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A theorem on the fixed point set of a unipotent transformation on the flag manifold

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Introduction.

Let V be a finite dimensional vector space over a field K. We denote by $G_k(V)$ the Grassmann manifold defined by the set of all k-dimensional subspaces of V. For an increasing sequence of natural numbers

$$1 \leqq k_1 < k_2 < \cdots < k_t < \dim V,$$

we denote by $\mathcal{F} = \mathcal{F}(k_1, \cdots, k_t; V)$ the flag manifold of type $(k_1, \cdots, k_t)$ defined by $\{(W_1, \cdots, W_t) \in G_{k_1}(V) \times \cdots \times G_{k_t}(V) \mid W_i \subset W_{i+1}, 1 \leqq i \leqq t\}$. For a nilpotent transformation N of V, put

$$\mathcal{F}^N = \{(W_i) \in \mathcal{F} \mid N(W_i) \subset W_i\}.$$

In this paper, we prove the following

THEOREM. *The variety $\mathcal{F}^N$ has a partition into a finite number of affine spaces and this partition is determined by the Young diagram associated to N.*

The partition is given by some inductive formula and is described precisely in § 1. The crucial point of the proof of the theorem is the proof in the case of Grassmann manifold $\mathcal{F}^N = \mathcal{F}(k; V)^N = G_k(V)^N$ and this is given in Proposition of § 3. If $t = \dim V - 1$, i.e. $\mathcal{F}$ is the manifold of complete flags, N. Spaltenstein [1] has proved, among other interesting results, a theorem below (see also R. Steinberg [2], 3.10) We remark that, by an appropriate identification, we can rewrite the theorem in the following form.

THEOREM′. *For any parabolic subgroup P of the general linear group $G = GL_n(K)$ and for any unipotent element u of G, the variety*

$$(G/P)_u = \{gP \mid u \cdot gP = gP\}$$

has a partition into a finite number of affine spaces and this partition is determined by the Young diagram associated to u.

Several consequences about the characters of the finite general linear groups have been deduced from our theorem, and they will be discussed in a subsequent paper.

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Notation.

In this paper, K is simply a field without assumptions on algebraically closedness or its characteristic. Let V be a vector space over K. If N is an endomorphism of V, we may consider V as a $K[N]$-module. Then we wirte (V, N) for V. If $\{x_\nu | \nu \in N\}$ is a set of generators of V, then we write $V = \langle x_\nu | \nu \in N\rangle$ or $V = \langle \cdots, x_\nu, \cdots \rangle$. We denote by $\boldsymbol{N}$ the set of all natural numbers. For $n \in \boldsymbol{N}$, let $\boldsymbol{A}^n$ be the n-dimensional affine space over K. Let X be a set. If $\{X_\nu\}$ is a family of subspaces of X, then $X = \coprod_\nu X_\nu$ means the direct sum decomposition of X. If X is finite, then $\sharp(X)$ denotes the number of its elements.

§1. Statement of the result.

1. We use a Jordan basis $\{w_{ij_i} | 1 \leqq j_i \leqq l_i\}$ of (V, N) satisfying the following requirement:

$$l_1 \leqq l_2 \leqq \cdots \leqq l_n,\ N w_{ij} = w_{i+1j} \quad \text{and} \quad N w_{nj} = 0 .$$

By making use of this basis, we may associate to (V, N) the Young diagram of degree $\dim V$, which will also be denoted by (V, N).

EXAMPLE 1. Let $\dim V = 10$. If N has two Jordan blocks of dimension 4 and one Jordan block of dimension 2, then we can write

$$(V, N) = \begin{array}{|c|c|c|c|} \hline w_{41} & w_{31} & w_{21} & w_{11} \\ \hline w_{42} & w_{32} & w_{22} & w_{12} \\ \hline w_{43} & w_{33} & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} .$$

2. For a natural number k and for (V, N), let $L_k(N)$ be the set of mappings

$$l : \{1, 2, \cdots, n\} \longrightarrow \{\text{all the subsets of } \boldsymbol{N}\}$$

