \leqq s)$. It is inductively proved that each N^i is finitely generated $(0 \leqq i \leqq s)$. Assume the converse. Since $(0 : x_0)_{E^0} \subseteqq N^0$, $(0 : x_0)_{E^0}$ is finitely generated. On the other hand $(0 : x_0)_{E^0}$ is injective as an $R/(x_0)$-module. So we have $(0 : x_0)_{E^0}=0$, because depth $R/(x_0) \neq 0$. Hence x_0 is a nonzero divisor of N and $d^0(N^0) \cong N/x_0N$. Suppose $x_0, x_1, \cdots, x_i$ is an N-sequence $(0 \leqq i < s)$. Since $(0 : (x_0, x_1, \cdots, x_{i+1}))_{E^{i+1}} \subseteqq N^{i+1}$, $(0 : (x_0, x_1, \cdots, x_{i+1}))_{E^{i+1}}$ is finitely generated. Hence it is a zero module and so x_{i+1} is a nonzero divisor of $(0 : (x_0, x_1, \cdots, x_i))_{E^{i+1}}$. Hence x_{i+1} is a nonzero divisor of N^{i+1} and so of $d^i(N^i)$. Since $d^i(N^i) \cong N/(x_0, x_1, \cdots, x_i)N$, $x_0, x_1, \cdots, x_{i+1}$ is an N-sequence. By induction the proof is completed.

COROLLARY 7. *Let $(x)=(x_0, x_1, \cdots, x_s)$ be an R-sequence $(s < \text{depth } R)$ and N a nonzero finitely generated R-module. Then $x_0, x_1, \cdots, x_s$ is an N-sequence if and only if $N=N^0$, $d^0(N^0) \cong N^1, \cdots, d^{s-1}(N^{s-1}) \cong N^s$ for the (x)-complex under N: $0 \longrightarrow N \xrightarrow{d^{-1}} N^0 \xrightarrow{d^0} N^1 \xrightarrow{d^1} \cdots$.*

PROOF. The 'if' part follows from the theorem. Suppose $x_0, x_1, \cdots, x_s$ is an N-sequence. Let $0 \longrightarrow N \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots$ be a minimal injective resolution of N. Since x_0 is a nonzero divisor of R and of E^0, the multiplication map by x_0 induces an automorphism of E^0. So the multiplication map by $x_0: N^0 \to d^{-1}(N)$ $(\cong N)$ is an isomorphism. In general, x_i is a nonzero divisor of $d^{i-1}(N^{i-1})$ $(\cong N/(x_0, x_1, \cdots, x_{i-1})N)$ and $\mathrm{Hom}(R/(x_0, x_1, \cdots, x_{i-1}), E^i)$ is an injective envelope of the $R/(x_0, x_1, \cdots, x_{i-1})$-module $d^{i-1}(N^{i-1})$ $(1 \leqq i \leqq s)$. Using the same argument as above, we have $N^i \cong d^{i-1}(N^{i-1})$ $(1 \leqq i \leqq s)$.

COROLLARY 8. *Let R be a Gorenstein local ring with a maximal ideal $\mathfrak{m}$. Then, for any maximal R-sequence (x), each term of the (x)-complex under R is finitely generated and its final nonzero term is isomorphic to the injective envelope of an $R/(x)$-module $R/\mathfrak{m}$.*

THEOREM 9. *Assume depth $R \leqq n+1$ for a non negative integer n. If there exists a sequence $(x)=(x_0, x_1, \cdots, x_n)$ of elements of $\mathfrak{m}$ such that the (x)-complex under R is acyclic with each term finitely generated, then* $\dim R \leqq n+1$.

