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On characteristic classes of conformal and projective foliations

Dedicated to Professor A. Komatu on his 70th birthday

By Shigeyuki MORITA[1]

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0. Introduction.

In this paper, we define characteristic classes for conformal and projective foliations and investigate the relationship of them with those for smooth foliations defined by Bott and Haefliger [4] and those for Riemannian foliations due to Lazarov and Pasternack [16] and Kamber and Tondeur [12] (see also [18]). For a construction of the characteristic classes of smooth foliations [2], Bott's vanishing theorem [1] concerning the Pontrjagin classes of the normal bundles played an important role. Also Pasternack's vanishing theorem for the Riemannian foliations [25] was the starting point of Lazarov-Pasternack theory. Similarly our motivation for the present work was the strong vanishing theorem of Nishikawa and Sato [22], which states that the ring generated by the Pontrjagin classes of the normal bundle of a conformal or projective foliation is trivial for cohomology degree>codimension. However we do not use this theorem in our construction. Instead, we follow the Bott-Haefliger approach [4] to the characteristic classes of smooth foliations (namely, à la Gelfand-Fuks theory——see [3]), and also the method of Kamber and Tondeur used in their theory of characteristic classes for foliated bundles [12] [13]. Thus just as the cohomology of some truncated Weil algebra of $\mathfrak{gl}(n; \boldsymbol{R})$ or $\mathfrak{so}(n)$ played the role of characteristic classes for smooth or Riemannian foliations, our characteristic classes also take the form of the cohomology of certain truncated Weil algebra of $\mathfrak{so}(n+1, 1)$ for the conformal case and of $\mathfrak{sl}(n+1; \boldsymbol{R})$ for the projective case, where $\mathfrak{gl}(n; \boldsymbol{R})$, $\mathfrak{so}(n)$, $\mathfrak{so}(n+1, 1)$ and $\mathfrak{sl}(n+1; \boldsymbol{R})$ are the Lie algebras of $GL(n; \boldsymbol{R})$, $SO(n)$, $SO(n+1, 1)$ and $PGL(n; \boldsymbol{R})$ respectively. The main point of our construction is the use of Cartan connection, by which we have also shown that there are other characteristic classes for Riemannian foliations which are not covered by Lazarov-

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Pasternack's nor Kamber-Tondeur's definitions (see [18]).

The present paper is organized as follows. In § 1 we define foliations associated with second order G-structures, in particular conformal and projective structures and in § 2 we recall the theory of Cartan connection and by using it, we construct a system of differential forms on the normal bundles of conformal or projective foliations. In § 3 we describe the main construction of the characteristic classes and in § 4, we determine the cohomology of some truncated Weil algebras. § 5 is devoted to the study of typical examples and in §§ 6 and 7, we compare our theory with smooth and Riemannian cases. In particular, we prove that the rigid classes for smooth foliations are all zero for conformal and projective foliations. Finally in § 8, we investigate the behaviour of our characteristic classes under deformations of foliations.

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1. Conformal and projective structures and foliations.

In this section we define conformal and projective foliations following Nishikawa and Sato [**22**] (see also Kobayashi [**14**] for a detailed description of the materials of this section).

Let M be a smooth manifold of dimension n and let $J_k(M)$ be the k-jet bundle of M. Thus as a set $J_k(M)$ consists of all the k-jets at 0 of all diffeomorphisms from open sets of $0 \in \boldsymbol{R}^n$ to open sets of M. The natural projection $\pi: J_k(M) \to M$ has a structure of principal bundle with structure group $G_k(n)$: the group of k-jets at 0 of all local diffeomorphisms of $\boldsymbol{R}^n$ fixing the origin 0. In this paper, we are only interested in the 2-jet bundle $J_2(M)$. On $J_2(M)$, there is defined a 1-form θ, called the canonical form, with values in $\mathfrak{a}(n; \boldsymbol{R})$, the Lie algebra of the group of all affine transformations of $\boldsymbol{R}^n$. In terms of the natural basis of $\mathfrak{a}(n; \boldsymbol{R}) = \boldsymbol{R}^n + \mathfrak{gl}(n; \boldsymbol{R})$, θ is represented by real valued 1-forms θ^i, $i=1, \cdots, n$ and θ^i_j, $i, j=1, \cdots, n$. We know the following equation,

$$d\theta^i = -\sum_j \theta^i_j \wedge \theta^j . \tag{1.1}$$

Now let $G \subset G_2(n)$ be a Lie subgroup. Then a G-principal subbundle P of $J_2(M)$ is called a G-structure of second order. A diffeomorphism $f: M \to N$ between two smooth manifolds M and N with given second order G-structures are said to be a G-diffeomorphism if $J_2(f)$, the 2-jet extension of f, sends the subbundle of $J_2(M)$ defining the structure to that of $J_2(N)$. Let us recall classical examples of second order G-structures, namely conformal and projective structures.

EXAMPLE 1.1. Let S^n be the Möbius space, i.e. it is the quadric in the real projective space P^{n+1} defined by

$$S^n=\{[x]\in P^{n+1}\,;\quad x={}^t(x_0,\cdots,x_{n+1}),\quad {}^txSx=0\}$$

where S is the matrix given by

$$S=\begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Then with respect to a natural metric on P^{n+1}, S^n is isometric to a Euclidean sphere in $\boldsymbol{R}^{n+1}$. Let $L=O(n+1,1)=\{X\in GL(n+2\,;\boldsymbol{R})\,;\ {}^tXSX=0\}$ acting on S^n as conformal transformations. This action is transitive and let L_0 be the isotropy subgroup at the origin $o={}^t[0,\cdots,0,1]$. Thus

$$L_0=\left\{\begin{pmatrix} a^{-1} & 0 & 0 \\ v & A & 0 \\ b & \xi & a \end{pmatrix}\in O(n+1,1)\,;\quad A\in O(n),\quad a\in\boldsymbol{R}^*,\quad \xi\in\boldsymbol{R}^n\right\}$$

and $L/L_0=S^n$. Consider a linear subspace

$$V=\left\{\begin{pmatrix} 0 & {}^tv & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}\in\mathfrak{l}\right\}$$

of $\mathfrak{l}$, where v is a column n-vector and $\mathfrak{l}=\mathfrak{so}(n+1,1)=\{X\in\mathfrak{gl}(n+2\,;\boldsymbol{R})\,;\ {}^tXS+SX=0\}$ is the Lie algebra of L. Then the mapping

$$\boldsymbol{R}^n=V\xrightarrow{\ \exp\ }L\longrightarrow L/L_0=S^n$$

is a diffeomorphism from a neighborhood of 0 to a neighborhood of the origin o so that it defines a coordinate system around o. Now we can consider each element g of L_0 as a transformation of L/L_0 fixing the origin. Moreover it can be seen that the 2-jet at the origin of g completely determines this element. Therefore by using the coordinate system described above, we can think that L_0 is a subgroup of $G_2(n)$. With these understood, an L_0-structure on a smooth manifold is called a conformal structure and an L_0-diffeomorphism is called a conformal diffeomorphism.

EXAMPLE 1.2. Let $L=PGL(n\,;\boldsymbol{R})=GL(n+1\,;\boldsymbol{R})/\text{center}$ acting transitively on the real projective space P^n and let L_0 be the isotropy subgroup at the origin $o={}^t[0,\cdots,0,1]$. Thus $L/L_0=P^n$ and

$$L_0=\left\{\begin{pmatrix} A & 0 \\ \xi & a \end{pmatrix}\in GL(n+1\,;\,\boldsymbol{R})\right\}\Big/\text{center},\ (A\in GL(n\,;\,\boldsymbol{R}))\,.$$

Consider a linear subspace

$$V=\left\{\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}\in\mathfrak{l}\right\}$$

of $\mathfrak{l}=\mathfrak{sl}(n+1\,;\,\boldsymbol{R})$, where v is a column n-vector. Then the mapping

$$\boldsymbol{R}^n=V\xrightarrow{\exp}L\longrightarrow L/L_0=P^n$$

defines a coordinate system around the origin o and just the same as in Example 1.1, L_0 can be considered as a subgroup of $G_2(n)$. With these understood, an L_0-structure on a smooth manifold is called a projective structure and an L_0-diffeomorphism is called a projective diffeomorphism.

Now let L/L_0 be as in Example 1.1 or Example 1.2. Then the orthogonal group $O(n)$ is naturally contained in L_0 as a subgroup. Let M be a Riemannian manifold. Then there is defined an $O(n)$-principal subbundle $O(M)$ of $J_2(M)$, where $O(n)$ is now considered as a subgroup of $G_2(n)$. Hence by enlarging the structure group to L_0, we obtain an L_0-principal subbundle P of $J_2(M)$. The conformal or projective structure thus defined will be called the underlying conformal or projective structure of the Riemannian manifold M.

Now we define conformal and projective foliations.

DEFINITION 1.3. Let M be a smooth manifold. A codimension n conformal (resp. projective) foliation F on M is a maximal family of submersions

$$f_\alpha : U_\alpha \longrightarrow \boldsymbol{R}^n_\alpha$$

from open sets U_α in M to the Euclidean n-space $\boldsymbol{R}^n_\alpha$ with a fixed conformal (resp. projective) structure α such that for each $x\in U_\alpha\cap U_\beta$, there is a local conformal (resp. projective) diffeomorphism $\gamma_{\beta\alpha}$: neighborhood of $f_\alpha(x)\to$neighborhood of $f_\beta(x)$ with $f_\beta=\gamma_{\beta\alpha}\circ f_\alpha$ on a neighborhood of x.

A Riemannian foliation F on M is defined similarly (it is enough to change conformal structure and conformal diffeomorphism in the above definition for Riemannian structure and isometry). By considering the underlying conformal or projective structure of a Riemannian structure, a Riemannian foliation has the underlying structure of conformal and projective foliations. Definition 1.3 can also be generalized to foliations associated with any second order G-structure.