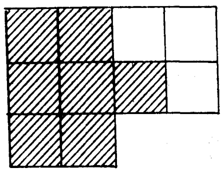
such that $l(i) \cap l(j) = \emptyset$ $(i \neq j)$, $l(i) \subset \{1, 2, \cdots, l_i\}$ and $\sum_{i=1}^{n} (n-i+1) \cdot \sharp(l(i)) = k$. We write the elements of $l(i)$ as follows:

$$l(i) = \{l(i)_1, l(i)_2, \cdots, l(i)_{d(i)}\},\ l(i)_1 < l(i)_2 < \cdots < l(i)_{d(i)} .$$

For $l \in L_k(N)$, put

$$M_l = \{N^h w_{il(i)_m} | 1 \leqq i \leqq n,\ 1 \leqq m \leqq d(i),\ 0 \leqq h \leqq n-i\} .$$

EXAMPLE 2. Let (V, N) be as in Example 1 and let $k=7$. If $l(1)=\emptyset$, $l(2)=\{l(2)_1\}=\{2\}$, $l(3)=\{l(3)_1, l(3)_2\}=\{1, 3\}$ and $l(4)=\emptyset$, then M_l is the collection of w_{ij} in ▩ on the following diagram:



3. DEFINITION. For $l \in L_k(N)$, let T_l be the set of vector spaces defined by

$$\{\langle N^h w_{il(i)_m} + \sum_{(2)} a_{imj} N^h w_{ij} + \sum_{(3)} b_{impq} N^h w_{pq} \mid (1)\rangle \mid a_{imj}, b_{impq} \in K\}$$

where the conditions (1), (2) and (3) are defined in the following:

(1) $1 \leqq i \leqq n$, $1 \leqq m \leqq d(i)$, $0 \leqq h \leqq n-i$,

(2) $j < l(i)_m$, $j \notin \bigcup_{1 \leq s \leq i} l(s)$,

(3) $p \geqq i+1$, $w_{pq} \notin M_l$.

For $l \in L_k(N)$, put

$$n(l) = \sum_{i=1}^{n} \left[\sum_{m=1}^{d(i)} (l(i)_m - \#\{1 \leqq j \leqq l(i)_m \mid j \in \bigcup_{1 \leq s \leq i} l(s)\}) + \#\{w_{pq} \notin M_l \mid p \geqq i+1\} \right].$$

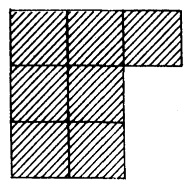
We shall prove that $G_k(V)^N = \coprod_{l \in L_k(N)} T_l$. Here T_l is a locally closed subset of $G_k(V)^N$ and isomorphic to $A^{n(l)}$ (§ 2, Lemma 2 and Remark).

4. Let l and M_l be as in 2. Put

$$V_l = \langle w_{ij} \mid w_{ij} \in M_l \rangle .$$

Then V_l is an N-stable subspace of V and $V_l \in T_l$.

EXAMPLE 3. If V, N and M_l are as in Example 2, then

$(V_l, N)=$ 

Let $\mathcal{F} = \mathcal{F}(k_1, \cdots, k_t; V)$ be the flag manifold of type $(k_1, \cdots, k_t)$ as in the introduction and let $\mathcal{F}_l = \mathcal{F}(k_1, \cdots, k_{t-1}; V_l)$ be the flag manifold of type $(k_1, \cdots, k_{t-1})$. We may also define the subvariety $\mathcal{F}_l^N$ for (V_l, N). Then we have the following theorem.

THEOREM. *The variety $\mathcal{F}^N$ has a partition into a finite number of affine spaces by the following recurrence formula:*

$$\mathcal{F}^N \cong \coprod_{l\in L_{k_t}(N)} T_l \times \mathcal{F}_l^N .$$

REMARK. We can change the Theorem in the following form. Let $L_k(N)$ and M_l be as in 2.