PROOF. Let $0 \longrightarrow R \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots$ be a minimal injective resolution of R. Then $(0: x_0)_{E^0}$ is embedded in the first term R^0 of the (x)-complex under R and so is finitely generated. If $(0: x_0)_{E^0} \neq 0$, we have depth $R/(x_0)=0$ and hence $\dim R \leqq 1$ [see **4**, Corollaire 1.3]. Assume $(0: x_0)_{E^0}=0$. Then x_0 is a nonzero divisor of R. Since $(0:(x_0, x_1))_{E^1}$ is embedded in R^1, it is finitely generated. If $(0:(x_0, x_1))_{E^1} \neq 0$, we have depth $R/(x_0, x_1)=0$ and so $\dim R \leqq 2$. If $(0:(x_0, x_1))_{E^1}=0$, x_1 is a nonzero divisor of $(0: x_0)_{E^1}$. Since $d^0(R^0) \cong R/x_0R$ is embedded in $(0: x_0)_{E^1}$, x_1 is a nonzero divisor of R/x_0R. Proceeding in this fashion, if $(0:(x_0, x_1, \cdots, x_i))_{E^i} \neq 0$ $(0 \leqq i \leqq n)$, we have $\dim R \leqq i+1$ and if $(0:(x_0, x_1, \cdots, x_i))_{E^i}=0$ for all i with $0 \leqq i \leqq n$, $x_0, x_1, \cdots, x_n$ is an R-sequence. Hence we have depth $R=n+1$ and so $(0:(x_0, x_1, \cdots, x_n))_{E^{n+1}} \neq 0$. Since $(0:(x_0, x_1, \cdots, x_n))_{E^{n+1}}$ $(\cong R^{n+1})$ is finitely generated by the hypothesis, we obtain $\dim R \leqq n+1$.

THEOREM 10. *Suppose a noetherian local ring R possesses a finitely generated module N of finite injective dimension such that, for any maximal R-sequence $(x)=(x_0, x_1, \cdots, x_r)$, some term of the (x)-complex under N is nonzero and finitely generated. Then R is Cohen-Macaulay.*

PROOF. Assume N^i is finitely generated where N^i is some term of the (x)-complex under $N: 0 \to N \to N^0 \to N^1 \to \cdots \to N^{r+1} \to 0$. If $i=r+1$, our statement obviously holds. When $i=r$, N^{r+1} is also finitely generated. Hence we suppose $i<r$. Let $0 \to N \to E^0 \to E^1 \to \cdots \to E^{r+1} \to 0$ be a minimal injective resolution of N. We have $(0:(x_0, x_1, \cdots, x_i))_{E^i}=0$. For, if $(0:(x_0, x_1, \cdots, x_i))_{E^i} \neq 0$, it is finitely generated since it is contained in N^i, and so depth $R/(x_0, x_1, \cdots, x_i)=0$, which is a contradiction. Therefore x_i is a nonzero divisor on $(0:(x_0, x_1, \cdots, x_{i-1}))_{E^i}$ and so we have $d^{i-1}(N^{i-1}) \neq 0$. Since $d^{i-1}(N^{i-1})$ is a submodule of the

finitely generated module N^i and of finite injective dimension as an $R/(x_0, x_1, \cdots, x_{i-1})$-module, its isomorphic module $N^{i-1}/d^{i-2}(N^{i-2})$ is a finitely generated $R/(x_0, x_1, \cdots, x_{i-1})$-module of finite injective dimension. Set $\mathfrak{a}=\mathrm{Ann}_{R/(x_0, x_1, \cdots, x_{i-1})}N^{i-1}/d^{i-2}(N^{i-2})$. If $\mathfrak{a}=0$, $R/(x_0, x_1, \cdots, x_{i-1})$ is Cohen-Macaulay [see **2**, Lemma (3.1), (3.3)]. Next assume $\mathfrak{a}\neq 0$. Then there is an element x'_i of R such that the residue class $\bar{x}'_i$ belongs to $\mathfrak{a}$ and x'_i is a nonzero divisor on $R/(x_0, x_1, \cdots, x_{i-1})$ [see **3**, Theorem 4.1]. For a maximal R-sequence $(x')=(x_0, x_1, \cdots, x_{i-1}, x'_i, x'_{i+1}, \cdots, x'_r)$, the i-th term $N^i_{(x')}$ of the (x')-complex under N is non finitely generated, for $N^{i-1}/d^{i-2}(N^{i-2})$ $(\neq 0)$ is embedded in $(0:(x_0, x_1, \cdots, x_{i-1}, x'_i))_{E^i}$ and so $(0:(x_0, x_1, \cdots, x_{i-1}, x'_i))_{E^i}\neq 0$. Thus, if i is a minimal non negative integer such that N^i is finitely generated, $R/(x_0, x_1, \cdots, x_{i-1})$ is Cohen-Macaulay or there is a maximal R-sequence (x') such that all $N^j_{(x')}$ are non finitely generated for $0\leqq j\leqq i$. When the second statement holds, repeating the above argument, we obtain that $R/(x_0, x_1, \cdots, x_k)$ is Cohen-Macaulay or there is a maximal R-sequence (x'') such that $N^{r+1}_{(x'')}$ is finitely generated. In either case, R is Cohen-Macaulay.

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