Now let F be a conformal (or projective) foliation on a smooth manifold M and let $J_2(F)$ be the 2-jet bundle of F; $J_2(F)|U_\alpha=f_\alpha^*(J_2(\boldsymbol{R}^n_\alpha))$. Since the diffeomorphism $\gamma_{\beta\alpha}$ sends the L_0-principal subbundle of $J_2(\boldsymbol{R}^n_\alpha)$ defining the structure α to that of $J_2(\boldsymbol{R}^n_\beta)$, there is defined a principal L_0 subbundle $P(F)$ of $J_2(F)$. Henceforth we refer to $P(F)$ as the conformal (resp. projective) normal bundle of F. If there is given a cross section $s: M \to P(F)$, then we say that the conformal (or projective) normal bundle of F is trivialized.

Generalizing Definition 1.3, we can speak of conformal (or projective) Haefliger structures and by a general theory of Haefliger [8], there are classifying spaces $BC\Gamma_n$, $BC\bar{\Gamma}_n$, $BP\Gamma_n$, $BP\bar{\Gamma}_n$ etc., where $BS\Gamma_n(BS\bar{\Gamma}_n)$ is the classifying space for codimension n conformal (resp. projective) Haefliger structures (with trivial normal bundles) according as $S=C$ (resp. P). Since L_0 is homotopy equivalent to $GL(n; \boldsymbol{R})$, we have the following fibration

$$BS\bar{\Gamma}_n \longrightarrow BS\Gamma_n \longrightarrow BGL(n; \boldsymbol{R}) \qquad (S=C,\ P).$$

The main purpose of this paper is to define certain elements in the real cohomology ring of these spaces.

2. Cartan connections.

In this section we recall a few facts from the theory of Cartan connections, in particular conformal and projective connections, which will be needed later. For a detailed description of the theory, we refer to Kobayashi [**14**] and Kobayashi and Nagano [**15**]. By using these facts we construct a system of differential forms on the conformal (or projective) normal bundle of conformal (or projective) foliations.

Let M be a smooth manifold of dimension n, L a Lie group, L_0 a closed subgroup of L with $\dim L/L_0=n$ and let P be a principal bundle over M with structure group L_0. An important example is the L_0-principal subbundle P of $J_2(M)$ where M is a smooth manifold with a conformal (or projective) structure and L/L_0 is as in Example 1.1 (or 1.2). Let $\mathfrak{l}_0$ be the Lie algebra of L_0. For each element $A \in \mathfrak{l}_0$, let us denote A^* for the fundamental vector field corresponding to A and also let us write R_a for the right action defined by an element $a \in L_0$.

Definition 2.1. A Cartan connection in the bundle P is an $\mathfrak{l}$ (=the Lie algebra of L)-valued 1-form θ on P satisfying the following conditions:

(i) $\theta(A^*)=A$ for every $A \in \mathfrak{l}_0$,

(ii) $(R_a)^*\theta = ad(a^{-1})\theta$ for every $a \in L_0$,

(iii) $\theta(X) \neq 0$ for every nonzero vector X of P.

The curvature form Θ of the Cartan connection θ is defined by the structure equation:

$$d\theta = -\frac{1}{2}[\theta, \theta] + \Theta .$$

Henceforth we specialize our consideration to the case when L/L_0 is as in Example 1.1 or 1.2. In these cases, the Lie algebra $\mathfrak{l}$ of L has a structure of graded Lie algebra $\mathfrak{l}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ with $[\mathfrak{g}_i, \mathfrak{g}_j]\subset\mathfrak{g}_{i+j}$ and $\mathfrak{l}_0=\mathfrak{g}_0+\mathfrak{g}_1$ described as follows. Conformal case: $\mathfrak{l}=\mathfrak{so}(n+1, 1)$,

$$\mathfrak{g}_{-1}=\left\{\begin{pmatrix} 0 & {}^tv & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}\right\}, \qquad \mathfrak{g}_0=\left\{\begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix}; \quad A\in\mathfrak{so}(n), \quad a\in\boldsymbol{R}\right\},$$

$$\mathfrak{g}_1=\left\{\begin{pmatrix} 0 & 0 & 0 \\ {}^t\xi & 0 & 0 \\ 0 & \xi & 0 \end{pmatrix}\right\},$$

where v is a column n-vector, ξ is a row n-vector. Let V be the n-dimensional vector space of column n-vectors, V^* its dual and let $\mathfrak{co}(n)$ be the Lie algebra of $CO(n)=\{A\in GL(n\,;\,\boldsymbol{R})\,;\,{}^tAA=cI_n \text{ for some } c>0\}$. Then under the identifications

$$\mathfrak{g}_{-1}\ni\begin{pmatrix} 0 & {}^tv & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}\longrightarrow v\in V, \qquad \mathfrak{g}_1\ni\begin{pmatrix} 0 & 0 & 0 \\ {}^t\xi & 0 & 0 \\ 0 & \xi & 0 \end{pmatrix}\longrightarrow\xi\in V^*,$$

$$\mathfrak{g}_0\ni\begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix}\longrightarrow A-aI_n\in\mathfrak{co}(n),$$

we can write $\mathfrak{l}=V+\mathfrak{co}(n)+V^*$.

Projective case: $\mathfrak{l}=\mathfrak{sl}(n+1\,;\,\boldsymbol{R})$,

$$\mathfrak{g}_{-1}=\left\{\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}\right\}, \qquad \mathfrak{g}_0=\left\{\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}; \ \text{Trace}\, A+a=0\right\}, \qquad \mathfrak{g}_1=\left\{\begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix}\right\},$$

where v and ξ are as in the conformal case. Under the identifications

$$\mathfrak{g}_{-1}\ni\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}\longrightarrow v\in V, \qquad \mathfrak{g}_1\ni\begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix}\longrightarrow\xi\in V^*,$$

$$\mathfrak{g}_0\ni\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}\longrightarrow A-aI_n\in\mathfrak{gl}(n\,;\,\boldsymbol{R}),$$

we can write $\mathfrak{l}=V+\mathfrak{gl}(n\,;\,\boldsymbol{R})+V^*$.

Hereafter we fix the natural bases $e_1, \cdots, e_n$ of V, $e^1, \cdots, e^n$ of V^* and e_j^i of $\mathfrak{gl}(n;\boldsymbol{R})$. Let M be an n-dimensional smooth manifold and let $P\subset J_2(M)$ be a conformal or projective structure on M. We denote $(\theta^i;\theta_j^i)$ for the restriction to P of the canonical form of $J_2(M)$ (see §1). It is a $\mathfrak{g}_{-1}+\mathfrak{g}_0$ valued 1-form on P. Now we can state a well-known theorem on normal conformal and projective connections which is very important to our construction.

THEOREM 2.2. *Let L/L_0 be as in Example 1.1 or 1.2 and let $P\subset J_2(M)$ be a conformal or projective structure on a smooth manifold M of dimension n ($\geqq 3$ for Example 1.1 and $\geqq 2$ for Example 1.2). Then there is a unique Cartan connection $\theta=\theta_{-1}+\theta_0+\theta_1=(\theta^i;\theta_j^i;\theta_j)$ such that*

(i) *$(\theta^i;\theta_j^i)$ is the canonical form,*

(ii) *The curvature $\Theta=(0;\Theta_j^i;\Theta_j)$ satisfies the following conditions:*

$$\sum K_{jil}^i=0, \qquad \text{where } \Theta_j^i=\frac{1}{2}\sum K_{jkl}^i\theta^k\wedge\theta^l,$$

in particular $\sum\Theta_i^i=0$.

This unique connection is called the normal conformal or projective connection in P according as L/L_0 is as in Example 1.1 or 1.2.

We have also

THEOREM 2.3. *Let P be as in Theorem 2.2. Then*

$$(\mathrm{I}) \qquad d\theta^i=-\sum\theta_k^i\wedge\theta^k,$$

$$(\mathrm{II})_c \qquad d\theta_j^i=-\sum\theta_k^i\wedge\theta_j^k-\theta^i\wedge\theta_j-\theta_i\wedge\theta^j+\delta_j^i\sum\theta_k\wedge\theta^k+\Theta_j^i,$$

$$(\mathrm{II})_p \qquad d\theta_j^i=-\sum\theta_k^i\wedge\theta_j^k-\theta^i\wedge\theta_j+\delta_j^i\sum\theta_k\wedge\theta^k+\Theta_j^i,$$

$$(\mathrm{III}) \qquad d\theta_j=-\sum\theta_k\wedge\theta_j^k+\Theta_j,$$

$$(\mathrm{IV}) \qquad \sum\Theta_j^i\wedge\theta^j=0,$$

$$(\mathrm{V}) \qquad \sum\theta^i\wedge\Theta_i=0, \qquad \text{where } \Theta_j=\frac{1}{2}\sum K_{jkl}\theta^k\wedge\theta^l.$$

Here c (resp. p) denotes conformal (resp. projective) case and of course (I)-(III) are the structure equations of the Cartan connection and (IV), (V) are the Bianchi identities.

Let $f: M\to N$ be a conformal or projective diffeomorphism and let $J_2(f)$: $P(M)\to P(N)$ be the induced diffeomorphism. Then from the uniqueness of the Cartan connection, we conclude that $J_2(f)$ sends the normal Cartan connection form of N to that of M. Now let F be a codimension n conformal (resp. projective) foliation on a smooth manifold M defined by submersions $f_\alpha: U_\alpha\to\boldsymbol{R}_\alpha^n$, and let $P(F)$ be the conformal (resp. projective) normal bundle of F (see §1). Then by the above remark, there are defined an $\mathfrak{l}$-valued 1-form $\theta=(\theta^i;\theta^i_j;\theta_j)$ and an $\mathfrak{l}$-valued 2-form $\Theta=(0;\Theta^i_j;\Theta_j)$ on $P(F)$ such that θ (resp. Θ) restricted to U_α is the pull back under f^*_α of the normal Cartan connection form on $P(\boldsymbol{R}^n_\alpha)$ (resp. the curvature form of the normal Cartan connection in $P(\boldsymbol{R}^n_\alpha)$). It is clear that these forms θ, Θ also satisfy the equations in Theorems 2.2 and 2.3.