DEFINITION$'$. For $l\in L_k(N)$, let T_l' be the set of vector spaces defined by

$$\{\langle N^h w_{il(i)_m} + \sum_{(2')} a_{imj} N^h w_{ij} + \sum_{(3)} b_{impq} N^h w_{pq} \,|\, (1)\rangle \,|\, a_{imj},\ b_{impq} \in K\} ,$$

where the conditions (1) and (3) are as in 3, Definition and the condition (2$'$) is defined by $j>l(i)_m$, $j\notin \bigcup_{1\leq s\leq i} l(s)$. For $l\in L_k(N)$, put

$$n'(l)=\sum_{i=1}^{n}\Big[\sum_{m=1}^{d(i)}(l_i-l(i)_m-\#\{l(i)_m\leqq j\leqq l_i \,|\, j\in \bigcup_{1\leq s\leq i} l(s)\})$$
$$+\#\{w_{pq}\notin M_l \,|\, p\geqq i+1\}\Big] .$$

Then T_l' is a locally closed subset of $G_k(V)^N$ and isomorphic to $A^{n'(l)}$ and further $G_k(V)^N = \coprod_{l\in L_k(N)} T_l'$.

Let V_l be as in 4. Then we also have $V_l\in T_l'$. Let $\mathcal{F}$ and $\mathcal{F}_l$ be the flag manifold as in the Theorem. Under these information, we have

THEOREM$'$. *The variety $\mathcal{F}^N$ has a partition into a finite number of affine spaces by the following recurrence formula:*

$$\mathcal{F}^N \cong \coprod_{l\in L_{k_t}(N)} T_l' \times \mathcal{F}_l^N .$$

The proof can be carried out mutatis mutandis.

§ 2. Preliminaries.

Let T_l be as in § 1, 3, Definition.

LEMMA 1. *The set T_l is a subset of $G_k(V)^N$.*

PROOF. Put

$$v_{im}=w_{il(i)_m}+\sum_{(2)} a_{imj} w_{ij} + \sum_{(3)} b_{impq} w_{pq} ,$$

where the summations (2), (3) are as in § 1, 3, Definition. Then we can write the element of T_l as

$$\langle N^h v_{im} \,|\, (1)\rangle .$$

It is obvious that $N(\langle N^h v_{im} \,|\, (1)\rangle)\subset \langle N^h v_{im} \,|\, (1)\rangle$. We shall prove that the set $\{N^h v_{im} \,|\, (1)\}$ is linearly independent. Assume

$$(*) \qquad \sum_{(1)} c_{imh} N^h v_{im} = 0 \qquad (c_{imh}\in K) .$$

We may assume $l(1)\neq\emptyset$. By (2) and (3), $\{v_{1m} \,|\, 1\leqq m\leqq d(1)\}$ is linearly independent.

Since $v_{1m} \notin \mathrm{Ker}\, N^{n-1}$ and

$$\sum c_{1m0} v_{1m} = -\sum_{i+h>1} c_{imh} N^h v_{im} \in \mathrm{Ker}\, N^{n-1},$$

we have $c_{1m0}=0$ for $1 \leqq m \leqq d(1)$. Thus the (*) becomes

$$\sum_{i+h>1} c_{imh} N^h v_{im} = 0.$$

By (2) and (3), $\{v_{2m} | 1 \leqq m \leqq d(2)\} \cup \{N v_{1m} | 1 \leqq m \leqq d(1)\}$ is linearly independent. Since v_{2m}, $N v_{1m} \notin \mathrm{Ker}\, N^{n-2}$ and

$$\sum_{i+h=2} c_{imh} v_{im} = -\sum_{i+h>2} c_{imh} N^h v_{im} \in \mathrm{Ker}\, N^{n-2},$$

we have $c_{2m0}=0$ for $1 \leqq m \leqq d(2)$ and $c_{1m1}=0$ for $1 \leqq m \leqq d(1)$. If we continue this procedure, we have

$$c_{imh}=0 \qquad \text{for any} \quad i, m \quad \text{and} \quad h.$$

Since $\sharp\{N^h v_{im} | (1)\} = \sum_{i=1}^{n} (n-i+1) d(i) = k$, the proof of the lemma is completed.