3. Construction of the characteristic classes.

Let F be a codimension n conformal (resp. projective) foliation on a smooth manifold M and let $P(F)$ be the conformal (resp. projective) normal bundle of F. In §2 we have constructed $\mathfrak{l}$-valued 1-form $\theta=(\theta^i;\theta^i_j;\theta_j)$ and 2-form $\Theta=(0;\Theta^i_j;\Theta_j)$ on $P(F)$ using the normal Cartan connection forms, where $\mathfrak{l}=\mathfrak{so}(n+1,1)$ (resp. $\mathfrak{sl}(n+1;\boldsymbol{R})$). Let $W(\mathfrak{l})$ be the Weil algebra of $\mathfrak{l}$, namely the tensor product of the exterior algebra $\Lambda\mathfrak{l}^*$ with the symmetric algebra $S\mathfrak{l}^*$ (cf. [5]). Then θ and Θ define a d. g. a. map

$$\phi:\ W(\mathfrak{l})\longrightarrow\Omega^*(P(F))$$

where $\Omega^*(P(F))$ is the de Rham complex of $P(F)$. Let ω^i, ω^i_j, ω_j, Ω^i, Ω^i_j, Ω_j $\in W(\mathfrak{l})$ be the universal connection and curvature forms (namely generators of $\Lambda^1\mathfrak{l}^*$ and $S^1\mathfrak{l}^*$) expressed with respect to the natural basis of $\mathfrak{l}$ (see §2). Then ϕ is defined by $\phi(\omega^i)=\theta^i$, $\phi(\omega^i_j)=\theta^i_j$, $\phi(\omega_j)=\theta_j$, $\phi(\Omega^i)=0$, $\phi(\Omega^i_j)=\Theta^i_j$, $\phi(\Omega_j)=\Theta_j$. In view of the equations in Theorems 2.2 and 2.3, which the forms θ, Θ satisfy, we define an ideal I of $W(\mathfrak{l})$ as the one generated by the following elements:

(3.1)
- (i) Ω^i,
- (ii) elements whose "length" l is greater than n, where l is defined by $l(\omega^i)=1,\ l(\Omega^i_j)=l(\Omega_j)=2,\ l(\omega^i_j)=l(\omega_j)=l(\Omega^i)=0$,
- (iii) $\sum\Omega^i_j\wedge\omega^j$,
- (iv) $\sum\Omega_i\wedge\omega^i$,
- (v) $\sum\Omega^i_i$.

Then we have $\phi(I)=0$ and it can be checked that I is a subcomplex of $W(\mathfrak{l})$. Therefore, writing $\widetilde{W}(\mathfrak{l})=W(\mathfrak{l})/I$, we obtain a d. g. a. map

$$\phi:\ \widetilde{W}(\mathfrak{l})\longrightarrow\Omega^*(P(F)).$$

If the normal bundle $P(F)$ of F is trivialized by a cross section $s: M\to P(F)$, then we have

$$H^*(\widetilde{W}(\mathfrak{l}))\longrightarrow H^*_{DR}(P(F))\xrightarrow{s^*}H^*_{DR}(M).$$

Since this construction is functorial, we finally obtain

$$\Phi : H^*(\widetilde{W}(\mathfrak{l}))\longrightarrow H^*(BS\Gamma_n\,;\,\boldsymbol{R})$$

where $S=C$ or P according as the foliation F is conformal or projective. In the general case, we have

$$H^*(\widetilde{W}(\mathfrak{l})_{O(n)})\longrightarrow H^*(BS\Gamma_n\,;\,\boldsymbol{R})$$

$$H^*(\widetilde{W}(\mathfrak{l})_{SO(n)})\longrightarrow H^*(BS\Gamma_n^+\,;\,\boldsymbol{R}),$$

where $\widetilde{W}(\mathfrak{l})_G$ ($G=O(n)$ or $SO(n)$) is the subcomplex of $\widetilde{W}(\mathfrak{l})$ consisting of G basic elements and $+$ denotes the oriented category.

This is our construction of the characteristic classes for conformal and projective foliations.

4. Cohomology of $\widetilde{W}(\mathfrak{l})$.

In this section we compute the cohomology of the truncated Weil algebra $\widetilde{W}(\mathfrak{l})$ defined in the previous section. First we begin with the conformal case. Conformal case: We define a decreasing filtration F^p on $\widetilde{W}(\mathfrak{so}(n+1,1))$ by

$$F^p=\{x\in\widetilde{W}(\mathfrak{so}(n+1,1))\,;\ \bar{l}(x)\geqq p\}$$

where $\bar{l}$ is the function on $W(\mathfrak{so}(n+1,1))$ defined by $\bar{l}(\omega^i_j)=0$, $\bar{l}(\omega^i)=\bar{l}(\omega_j)=1$ and $\bar{l}(\Omega^i_j)=\bar{l}(\Omega_j)=2$ (note that $\bar{l}$ is different from the length function l since $\bar{l}(\omega_j)=1\neq l(\omega_j)=0$). Let $\{E_r^{p,q},d_r\}$ be the spectral sequence associated with this filtration. Now define M_p to be the linear subspace of $\widetilde{W}(\mathfrak{so}(n+1,1))$ spanned by the elements $x\in W(\mathfrak{so}(n+1,1))$ with, $i(X)x=0$ for all $X\in\mathfrak{co}(n)\subset\mathfrak{so}(n+1,1)$ and $\bar{l}(x)=$degree $x=p$, where $i(X)$ is the inner product with respect to X. $\mathfrak{co}(n)$ acts on M_p by the Lie derivation and thus M_p is a $\mathfrak{co}(n)$-module. If we denote $C^q(\mathfrak{co}(n)\,;\,M_p)$ for the set of q-cochains on $\mathfrak{co}(n)$ with coefficient in M_p, then it is easy to see that

$$E_0^{p,q}\cong C^q(\mathfrak{co}(n)\,;\,M_p)\,.$$

Moreover, from the forms of the differentials of ω^i, ω^i_j, ω_j, Ω^i_j, Ω_j, (cf. Theorem 2.3 and $d\Omega^i_j=\Omega^i_k\wedge\omega^k_j-\omega^i_k\wedge\Omega^k_j-\omega^i\wedge\Omega_j+\Omega_i\wedge\omega^j$, $d\Omega_j=\Omega_k\wedge\omega^k_j-\omega_k\wedge\Omega^k_j$) and the action of $\mathfrak{co}(n)$ on M_p, it can be shown that the following diagram is commutative up to sign:

$$\begin{array}{ccc} E_0^{p,q} & \cong & C^q(\mathfrak{co}(n)\,;\,M_p) \\ {\scriptstyle d_0}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ E_0^{p,q+1} & \cong & C^{q+1}(\mathfrak{co}(n)\,;\,M_p) \end{array}$$

where d is the differential of the complex $C^*(\mathfrak{co}(n)\,;\,M_p)$. Therefore we obtain

$$E_1^{p,q} \cong H^q(\mathfrak{co}(n)\,;\, M_p)\,. \tag{4.1}$$

Now $\mathfrak{co}(n)=\mathfrak{so}(n)+\boldsymbol{R}$ and let $I\in\mathfrak{co}(n)$ be the identity matrix and $L(I)$: the Lie derivative with respect to I. Then it is easy to see that M_p splits as a direct sum of eigenspaces V_λ for I with $L(I)|V_\lambda$=multiplication by $\lambda\in\boldsymbol{R}$. Therefore by an argument in [20] (Corollary IV.2.2., also cf. [10]), we have

$$H^q(\mathfrak{co}(n)\,;\, M_p)\cong H^q(\mathfrak{co}(n))\otimes M_p^{\mathfrak{co}(n)}\,. \tag{4.2}$$

Next we determine $M_p^{\mathfrak{co}(n)}$. For this let us define elements $d_1, d_2, d_4, \cdots$ of $\widetilde{W}(\mathfrak{so}(n+1, 1))$ and if n is even $(n=2m)$, $\chi_{s,0}$ $(s=0, \cdots, m)$, $\chi_{s,1}$ $(s=1, \cdots, m-1)$ $\in\widetilde{W}(\mathfrak{so}(n+1, 1))$ by

$$\begin{aligned}
&d_1=\sum_k \omega_k\wedge\omega^k\,,\\
&d_2=\mathrm{Trace}\,((\Omega^i_j)^2)\,,\\
&d_4=\mathrm{Trace}\,((\Omega^i_j)^4),\ \cdots\cdots,\\
&\chi_{s,t}=\sum_\sigma \mathrm{sgn}\,(\sigma)\omega^{\sigma(1)}\wedge\cdots\wedge\omega^{\sigma(s+t)}\wedge\omega_{\sigma(s+t+1)}\wedge\cdots\wedge\omega_{\sigma(2s+t)}\\
&\qquad\wedge\Omega_{\sigma(2s+t+1)}\wedge\cdots\wedge\Omega_{\sigma(2s+2t)}\wedge\Omega^{\sigma(2s+2t+1)}_{\sigma(2s+2t+2)}\wedge\cdots\wedge\Omega^{\sigma(n-1)}_{\sigma(n)}\,.
\end{aligned} \tag{4.3}$$

Note that $\chi_{s,t}=0$ for $t\geqq 2$. We also set

$$\begin{aligned}
\chi=&\sum_\sigma \mathrm{sgn}\,(\sigma)(-\omega^{\sigma(1)}\wedge\omega_{\sigma(2)}-\omega_{\sigma(1)}\wedge\omega^{\sigma(2)}+\Omega^{\sigma(1)}_{\sigma(2)})\cdots\\
&\qquad(-\omega^{\sigma(n-1)}\wedge\omega_{\sigma(n)}-\omega_{\sigma(n-1)}\wedge\omega^{\sigma(n)}+\Omega^{\sigma(n-1)}_{\sigma(n)})\\
=&\chi_{0,0}-2m\chi_{1,0}+4\binom{m}{2}\chi_{2,0}-8\binom{m}{3}\chi_{3,0}+\cdots+(-1)^m2^m\chi_{m,0}\,.
\end{aligned} \tag{4.4}$$

The degrees of these elements are given by degree$(d_i)=2i$, degree$(\chi_{s,0})=n$, degree$(\chi_{s,1})=n+1$ and the lengths are $l(d_1)=1$, $l(d_{2i})=4i$ (thus $d_{2i}=0$ if $4i>n$), $l(\chi_{s,0})=n-s$, $l(\chi_{s,1})=n-s+1$. The differentials are given by

$$\begin{aligned}
&dd_1=dd_{2i}=0 \qquad \left(i=1, 2, \cdots, \left[\frac{n}{4}\right]\right),\\
&d\chi_{s,0}=-s\chi_{s-1,1}-2(m-s)\chi_{s,1}\,,\\
&d\chi_{s,1}=0\,.
\end{aligned} \tag{4.5}$$

In particular $\chi_{s,1}$ $(s=1, \cdots, m-1)$ is the d-image of a linear combination of $\chi_{s,0}$ $(s=0, \cdots, m)$ and $\chi_{0,0}$, χ are the only d-closed elements among linear combinations of $\chi_{s,0}$. Now by applying Weyl's theorem on the $SO(n)$-invariants [27] (see also [20] §19) and by studying the action of $L(I)$, we conclude that the algebra $\bigoplus_p M_p^{\mathfrak{co}(n)}$ is multiplicatively generated by $d_1, d_2, \cdots d_{2[\frac{n}{4}]}$, $\chi_{s,0}$ $(s=0, \cdots, m)$ and $\chi_{s,1}$ $(s=1, \cdots, m-1)$. Among these generators, there are the following relations.

$$
(4.6)\qquad \begin{aligned} &\text{(i)}\quad \chi_{0,0}^2 =(-1)^{\frac{m(m-1)}{2}}2^n m!d_1^n\,,\\ &\text{(ii)}\quad d_1\chi_{s,t}=0 \qquad \text{for all } s,\ t\,. \end{aligned}
$$

(i) is easy to check. We prove (ii). Since d_1 is closed and $\chi_{s,1}$ is the d-image of a linear combination of $\chi_{s,0}$, it is enough to prove $d_1\chi_{s,0}=0$ for $s=0, \cdots, m$. If $s=m$, then clearly $d_1\chi_{m,0}=0$. So we assume that $s\neq m$. For each $l=1, \cdots, n$, let

$$
K^l=\sum_{i,j,k}\operatorname{sgn}(i,\ j,\ k)\omega^{i(1)}\cdots\omega^{i(s)}\omega_{j(1)}\cdots\omega_{j(s+1)}\Omega_{k(2)}^{k(1)}\cdots\Omega_{k(2t)}^{k(2t-1)}\,,
$$

where $i=(i(1), \cdots, i(s))$, $j=(j(1), \cdots, j(s+1))$, $k=(k(1), \cdots, k(2t))$ and the sum ranges over all $(i,\ j,\ k)$ such that $\{i(1), \cdots, i(s), j(1), \cdots, j(s+1), k(1), \cdots, k(2t)\}=\{1, \cdots, l-1, l+1, \cdots, n\}$ and $i(1)<\cdots<i(s)$, $j(1)<\cdots<j(s+1)$, $k(1)<k(3)<\cdots<k(2t-1)$, $k(2p-1)<k(2p)$ $(p=1, \cdots, t=m-s-1)$. $\operatorname{sgn}(i,\ j,\ k)$=sign of the permutation $\begin{pmatrix}1 \cdots & l-1\,l+1 \cdots & n\\ i(1)\cdots i(s)j(1)\cdots j(s+1)k(1)\cdots k(2t)\end{pmatrix}$. Here in the description of K^l and also in the following calculation, we omit the symbol $\wedge$ for simplicity. Now for each $m(\neq l)$, K^l can be expressed as

$$
(4.7)\qquad K^l=\omega_m L_m^l+\omega^m M_m^l+\sum_{k\neq l,m}N_{mk}^l\Omega_k^m
$$

for some L_m^l, M_m^l, N_{mk}^l. Then calculation shows

$$
(4.8)\qquad d_1\chi_{s,0}=2^{m-s}(m-s)!\,(s!)^2\sum_{l=1}^n(-1)^{l+1}\Big(\sum_{m\neq l}\omega_m L_m^l\Omega_m^l\omega^m\Big)\,.
$$

Now the expression

$$
(4.9)\qquad P_{lm}=\sum_p\Big(\sum_{k\neq l,m}(-1)^{l+1}N_{mk}^l\Omega_k^p\Omega_m^l\omega^p\Big)
$$

is contained in the ideal I (see (3.1) (iii)) and hence is 0 in $\widetilde{W}(\mathfrak{so}(n+1,\ 1))$. We have

$$
(4.10)\qquad \begin{aligned} P_{lm}&=\sum_{k\neq l,m}(-)^{l+1}N_{mk}^l\Omega_k^m\Omega_m^l\omega^m+\sum_{p\neq m}\Big(\sum_{k\neq l,m}(-1)^{l+1}N_{mk}^l\Omega_l^m\Omega_p^k\omega^p\Big)\\ &=\sum_{k\neq l,m}(-1)^{l+1}N_{mk}^l\Omega_k^m\Omega_m^l\omega^m+\sum_{\substack{k\neq l,m\\ p\neq m}}(-1)^k N_{ml}^k\Omega_l^m\Omega_p^k\omega^p\,, \end{aligned}
$$

here we have used the relation $N_{mk}^l=(-1)^{l-k+1}N_{ml}^k$. Combining (4.8) and (4.10), we obtain

$$
(4.11)\qquad \begin{aligned} d_1\chi_{s,0}&=d_1\chi_{s,0}+\sum_{l,m}P_{lm}\\ &=2^{m-s}(m-s)!\,(s!)^2\sum_{l=1}^n\Big[\sum_{m\neq l}\{(-1)^{l+1}\omega_m L_m^l\\ &\qquad+(-1)^{l+1}\sum_{k\neq l,m}N_{mk}^l\Omega_k^m+\sum_{\substack{p\neq l,m\\ q\neq l}}(-1)^l N_{pq}^l\Omega_q^p\}\,\Omega_m^l\omega^m\Big]\,. \end{aligned}
$$

Since $\sum_{p,q\neq l}(-1)^l N^l_{pq}\Omega^p_q$ is independent of m, we have

$$(4.12)\qquad d_1\chi_{s,0}=2^{m-s}(m-s)!\,(s!)^2\sum_{l=1}^n\{\sum_{m\neq l}(-1)^{l+1}\omega_m L^l_m +2(-1)^{l+1}\sum_{k\neq l,m}N^l_{mk}\Omega^m_k\}\,\Omega^l_m\omega^m\,.$$

From this we conclude

$$2d_1\chi_{s,0}=2^{m-s}(m-s)!\,(s!)^2\sum_{l=1}^n\sum_{m\neq l}2K^l\Omega^l_m\omega^m =0\,.$$

This proves (ii). Now it is not difficult to see that (4.6) is the only relation among the generators other than the truncation by the length. (We omit the proof because it consists of tedious calculations). Thus we have determined the algebra $\oplus M_p^{\mathfrak{co}(n)}$. This algebra has a differential $d: M_p^{\mathfrak{co}(n)}\to M_{p+1}^{\mathfrak{co}(n)}$ which is the restriction of that of $\widetilde{W}(\mathfrak{so}(n+1,1))$ and the following diagram is commutative.

$$\begin{array}{ccc} E_1^{p,q} & \cong H^q(\mathfrak{co}(n))\otimes M_p^{\mathfrak{co}(n)} \\ \downarrow d_1 & \downarrow (-1)^q 1\otimes d \\ E_1^{p+1,q} & \cong H^q(\mathfrak{co}(n))\otimes M_{p+1}^{\mathfrak{co}(n)}\,. \end{array}$$

Now we define elements $k_1, k_2, k_4, \cdots$ and if n is even $k_\chi\in\widetilde{W}(\mathfrak{so}(n+1,1))$ as follows. k_1 is defined to be $1/n\sum_i\omega^i_i$ so that $dk_1=d_1$. Let $\pi: W(\mathfrak{so}(n+1,1))\to\widetilde{W}(\mathfrak{so}(n+1,1))$ be the projection and let f_i be the invariant polynomial of $SO(n+1,1)$ defined by $f_i(X)=\mathrm{Trace}\,(X^i)$ for $X\in\mathfrak{so}(n+1,1)$. If we consider that f_i is an element of $W(\mathfrak{so}(n+1,1))$, then it can be seen that $\pi(f_{2i})=d_{2i}$. (Recall that we have killed Trace (Ω^i_j), see (3.1) (v)). Now let $Tf_i\in W(\mathfrak{so}(n+1,1))$ be the Chern-Simons' transgression form of f_i (cf. [6]). We set

$$(4.13)\qquad k_{2i}=\pi(Tf_{2i})\,.$$

Then clearly $dk_{2i}=d_{2i}$. Next we define k_χ for even n. Let $W(\mathfrak{so}(n))$ be the Weil algebra of $\mathfrak{so}(n)$. We define a map $g: W(\mathfrak{so}(n))\to W(\mathfrak{so}(n+1,1))$ by $g(\bar\omega^i_j)=\omega^i_j$ and $g(\bar\Omega^i_j)=-\omega^i\wedge\omega_j-\omega_i\wedge\omega^j+\Omega^i_j$, where $\bar\omega^i_j$, $\bar\Omega^i_j$ are the universal connection and curvature forms of $\mathfrak{so}(n)$ expressed in terms of the natural basis of $\mathfrak{so}(n)$. Then it can be checked that g is actually a d.g.a. map. Let $\bar\chi\in W(\mathfrak{so}(n))$ be the Euler class and let $T\bar\chi\in W(\mathfrak{so}(n))$ be the transgression form of $\bar\chi$. By definition of χ, we have $g(\bar\chi)=c\chi$ ($c=\dfrac{(-1)^m}{2^{n}\,{}^m m!}$, see (4.4)). Now we define $k_\chi=\pi\Big(g\Big(\dfrac{1}{c}T\bar\chi\Big)\Big)$. Then clearly we have $dk_\chi=\chi$.