LEMMA 2. *Under a mapping* $(\cdots, a_{imj}, \cdots, b_{impq}, \cdots) \mapsto \langle N^h w_{il(i)_m} + \sum_{(2)} a_{imj} N^h w_{ij} + \sum_{(3)} b_{impq} N^h w_{pq} | (1) \rangle$, *we have an isomorphism* $A^{n(l)} \simeq T_l$, *where* $n(l)$ *is a natural number defined in* § 1, 3.

PROOF. We have to prove that the mapping is injective. Put

$$v_{im} = w_{il(i)_m} + \sum_{(2)} a_{imj} w_{ij} + \sum_{(3)} b_{impq} w_{pq},$$

$$v'_{im} = w_{il(i)_m} + \sum_{(2)} a'_{imj} w_{ij} + \sum_{(3)} b'_{impq} w_{pq},$$

$$(a_{imj}, b_{impq}, a'_{imj}, b'_{impq} \in K).$$

We have to prove that if $\langle N^h v_{im} | (1) \rangle = \langle N^h v'_{im} | (1) \rangle$, then $a_{imj} = a'_{imj}$ and $b_{impq} = b'_{impq}$ for any i, m, j, p, q. Assume

$$(*) \qquad v'_{i_0 m_0} = \sum_{(1)} c_{mhi} N^h v_{im} \qquad (c_{mhi} \in K).$$

We shall prove $c_{m_0 0 i_0}=1$ and $c_{mhi}=0$ for $(m, h, i) \neq (m_0, 0, i_0)$. We may assume $l(1) \neq \emptyset$.

Assume $i < i_0$. In (*), there is only one $N^h v_{im}$ which contains $N^h w_{1l(1)_m}$. Hence $c_{mh1}=0$. If $2 < i_0$, by $c_{mh1}=0$, there is only one $N^h v_{im}$ in (*) which contains $N^h w_{2l(2)_m}$. Hence $c_{mh2}=0$. If $3 < i_0$, by $c_{mh1}=c_{mh2}=0$, there is only one $N^h v_{im}$ in (*) which contains $N^h w_{3l(3)_m}$. Hence $c_{mh3}=0$. In this way, if we continue this procedure for $i=4, 5, \cdots, i_0-1$, we have

$$c_{mhi}=0 \qquad \text{for} \quad i < i_0.$$

Assume $i=i_0$. Since $c_{mhi}=0$ for $i<i_0$, there is only one $N^h v_{im}$ in $(*)$ which contains $w_{i_0 l(i_0)_m}$ $(m\neq m_0)$ and $c_{m_0 0 i_0} w_{i_0 l(i_0)_{m_0}} = w_{i_0 l(i_0)_{m_0}}$. Hence $c_{m_0 0 i_0}=1$ and

$$c_{mhi_0}=0\ (m\neq m_0), \qquad c_{m_0 h i_0}=0\ (h>0).$$

Assume $i>i_0$. Since $c_{mhi}=0$ for $i<i_0$ or $i=i_0$ $(m\neq m_0)$ and $c_{m_0 h i_0}=0$ for $h>0$, there is only one $N^h v_{im}$ in $(*)$ which contains $N^h w_{i_0+1 l(i_0+1)_m}$. Hence $c_{mhi_0+1}=0$. If we continue this procedure for $i=i_0+2, i_0+3, \cdots, n$, then we have $c_{mhi_0+2}=0, c_{mhi_0+3}=0, \cdots, c_{mhn}=0$. Therefore, $v'_{i_0 m_0}=v_{i_0 m_0}$. Hence

$$a'_{i_0 m_0 j}=a_{i_0 m_0 j} \quad \text{and} \quad b'_{i_0 m_0 pq}=b_{i_0 m_0 pq}.$$

Since i_0 and m_0 are arbitrary, we have $a'_{imj}=a_{imj}$ and $b'_{impq}=b_{impq}$ for any i, m, j, p, q. For a fixed i, the number of w_{ij} in $\sum_{(2)}$ is

$$l(i)_m - \#\{1\leqq j\leqq l(i)_m \mid j\in \bigcup_{1\leqq s\leqq i} l(s)\}$$

and the number of w_{pq} in $\sum_{(3)}$ is

$$\#\{w_{pq}\notin M_l \mid p\geqq i+1\}.$$

This completes the proof of the lemma.

Let $V=\langle v_i \mid i=1, \cdots, l\rangle$ be an l-dimensional vector space over K, For an increasing sequence of natural numbers: $1\leqq s_1<s_2<\cdots<s_d\leqq l$, put

$$S_{s_1,\dots,s_d}=\{\langle v_{s_m}+\sum_{I_m} a_{mi}v_i \mid 1\leqq m\leqq d\rangle \mid a_{mi}\in K\},$$

where I_m is a condition: $i<s_m, i\neq s_1, \cdots, s_{m-1}$. The next lemma gives a well-known cellular decomposition of the Grassmann manifold.

LEMMA 3. *We have* $G_d(V)=\coprod_{1\leqq s_1<\cdots<s_d\leqq l} S_{s_1,\dots,s_d}$ *and* $A^e\simeq S_{s_1,\dots,s_d}\left(e=\sum_{m=1}^{d}(s_m-m)\right)$ *under a mapping*

$$(\cdots, a_{mi}, \cdots)\longmapsto \langle v_{s_m}+\sum_{I_m} a_{mi}v_i \mid 1\leqq m\leqq d\rangle.$$

REMARK. Let V be a vector space as in §1, 1. We now arrange the basis $\{w_{ij_i} \mid 1\leqq j_i\leqq l_i\}$ of V in the following way

$$w_{n1}, \cdots, w_{nl_n}, \cdots, w_{21}, \cdots, w_{2l_2}, w_{11}, \cdots, w_{1l_1}.$$

Put

$$S_l=\{\langle N^h w_{il(i)_m}+\sum_{(2)} a^{(h)}_{imj}N^h w_{ij}+\sum_{(3)} b^{(h)}_{impq}N^h w_{pq} \mid (1)\rangle \mid a^{(h)}_{imj}, b^{(h)}_{impq}\in K\}.$$

Then S_l is an object similar to the ones in Lemma 3, and we have

$$S_l\cap G_k(V)^N=T_l.$$

Therefore, T_l is a locally closed subset of $G_k(V)^N$.

§ 3. Proof of the Theorem.

Let V be a finite dimensional vector space over a field K. For a nilpotent transformation N of V, put

$$G_k(V)^N=\{W\in G_k(V)\mid N(W)\subset W\}.$$

If $N^\nu=0$ and $N^{\nu-1}\neq 0$, then

$$G_k(\mathrm{Ker}\,N)\subset G_k(V)^N\subset G_k(\mathrm{Ker}\,N^{\min(k,\nu)}).$$

By this inclusion formula, we have

$$G_k(V)^N=G_k(\mathrm{Ker}\,N^n)^N,$$

where $n=\min(k, \nu)$. Therefore, we may assume

$$V=\mathrm{Ker}\,N^n.$$

In particular, if $k=1$, then

$$G_1(V)^N=G_1(\mathrm{Ker}\,N)=\boldsymbol{P}(\mathrm{Ker}\,N).$$

Let r be a quotient homomorphism of vector spaces:

$$V\longrightarrow V/\mathrm{Ker}\,N^{n-1}.$$

Let

$$\{d_1, \cdots, d_m\}=\{\dim r(W)\neq 0\mid W\in G_k(V)^N\},$$

and let

$$D_i=\{W\in G_k(V)^N\mid \dim r(W)=d_i\}, \qquad i=1, \cdots, m.$$

Then we have the following partition:

$$G_k(V)^N-G_k(\mathrm{Ker}\,N^{n-1})^N=\coprod_{1\leq i\leq m} D_i.$$

PROPOSITION. *Let T_l be the set defined in* § 1, 3. *Then we have*

$$G_k(V)^N=\coprod_{l\in L_k(N)} T_l,$$

where $L_k(N)$ is as in § 1, 2.

PROOF. We assume that for any N-stable proper subspace of V, this proposition is proved for appropriately defined l, s.