With the above preparation, we define a d. g. a. CW_n as to be the subalgebra of $\widetilde{W}(\mathfrak{so}(n+1, 1))$ generated by the elements k_i, k_χ, d_i and $\chi_{s,t}$. Thus we can express CW_n as follows.

$$
\begin{aligned}
CW_n &= E(k_1, k_2, k_4, \cdots, k_{n-1})\otimes\hat{\boldsymbol{R}}[d_1, d_2, d_4, \cdots, d_{2[\frac{n}{4}]}] && n \text{ odd}, \\
(4.14) \qquad &= E(k_1, k_2, k_4, \cdots, k_{n-2}, k_\chi)\otimes\hat{\boldsymbol{R}}[d_1, d_2, d_4, \cdots, d_{2[\frac{n}{4}]}, \chi_{0,0}, \\
&\qquad \cdots, \chi_{m,0}, \chi_{1,1}, \cdots, \chi_{m-1,1}] && n \text{ even},
\end{aligned}
$$

where E denotes the exterior algebra. $\boldsymbol{R}[\ ]$ is the polynomial algebra and $\hat{\boldsymbol{R}}[\ ]$ denotes the quotient algebra of $\boldsymbol{R}[\ ]$ modulo an ideal J described as follows. J for odd n is the ideal generated by (i) elements whose length$>n$ and J for even n is the one generated by (i) and (ii) $\chi_{0,0}^2-(-1)^{\frac{m(m-1)}{2}}2^n m!\, d_1^n$ (iii) $d_1\chi_{s,0}$ $(s=0, \cdots, m)$ and $d_1\chi_{s,1}$ $(s=1, \cdots, m-1)$, (see (4.6)). Then by a spectral sequence argument using the above results, we obtain

THEOREM 4.1. $H^*(\widetilde{W}(\mathfrak{so}(n+1, 1)))\cong H^*(CW_n)$.

The d. g. a. CW_n plays a similar role as the complex W_n (cf. [4]) does for the characteristic classes of smooth foliations. For even n, CW_n is rather complicated. However we note that it has a simple subcomplex $CW_n'=E(k_1, k_2, k_4, \cdots, k_{n-2}, k_\chi)\otimes\hat{\boldsymbol{R}}[d_1, d_2, d_4, \cdots, d_{2[\frac{n}{4}]}, \chi]$.

The above calculation can also be done for the general case (when the normal bundle is not trivial) and the projective case. Since this calculation is similar to and even simpler than the conformal case with trivial normal bundles, we only state the results. We define various d. g. a.'s CWO_n, $CWSO_n$, PW_n, PWO_n and $PWSO_n$ as follows.

$$
\begin{aligned}
CWO_n &= E(k_1)\otimes\hat{\boldsymbol{R}}[d_1, d_2, d_4, \cdots, d_{2[\frac{n}{4}]}], \\
CWSO_n &= CWO_n && n \text{ odd}, \\
&= E(k_1)\otimes\hat{\boldsymbol{R}}[d_1, d_2, d_4, \cdots, d_{2[\frac{n}{4}]}, \chi_{0,0}, \cdots, \chi_{m,0}, \\
&\qquad \chi_{1,1}, \cdots, \chi_{m-1,1}] && n \text{ even}, \\
(4.15) \qquad PW_n &= E(k_1, k_2, \cdots, k_n)\otimes\hat{\boldsymbol{R}}[d_1, d_2, \cdots d_{[\frac{n}{2}]}], \\
PWO_n &= E(k_1, k_3, \cdots k_{2t+1})\otimes\hat{\boldsymbol{R}}[d_1, d_2, \cdots, d_{[\frac{n}{2}]}], \\
PWSO_n &= PWO_n && n \text{ odd}, \\
&= PWO_n(\chi)/(\chi^2-d_1^n) && n \text{ even},
\end{aligned}
$$

where CWO_n, $CWSO_n$ are considered to be subcomplexes of CW_n and k_i, d_i for the projective case are defined similarly as in the conformal case; $d_1=\sum\omega_k\wedge\omega^k$,

$d_i=\text{Trace}\,((\Omega^i_j)^i)$ $(i=2,3,\cdots,n)$, $k_1=\dfrac{1}{n+1}\sum \omega^i_i$ so that $dk_1=d_1$, $k_i=\pi(Tf_i)$ $(i=2, 3,\cdots,n)$, where $\pi: W(\mathfrak{sl}(n+1\,;\,\boldsymbol{R}))\to\widetilde{W}(\mathfrak{sl}(n+1\,;\,\boldsymbol{R}))$ is the projection and $f_i\in W(\mathfrak{sl}(n+1\,;\,\boldsymbol{R}))$ is the invariant polynomial of $SL(n+1\,;\,\boldsymbol{R})$ defined by $f_i(X)=\text{Trace}\,(X^i)$ for $X\in\mathfrak{sl}(n+1\,;\,\boldsymbol{R})$. Thus $dk_i=d_i$ for $i=2,\cdots,n$. $\hat{\boldsymbol{R}}[\ \ ]$ in PW's denotes truncation by the ideal generated by elements whose length$>n$ and $2t+1$ in PWO_n is the greatest odd integer$\leqq n$. With these understood we have

THEOREM 4.2.
$$H^*(\widetilde{W}(\mathfrak{so}(n+1,1)))_{G(n)})\cong H^*(CWG_n)\,,$$
$$H^*(\widetilde{W}(\mathfrak{sl}(n+1\,;\,\boldsymbol{R})))\cong H^*(PW_n)\,,$$
$$H^*(\hat{W}(\mathfrak{sl}(n+1\,;\,\boldsymbol{R}))_{G(n)})\cong H^*(PWG_n)\,,$$
were $G=O$ or SO.

Combining the results in §3 and Theorems 4.1, 4.2, we have constructed characteristic classes

$$\begin{aligned} &H^*(SW_n) \longrightarrow H^*(B\bar{S\Gamma}_n\,;\,\boldsymbol{R})\,,\\ \Phi:\ &H^*(SWO_n) \longrightarrow H^*(BS\Gamma_n\,;\,\boldsymbol{R})\,,\\ &H^*(SWSO_n) \longrightarrow H^*(BS\Gamma^+_n\,;\,\boldsymbol{R})\,,\end{aligned}$$

where $S=C$ or P.

By using the analysis of Vey (see [9]), we can determine generators of the cohomology of each d. g. a. defined above as follows. Let $I=(i(1),\cdots,i(s))$ and $J=(j(1),\cdots,j(t))$ be s- and t-tuples of positive integers with $i(1)<\cdots<i(s)$ and $j(1)\leqq\cdots\leqq j(t)$. We denote k_Id_J for $k_{i(1)}\cdots k_{i(s)}d_{j(1)}\cdots d_{j(t)}$ and $l(i,J)$ for $l(d_id_J)$ (we understand that $l(d_id_J)=\infty$ if $d_id_J=0$). Then we have

PROPOSITION 4.3. (I) *A basis for $H^*(CW_n)$ (n: odd) is given by the classes of $k_Id_J\in CW_n$ which satisfy $l(i(1),J)>n$ and $i(1)\leqq j(1)$ if $J\neq\emptyset$.*

(I)′ *$H^*(CW'_n)$ ($n=2m$) is generated (as an $\boldsymbol{R}$-vector space) by the following elements.*

(i) *the classes of $k_Id_J\in CW'_n$ which satisfy $l(i(1),J)>n$ and $i(1)\leqq j(1)$ if $J\neq\emptyset$.*

(ii) *the classes of $(rk_1d_1^{n-1}-k_\chi\chi)k_Id_J\in CW'_n$ which satisfy $1\notin J$, $i(1)\leqq j(1)$ if $J\neq\emptyset$ and $l(i(1),J)>m$. (Here $r=(-1)^{\frac{m(m-1)}{2}}2^nm!$ so that $\chi^2=rd_1^n$).*

(iii) *the classes of $k_Id_J\chi\in CW'_n$ which satisfy $1\notin J$, $i(1)\leqq j(1)$ if $J\neq\emptyset$, and $d(k_Id_J\chi)=0$.*

(iv) *the classes of $k_Ik_\chi d_J\in CW'_n$ which satisfy $i(1)\leqq j(1)$ and $d(k_Ik_\chi d_J)=0$.*

(II) *A basis for $H^*(CWO_n)$ is given by the following elements.*

(i) *the classes of $k_Id_J\in CWO_n$ with $l(1,J)>n$.*

(ii) *the classes of $d_J\in CWO_n$ with $1\notin J$.*

(III) *A basis for* $H^*(CWSO'_n)$ $(n=2m)$ *is given by the following elements.* ($CWSO'_n$ *is defined to be* $CW'_n \cap CWSO_n$).

(i) *the classes of* $k_1 d_J \in CWSO'_n$ *with* $l(1, J) > n$.

(ii) *the classes of* $k_1 d_J \chi \in CWSO'_n$ *with* $J=\emptyset$ *or* $1 \notin J$ *and* $l(1, J) > m$.

(iii) *the classes of* d_J *and* $d_J \chi \in CWSO'_n$ *with* $1 \notin J$.

REMARK 4.4. The classes in (I)$'$ are not linearly independent due to the relations $d_1 \chi = 0$ and $\chi^2 = r d_1^n$. For example the class of $k_1 k_{i(2)} \cdots k_{i(s)} d_1^n$ is zero if $2i(2) > m$.

PROPOSITION 4.5. *Bases for* $H^*(PW_n)$, $H^*(PWO_n)$ *and* $H^*(PWSO_n)$ *are given as follows.*

(I) $H^*(PW_n)$; *the classes of* $k_I d_J \in PW_n$ *which satisfy* $l(i(1), J) > n$ *and* $i(1) \leqq j(1)$ *if* $J \neq \emptyset$.

(II) $H^*(PWO_n)$; (i) *the classes of* $k_I d_J \in PWO_n$ *which satisfy* $l(i(1), J) > n$ *and* $i(1) \leqq j(l)$ ($j(l)$ *is the smallest odd integer in* J).

(ii) *the classes of* $d_J \in PWO_n$ *such that* $j(1), \cdots, j(t)$ *are all even.*

(III) $H^*(PWSO_n)$ $(n=2m)$; (i) *the classes in* (II).

(ii) *the classes of* $k_I d_J \chi \in PWSO_n$ *which satisfy* $l(i(1), J) > n$ *and* $i(1) \leqq j(l)$.

(iii) *the classes of* d_J *and* $d_J \chi \in PWSO_n$ *such that* $j(1), \cdots, j(t)$ *are all even.*

5. Examples.

In this section, we study some examples of conformal and projective foliations and determine our characteristic classes of them. These examples have been investigated by several authors (cf. [11] [13] [28] [29]).

EXAMPLE 5.1. Let L/L_0 be as in Example 1.1. Then the projection $\pi: L \to L/L_0 = S^n$ defines a codimension n conformal foliation F on L, namely F is the pull back under the map π of the natural conformal structure on S^n. The conformal normal bundle of F is just $\pi^*(L) = \{(x, y) \in L \times L\,;\ \pi(x) = \pi(y)\}$ and we have a commutative diagram

$$\begin{tikzcd}
\pi^*(L) \arrow[r, "p_2"] \arrow[d, "p_1"] & L \arrow[d, "\pi"] \\
L \arrow[r, "\pi"] & S^n ,
\end{tikzcd}$$

where p_i $(i=1, 2)$ is induced from the i-th projection $L \times L \to L$. Let $s: L \to \pi^*(L)$ be a cross section defined by $s(x) = (x, x)$ $(x \in L)$, then clearly $p_2 \circ s = \mathrm{id}$. Now the foliation F is invariant under the left action of L. Therefore if $\Gamma \subset L$ is a torsionfree discrete subgroup with compact quotient, then F induces a codimension n conformal foliation on the compact manifold $\Gamma \backslash L$. The conformal normal bundle of this foliation is also trivial since s is L-equivariant. Thus we have the characteristic classes $\Phi: H^*(CW_n)\to H_{DR}^*(M)$. We will determine this Φ. Let $\Omega^*(\mathfrak{l})$ be the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{l}$. Then it can be identified with the differential complex $\Omega_{Inv}^*(L)$ of left invariant differential forms on L. We have an inclusion $\xi: \Omega^*(\mathfrak{l}) =\Omega_{Inv}^*(L)\to\Omega^*(M)$. The induced homomorphism $\xi_*: H^*(\mathfrak{l})\to H_{DR}^*(M)$ is injective. Now let $\eta: CW_n\to\Omega^*(\mathfrak{l})$ be the d. g. a. map defined to be the composition $CW_n\xrightarrow{i}\hat{W}(\mathfrak{l})\xrightarrow{\pi}\Omega^*(\mathfrak{l})$, where i is the inclusion and π is the projection. Then we have $\Phi=(\xi\circ\eta)_*$ and since ξ_* is injective, we will determine $\eta_*: H^*(CW_n)\to H^*(\mathfrak{l})$. By the definitions of k_i, k_χ and d_1, we have the following.

(5.1)
(i) If n is odd, then the forms $\eta(k_2), \eta(k_4), \cdots, \eta(k_{n-1}), \eta(k_1d_1^n)$ are all closed and the product of all of these elements is the volume form of L.
(ii) If n is even, then the forms $\eta(k_2), \eta(k_4), \cdots, \eta(k_{n-2}), \eta(k_1\chi)$ and $\eta(rk_1d_1^{n-1}-k_\chi\chi)$ are all closed and the product of all of these elements is the volume form of L.

Now clearly (5.1) together with Proposition 4.3 determines η_*.

EXAMPLE 5.2. In the above example, let $K(=SO(n))\subset L_0$ be a maximal compact subgroup of L_0 and let $\pi: L'/K\to L/L_0=S^n$ be the projection (L' is the identity component of L). Then π induces an oriented codimension n conformal foliation on a compact manifold $N=\Gamma\backslash L'/K$ (we may think that N is the unit tangent sphere bundle of a compact $(n+1)$-dimensional manifold with negative constant curvature). We determine the characteristic classes $\Phi: H^*(CWSO_n)\to H_{DR}^*(N)$. Let $\mathfrak{k}$ be the Lie algebra of K and let $\Omega^*(\mathfrak{l}, \mathfrak{k})$ be the Chevalley-Eilenberg complex of $\mathfrak{l}$ relative to the subalgebra $\mathfrak{k}$. Then as before, we have an inclusion $\xi': \Omega^*(\mathfrak{l}, \mathfrak{k})=\Omega_{Inv}^*(L'/K)\to\Omega^*(N)$. The induced homomorphism $\xi'_*: H^*(\mathfrak{l}, \mathfrak{k})\cong H^*(T_1(S^{n+1});\boldsymbol{R})\to H_{DR}^*(N)$ is injective, where $T_1(S^{n+1})$ is the unit tangent sphere bundle of S^{n+1}. Let $\eta': CWSO_n\to\Omega^*(\mathfrak{l}, \mathfrak{k})$ be the d. g. a. map induced from the projection $\widetilde{W}(\mathfrak{l})_K\to\Omega^*(\mathfrak{l}, \mathfrak{k})$. Then we have $\Phi=(\xi'\circ\eta')_*$ and in view of Proposition 4.3, $\eta'_*: H^*(CWSO_n)\to H^*(\mathfrak{l}, \mathfrak{k})$ is determined by the following.

(5.2)
(i) If n is odd, then the class of $\eta'(k_1d_1^n)$ is the generator of $H^{2n+1}(\mathfrak{l}, \mathfrak{k})$.
(ii) If n is even, then the classes of $\eta'(\chi)$, $\eta'(k_1\chi)$ and $\eta'(k_1d_1^n)$ are the generators of $H^n(\mathfrak{l}, \mathfrak{k})$, $H^{n+1}(\mathfrak{l}, \mathfrak{k})$ and $H^{2n+1}(\mathfrak{l}, \mathfrak{k})$ respectively.

Next we consider projective foliations.

EXAMPLE 5.3. Let L/L_0 be as in Example 1.2. Then just the same as in Example 5.1, the projection $\pi: L\to L/L_0=P^n$ defines a codimension n projective foliation on a compact manifold $M=\Gamma\backslash L$ with trivial normal bundle, where $\Gamma\subset L$ is a torsionfree discrete subgroup of L with compact quotient. We determine the characteristic classes $\Phi: H^*(PW_n)\to H_{DR}^*(M)$. Let $\xi: \Omega^*(\mathfrak{l})$

$=\Omega^*_{Inv}(L)\to\Omega^*(M)$ be the inclusion and let $\eta: PW_n\to\Omega^*(\mathfrak{l})$ be the d. g. a. map induced from the projection $\widetilde{W}(\mathfrak{l})\to\Omega^*(\mathfrak{l})$. Then we have $\Phi=(\xi\circ\eta)_*$, ξ_* is injective in this case also and η_* is determined by the following.

(5.3) The forms $\eta(k_2), \cdots, \eta(k_n), \eta(k_1d_1^n)$ are all closed and the product of all of these elements is the volume form of L.

This follows from the definition of k_i and d_1.

EXAMPLE 5.4. In the above Example, let $K(=SO(n))\subset L_0$ be a maximal compact subgroup of L_0 and let $\pi: L/K\to L/L_0=P^n$ be the projection. Then π induces an oriented codimension n projective foliation on a compact manifold $N=\Gamma\backslash L/K$. We determine the characteristic classes $\Phi: H^*(PWSO_n)\to H^*_{DR}(N)$ of this foliation. Let $\xi': \Omega^*(\mathfrak{l}, \mathfrak{k})\to\Omega^*(N)$ be the inclusion and let $\eta': PWSO_n\to\Omega^*(\mathfrak{l}, \mathfrak{k})$ be the d. g. a. map induced from the projection $\widetilde{W}(\mathfrak{l})_K\to\Omega^*(\mathfrak{l}, \mathfrak{k})$. Then $\Phi=(\xi'\circ\eta')_*$ and ξ'_* is injective. We have

(5.4)
- (i) If n is odd, then the forms $\eta'(k_3), \eta'(k_5), \cdots, \eta'(k_n), \eta'(k_1d_1^n)$ are all closed and the product of all of these elements is the volume form of L/K.
- (ii) If n is even, then the forms $\eta'(k_3), \eta'(k_5), \cdots, \eta'(k_{n-1}), \eta'(\chi), \eta'(k_1d_1^n)$ are all closed and the product of all of these elements is the volume form of L/K.