Step 1. Let $D=D_i$ be as above. Put $d=d_i$. Let r_0 be a morphism defined by

$$D\longrightarrow G_d(V/\mathrm{Ker}\,N^{n-1}) \qquad (W\longmapsto r(W)).$$

Any element W of D has d linearly independent vectors which are not contained in Ker N^{n-1} and by the N-stability of W, W has at least nd linearly independent vectors. Hence $k\geqq nd$ and therefore, r_0 is surjective. Let $\{w_{ij_i}\mid 1\leqq j_i\leqq l_i\}$ be a basis of V as in § 1, 1. Put $w_m=w_{1m}$ $(1\leqq m\leqq l_1=l)$. Then

$$V/\mathrm{Ker}\,N^{n-1}=\langle r(w_m)\mid 1\leqq m\leqq l\rangle .$$

Let $S_{s_1,\dots,s_d}$ be as in § 2, Lemma 3 (change v_m to $r(w_m)$). If $W\in r_0^{-1}(S_{s_1,\dots,s_d})$, then by taking account of the N-stability of W, we can write

$$W=\langle \{N^h u_{s_m}\mid 0\leqq h\leqq n-1,\ 1\leqq m\leqq d\},\ \{\sum_{i_t\geqq 2,\, j_t} c_{i_t j_t} N^\alpha w_{i_t j_t}\mid \alpha,\ t\}\rangle ,$$

where

$$u_{s_m}=w_{s_m}+\sum_{i\in I_m} a_{mi}w_i+\sum_{p\geqq 2} b_{mpq}w_{pq} ,$$

(a_{mi}, b_{mpq}, $c_{i_t j_t}\in K$ and I_m is a condition: $i<s_m$, $i\neq s_1, \cdots, s_{m-1}$). Let $h(u_{s_m})$ be the minimum h such that

$$b_{mpq}\neq 0 \quad\text{and}\quad N^h w_{s_{m'}}=w_{pq} \qquad \text{for some } m'\ (1\leqq m'\leqq d).$$

Then $h(u_{s_m})\leqq n-1$. Put

$$u'_{s_m}=u_{s_m}-\sum_{N^h w_{s_{m'}}=w_{pq}} b_{mpq}N^h u_{s_{m'}} , \qquad \text{where} \quad h=h(u_{s_m}).$$

By the definition of I_m, we have

$$h(u'_{s_m})>h(u_{s_m}).$$

Replacing u_{s_m} by u'_{s_m} in the above generators of W and continuing this procedure if necessary, we can take u_{s_m} to have the following form:

$$u_{s_m}=w_{s_m}+\sum_{i\in I_m} a_{mi}w_i+\sum_{p\geqq 2,\ w_{pq}\notin M_1} b_{mpq}w_{pq} ,$$

where

$$M_1=\{N^h w_{s_m}\mid 1\leqq m\leqq d,\ 0\leqq h\leqq n-1\}.$$

Similarly, we can take $\sum_{i_t\geqq 2,\, j_t} c_{i_t j_t}w_{i_t j_t}$ to have the following form:

$$\sum_{i_t\geqq 2,\, j_t;\, w_{i_t j_t}\notin M_1} c_{i_t j_t}w_{i_t j_t} .$$

Put

$$V_1=\langle w_{ij}\mid i\geqq 2,\ w_{ij}\notin M_1\rangle .$$

Then, we have

$$W\cap V_1=\langle \sum_{i_t\geqq 2,\, j_t;\, w_{i_t j_t}\notin M_1} c_{i_t j_t}N^\alpha w_{i_t j_t}\mid \alpha,\ t\rangle .$$

We remark that $N(W\cap V_1)\subset W\cap V_1$ and $\dim(W\cap V_1)=k-nd$.