This determines η'_*.

6. Relation with Riemannian case.

Let F be a codimension n Riemannian foliation on a smooth manifold M and let $O(F)$ be the orthonormal frame bundle of F. It is a subbundle of $J_2(F)$. If we restrict the $\mathfrak{gl}(n; \boldsymbol{R})$-component of the canonical form on $J_2(F)$ to $O(F)$, then we obtain an $\mathfrak{so}(n)$-valued 1-form θ^i_j on $O(F)$. This defines a d. g. a. map

$$\phi: W(\mathfrak{so}(n)) \longrightarrow \Omega^*(O(F)).$$

Now recall that we denote $\bar{\omega}^i_j$, $\bar{\Omega}^i_j$ for the universal connection and curvature forms of $\mathfrak{so}(n)$. Then ϕ has a kernel I: the ideal of $W(\mathfrak{so}(n))$ generated by the elements $\Omega^{i(1)}_{j(1)}\cdots\Omega^{i(l)}_{j(l)}$ with $l>\frac{n}{2}$. Thus we have a map $\phi: \widetilde{W}(\mathfrak{so}(n))\to\Omega^*(O(F))$, where $\widetilde{W}(\mathfrak{so}(n))=W(\mathfrak{so}(n))/I$. Now if there is given a cross section $s: M\to O(F)$, then we have $H^*(\widetilde{W}(\mathfrak{so}(n))\longrightarrow H^*_{DR}(O(F))\xrightarrow{s^*} H^*_{DR}(M)$. Since this construction is functorial, we obtain a homomorphism:

$$\Phi: H^*(\widetilde{W}(\mathfrak{so}(n))) \longrightarrow H^*(BR\bar{\Gamma}_n: \boldsymbol{R})$$

where $BR\bar{\Gamma}_n$ is the classifying space for codimension n Riemannian Haefliger structures with trivial normal bundles. This is the definition of characteristic classes of Riemannian foliation due to Kamber and Tondeur (see [12], $\widetilde{W}(\mathfrak{so}(n)) = W(\mathfrak{so}(n))_n$ in their terminology). It is also equivalent to the definition given by Lazarov and Pasternack in [16]: Let $\bar{f}_{2k} \in I(SO(n))$ be defined by $\bar{f}_{2k}(X) = \mathrm{Trace}\,(X^{2k})$ for $X \in \mathfrak{so}(n)$ and let $T\bar{f}_{2k} \in W(\mathfrak{so}(n))$ be the transgression form of $\bar{f}_{2k}$. We also consider $\bar{f}_{2k}$ to be an element of $W(\mathfrak{so}(n))$. Let $c_{2k} = \pi(\bar{f}_{2k})$, $h_{2k} = \pi(T\bar{f}_{2k})$ and let $h_{\bar{\chi}} = \pi(T\bar{\chi})$, where $\bar{\chi}$ is the Euler form and $\pi: W(\mathfrak{so}(n)) \to \widetilde{W}(\mathfrak{so}(n))$ is the projection. Let RW_n be the subcomplex of $\widetilde{W}(\mathfrak{so}(n))$ generated by h_{2k}, c_{2k} and if n is even, also by $h_{\bar{\chi}}$, $\bar{\chi}$. We may write

$$
\begin{aligned}
RW_n &= E(h_2, h_4, \cdots, h_{n-1}) \otimes \hat{\boldsymbol{R}}[c_2, c_4, \cdots, c_{n-1}], && n \text{ odd}, \\
&= E(h_2, h_4, \cdots, h_{n-2}, h_{\bar{\chi}}) \otimes \hat{\boldsymbol{R}}[c_2, c_4, \cdots, c_{n-2}, \bar{\chi}], && n \text{ even},
\end{aligned}
$$

thus RW_n is essentially the differential complex defined by Lazarov and Pasternack (here we are using the trace classes). The inclusion $RW_n \to W(\mathfrak{so}(n))$ induces an isomorphism on cohomology and we have

PROPOSITION 6.1 (see Proposition 4.1 in [16]). *$H^*(RW_n)$ is generated (as an $\boldsymbol{R}$-vector space) by the classes of*

(I) *$h_I c_J$ with $2(i(1)+|J|)>n$ and $i(1) \leqq j(1)$, for n odd,*

(II) (i) *$h_I c_J$, $h_{\bar{\chi}} h_I c_J$ with $2(i(1)+|J|)>n$ and $i(1) \leqq j(1)$,*

(ii) *$h_I \bar{\chi}$, $h_{\bar{\chi}} h_I \bar{\chi}$,* for n even.

Here we are using a similar notation as before and $|J| = \sum j(l)$.

Now let $\mathfrak{l} = \mathfrak{so}(n+1, 1)$ or $\mathfrak{sl}(n+1; \boldsymbol{R})$ and let $i: \mathfrak{so}(n) \to \mathfrak{l}$ be the natural inclusion. It induces a d. g. a. map $i^*: W(\mathfrak{l}) \to W(\mathfrak{so}(n))$ and it is easy to see that i^* sends the truncation ideal of $W(\mathfrak{l})$ to that of $W(\mathfrak{so}(n))$. Hence we have a map $i^*: \widetilde{W}(\mathfrak{l}) \to \widetilde{W}(\mathfrak{so}(n))$. Then the following diagram is commutative:

$$
\begin{array}{ccc}
H^*(\widetilde{W}(\mathfrak{l})) & \xrightarrow{\quad\Phi\quad} & H^*(BS\bar{\Gamma}_n; \boldsymbol{R}) \\
\Big\downarrow{\scriptstyle i^*} & & \Big\downarrow \\
H^*(\widetilde{W}(\mathfrak{so}(n))) & \xrightarrow{\quad\Phi\quad} & H^*(BR\bar{\Gamma}_n; \boldsymbol{R})
\end{array}
\tag{6.1}
$$

where $S=C$ or P according as $\mathfrak{l} = \mathfrak{so}(n+1, 1)$ or $\mathfrak{sl}(n+1; \boldsymbol{R})$ respectively and the vertical arrow on the right hand side is the natural homomorphism induced from the forgetful map $BR\bar{\Gamma}_n \to BS\bar{\Gamma}_n$. The commutativity of (6.1) follows from the definitions of Φ. Here we only remark the following. Let M be a Riemannian manifold and let $P(M) \subset J_2(M)$ be the L_0 principal subbundle of $J_2(M)$ corresponding to the underlying conformal or projective structure of M. Then the Riemannian connection form in the orthonormal frame bundle $O(M)$ of M defines a Cartan connection in $P(M)$ (by enlarging the structure group). In general, this Cartan connection is not normal. However it has the property that Trace(Θ^i_j) restricted to $O(M)$ is zero (Θ^i_j is the $\mathfrak{g}_0$-component of the curvature of this Cartan connection) and any connection on the line segment (in the space of connections) joining this connection and the unique normal Cartan connection has this property. This is enough to prove the commutativity of (6.1).

Now we have the following.

THEOREM 6.2. (i) *The homomorphism* $i^*: H^*(\widetilde{W}(\mathfrak{so}(n+1, 1)))\rightarrow H^*(\widetilde{W}(\mathfrak{so}(n)))$ *is epimorphic, i.e. the characteristic classes of Riemannian foliations defined by Kamber-Tondeur* [12] *and Lazarov-Pasternack* [16] *can be defined already in the conformal context.*

(ii) *The homomorphism* $i^*: H^*(\widetilde{W}(\mathfrak{sl}(n+1\,; \boldsymbol{R})))\rightarrow H^*(\widetilde{W}(\mathfrak{so}(n)))$ *is epimorphic modulo those classes which contain* $\tilde{\chi}$ *or* $h_{\tilde{\chi}}$.

PROOF. (i) We check that each class in Proposition 6.1 is in the image of i^*. If $h_I c_J$ is as in Proposition 6.1, then $k_I d_J$ is closed and $i^*(k_I d_J)=h_I c_J$. Next we assume that n is even. For the classes of $h_{\tilde{\chi}} h_I c_J$, $h_I\tilde{\chi}$, $h_{\tilde{\chi}} h_I\tilde{\chi}$, we consider the forms $k_\chi k_I d_J$, $k_I\chi_{0,0}$, $k_\chi k_I\chi_{0,0}$ (see (4.3) for the definition of $\chi_{0,0}$). Clearly $k_I\chi_{0,0}$ and $k_\chi k_I\chi_{0,0}$ are closed and $i^*(k_I\chi_{0,0})=ch_I\tilde{\chi}$, $i^*(k_\chi k_I\chi_{0,0})=c^2 h_\chi h_I\chi$. (Note that $k_I\chi$ is not necessarily closed. This is the reason why we use $\chi_{0,0}$.) Now we consider the class of $h_{\tilde{\chi}} h_I c_J$. Since $h_{\tilde{\chi}} c_J$ is cohomologous to 0 or $h_I\tilde{\chi}$, in the expression $h_{\tilde{\chi}} h_I c_J$, we may assume that $I\neq\emptyset$. Then the conditions $i(1)\leqq j(1)$ and $2(i(1)+|J|)>n$ imply $2|J|>\frac{n}{2}$ and this in turn implies that $d(k_\chi k_I d_J)=0$. Since $i^*(k_\chi k_I d_J)=ch_{\tilde{\chi}} h_I c_J$, we are done.

(ii) By the same argument as in (i), the classes of $h_I c_J$ is in the image of i^*. In view of Proposition 6.1, this proves (ii). q.e.d.