Step 2. We can now consider the following morphism

$$r_1: r_0^{-1}(S_{s_1,\dots,s_d})\longrightarrow G_{k-nd}(V_1)^N \qquad (W\longmapsto W\cap V_1).$$

By the definition of V_1, this morphism is surjective. By the induction hypothesis

$$G_{k-nd}(V_1)=\coprod_{l'\in L_{k-nd}(N|V_1)} T_{l'} ,$$

where $T_{l'}$ is defined for V_1 as T_l for V. Let $l \in L_k(N)$ be such that $l(1)=\{s_1, \cdots, s_d\}$ and $l(i)=l'(i-1)$ for $i \geqq 2$. If $W \in r_1^{-1}(T_{l'})$, then by Step 1, we can take

$$W=\langle \{N^h u_{s_m} \mid 0 \leqq h \leqq n-1,\ 1 \leqq m \leqq d\},\ \{N^{h'} v_{im} \mid (1')\} \rangle ,$$

where

$$u_{s_m}=w_{s_m}+\sum_{i \in I_m} a_{mi} w_i + \sum_{p \geq 2,\ w_{pq} \notin M_1} b_{mpq} w_{pq} ,$$

$$v_{im}=w_{il'(i)_m}+\sum_{(2')} a_{imj} w_{ij}+\sum_{(3')} b_{impq} w_{pq} ,$$

(1') $2 \leqq i \leqq n,\ 1 \leqq m \leqq d(i),\ 0 \leqq h' \leqq n-i,$

(2') $j < l'(i)_m,\ j \notin \bigcup_{2 \leq s \leq i} l'(s),$

(3') $p \geqq i+1,\ w_{pq} \notin M_{l'}=\{N^{h'} w_{il'(i)_m} \mid 2 \leqq i \leqq n,\ 1 \leqq m \leqq d(i),\ 0 \leqq h' \leqq n-i\}.$

We remark that $M_1 \cup M_{l'}=M_l$ (§ 1, 2). Let $h'(u_{s_m})$ be the minimum h' such that $b_{mpq} \neq 0$ and $N^{h'} w_{il'(i)_{m'}}=w_{pq}$ for some i and m' (i and m' are as in (1')). Then $h'(u_{s_m}) \leqq n-2$, Put

$$u'_{s_m}=u_{s_m}-\sum_{N^{h'} w_{il'(i)_{m'}}=w_{pq}} b_{mpq} N^{h'} v_{im'} , \qquad \text{where} \quad h'=h'(u_{s_m}) .$$

By the definition of (2'), we have

$$h'(u'_{s_m}) > h'(u_{s_m}) .$$

Replacing u_{s_m} by u'_{s_m} in the basis of W and continuing this procedure if necessary, we can take u_{s_m} to have the following form:

$$u_{s_m}=w_{s_m}+\sum_{i \in I_m} a_{mi} w_i + \sum_{p \geq 2,\ w_{pq} \notin M_l} b_{mpq} w_{pq} .$$

Hence

$$r_1^{-1}(T_{l'})=T_l ,$$

where $l \in L_k(N)$ is determined by $l' \in L_{k-nd}(N)$ as above.

Step 3. By Step 2,

$$r_0^{-1}(S_{s_1,\dots,s_d})=\amalg\, T_l ,$$

where the summation runs over $l \in L_k(N)$ such that $l(1)=\{s_1, \cdots, s_d\}$. Therefore

$$D=\coprod_{l \in L_k(N),\ \#(l(1))=d} T_l .$$

By the formula

$$G_k(V)^N-G_k(\operatorname{Ker} N^{n-1})^N=\coprod_{1 \leq i \leq m} D_i ,$$

the proof of the proposition is completed.

We are now going into the Theorem in § 1. We assume that $k_l=k$ and $l \in L_k(N)$. If $W \in T_l$, then the projection

$$f: V \longrightarrow V_l$$

induces an N-module isomorphism

$$f_W : W \xrightarrow{\simeq} V_l .$$

We consider the projection

$$\pi : \mathscr{F} \longrightarrow G_k(V) \qquad ((W_1, \cdots, W_t) \longmapsto W_t).$$

Then we have the following trivialization:

$$\pi^{-1}(T_l) \xrightarrow{\simeq} T_l \times \mathscr{F}_l \qquad (x=(W_i) \longmapsto (\pi(x), (f_{W_t}(W_1), \cdots, f_{W_t}(W_{t-1}))) .$$

Under this trivialization, we have

$$\pi^{-1}(T_l) \cap \mathscr{F}^N \xrightarrow{\simeq} T_l \times \mathscr{F}_l^N .$$

Thus the Theorem.

References

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