REMARK 6.3. In [18], we showed that there are other characteristic classes of Riemannian foliations which are not covered by Kamber-Tondeur-Lazarov-Pasternack definition. These classes can not be defined in the conformal nor projective context.

7. Relation with smooth case.

First we recall the construction of characteristic classes of smooth foliations briefly (cf. [2] [3] [4] [11]). Let F be a smooth foliation on a smooth manifold M and let $J_2(F)$ be the 2-jet bundle of F. On $J_2(F)$, we have the canonical form (θ^i, Θ^i_j) and there is defined a d.g.a. map $\phi: W(\mathfrak{gl}(n\,; \boldsymbol{R}))\rightarrow\Omega^*(J_2(F))$ such that $\phi(\bar{\omega}^i_j)=\theta^i_j$ and $\phi(\bar{\Omega}^i_j)=d\theta^i_j+\sum\theta^i_k\wedge\theta^k_j$, where $\bar{\omega}^i_j$, $\bar{\Omega}^i_j$ are the universal connection and curvature forms of $\mathfrak{gl}(n\,; \boldsymbol{R})$ in terms of the natural basis. Now

ϕ has a kernel I: the ideal generated by monomials on Ω^i_j with degree$>2n$. Set $\widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R}))=W(\mathfrak{gl}(n\,;\,\boldsymbol{R}))/I$ and assume that there is given a cross section $s: M\to J_2(F)$. Then we have a homomorphism $\Phi: H^*(\widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R})))\longrightarrow H^*_{DR}(J_2(F)) \xrightarrow{s^*} H^*_{DR}(M)$. Since this construction is functorial, we obtain

$$\Phi: H^*(\widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R})))\longrightarrow H^*(B\Gamma_n\,;\,\boldsymbol{R})\,. \tag{7.1}$$

In the general case when the normal bundle is not necessarily trivial, we have

$$\begin{aligned}\Phi&: H^*(\widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R}))_{O(n)})\longrightarrow H^*(B\Gamma_n\,;\,\boldsymbol{R})\,,\\ \Phi&: H^*(\widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R}))_{SO(n)})\longrightarrow H^*(B\Gamma_n^+\,;\,\boldsymbol{R})\,.\end{aligned} \tag{7.2}$$

These are the characteristic classes of smooth foliations. In view of the constructions above and in §3, we define a d.g.a. map $f_s: W(\mathfrak{gl}(n\,;\,\boldsymbol{R}))\to W(\mathfrak{l})$ as follows, where $s=c$ or p according as $\mathfrak{l}=\mathfrak{so}(n+1,\,1)$ or $\mathfrak{sl}(n+1\,;\,\boldsymbol{R})$ respectively.

$$\begin{aligned}&f_s(\bar{\omega}^i_j)=\omega^i_j \qquad (s=c \text{ or } p)\,,\\ &f_c(\bar{\Omega}^i_j)=-\omega^i\wedge\omega_j-\omega_i\wedge\omega^j+\delta^i_j\sum\omega_k\wedge\omega^k+\Omega^i_j\,,\\ &f_p(\bar{\Omega}^i_j)=-\omega^i\wedge\omega_j+\delta^i_j\sum\omega_k\wedge\omega^k+\Omega^i_j\,.\end{aligned} \tag{7.3}$$

Then it can be checked that f_s commutes with the differentials and it sends the truncation ideal of $W(\mathfrak{gl}(n\,;\,\boldsymbol{R}))$ to that of $W(\mathfrak{l})$. Therefore f_s induces a d.g.a. map $\tilde{f}_s: \widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R}))\to\widetilde{W}(\mathfrak{l})$ and the following diagram is commutative.

$$\begin{array}{ccc} H^*(\widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R}))) & \longrightarrow & H^*(B\Gamma_n\,;\,\boldsymbol{R}) \\ \big\downarrow{\scriptstyle (\tilde{f}_s)_*} & & \big\downarrow \\ H^*(\widetilde{W}(\mathfrak{l})) & \longrightarrow & H^*(BS\Gamma_n\,;\,\boldsymbol{R})\,, \end{array} \tag{7.4}$$

where $S=C$ or P and the vertical arrow on the right hand side is induced from the forgetful map $BS\Gamma_n\to B\Gamma_n$. The map $\tilde{f}_s$ commutes also with the $O(n)$- or $SO(n)$-action. Therefore we have a d.g.a. map $\tilde{f}_s: \widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R}))_{G(n)} \to\widetilde{W}(\mathfrak{l})_{G(n)}$ ($G=O$ or SO) and commutative diagrams.

$$\begin{array}{ccc} H^*(\widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R}))_{O(n)}) & \longrightarrow & H^*(B\Gamma_n\,;\,\boldsymbol{R}) \\ \big\downarrow{\scriptstyle (\tilde{f}_s)_*} & & \big\downarrow \\ H^*(\widetilde{W}(\mathfrak{l})_{O(n)}) & \longrightarrow & H^*(BS\Gamma_n\,;\,\boldsymbol{R})\,, \\ H^*(\widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R}))_{SO(n)}) & \longrightarrow & H^*(B\Gamma_n^+\,;\,\boldsymbol{R}) \\ \big\downarrow{\scriptstyle (\tilde{f}_s)_*} & & \big\downarrow \\ H^*(\widetilde{W}(\mathfrak{l})_{SO(n)}) & \longrightarrow & H^*(BS\Gamma_n^+\,;\,\boldsymbol{R})\,. \end{array} \tag{7.5}$$

In the sequel, we determine the homomorphisms $(\tilde{f}_s)_*$.

Let $s_i \in I(GL(n\,;\,\boldsymbol{R}))$ be given by $s_i(X)=\mathrm{Trace}\,(X^i)$ for $X\in\mathfrak{gl}(n\,;\,\boldsymbol{R})$ and let $u_i=Ts_i$: the transgression form of s_i. Then u_i and s_i can be considered as elements of $W(\mathfrak{gl}(n\,;\,\boldsymbol{R}))$ and we use the same letteres for their images in $\widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R}))$. Let W_n be the subalgebra of $\widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R}))$ generated by u_i and s_i $(i=1, \cdots, n)$. Then we can denote

$$W_n=E(u_1, \cdots, u_n)\otimes\boldsymbol{R}[s_1, \cdots, s_n]$$

and the inclusion $W_n\subset\widetilde{W}(\mathfrak{gl}(n\,;\,\boldsymbol{R}))$ induces an isomorphism on cohomology. We also have similar differential complexes WO_n and WSO_n for $O(n)$- and $SO(n)$-basic elements. (For details, see [4]). Now first of all, we determine $f_s(s_i)$. Recall that $f_i\in I(L)$ was defined as $f_i(X)=\mathrm{Trace}\,(X^i)$ for $X\in\mathfrak{l}$. Let I' be the ideal of $W(\mathfrak{l})$ generated by the elements (3.1), (i) and (iii)-(v). Then we have

LEMMA 7.1. (i) $f_c(s_i)\equiv(n+2-2^i)d_1^i+\binom{i}{2}d_1^{i-2}f_2+\cdots+\binom{i}{i-1}d_1f_{i-1}+f_i(\mathrm{mod}\,I')$,

(ii) $f_p(s_i)\equiv(n+1)d_1^i+\binom{i}{2}d_1^{i-2}f_2+\cdots+\binom{i}{i-1}d_1f_{i-1}+f_i\ (\mathrm{mod}\,I')$.

PROOF. First we note that if $X\in\mathfrak{co}(n)\subset\mathfrak{so}(n+1, 1)$, then

$$\begin{aligned} f_k(X)&=\mathrm{Trace}\,\{(X-\frac{1}{n}\,\mathrm{Trace}\,(X)I_n)^k\}+2\Big(\frac{1}{n}\,\mathrm{Trace}\,(X)\Big)^k, &&\text{for } k \text{ even},\\ &=0, &&\text{for } k \text{ odd}. \end{aligned}$$

Therefore we obtain

$$f_k\equiv\mathrm{Trace}\,\{(\Omega_j^i)^k\}\quad(\mathrm{mod}\,I'). \tag{7.6}$$

Now if we set $R_j^i=-\omega^i\wedge\omega_j-\omega_i\wedge\omega^j+\delta_j^id_1$ (thus $f_c(\bar{\Omega}_j^i)=R_j^i+\Omega_j^i$) and also $R=(R_i^i)$ and $\Omega=(\Omega_j^i)$, then it is easy to show the following by induction on k.

$$(R^k)_j^i=\delta_j^id_1^k+d_1^{k-1}(-\omega^i\wedge\omega_j-(2^k-1)\omega_i\wedge\omega^j). \tag{7.7}$$

Hence we have

$$\mathrm{Trace}\,(R^k)=(n+2-2^k)d_1^k. \tag{7.8}$$

If we set $A=(A_j^i)$ with $A_j^i=\omega_i\wedge\omega^j$, then

$$\Omega R\equiv-\Omega A+d_1\Omega\ (\mathrm{mod}\,I')\quad\text{and}\quad A\Omega\equiv0\ (\mathrm{mod}\,I').$$

Therefore we obtain

$$\mathrm{Trace}\,(R^l\Omega^k)\equiv d_1^l\,\mathrm{Trace}\,(\Omega^k)\ (\mathrm{mod}\,I'). \tag{7.9}$$

Since $f_c(s_i)=\mathrm{Trace}\,\{(R+\Omega)^i\}$, (7.8) and (7.9) prove (i). The projective case (ii) can be proved similarly and we omit it. q. e. d.

In view of Lemma 7.1, we define d.g.a. maps $g_c\colon W_n\to CW_n$ and $g_p\colon W_n\to PW_n$ by

$$\begin{aligned}
&g_c(s_i)=(n+2-2^i)d_1^i+\binom{i}{2}d_1^{i-2}d_2+\cdots+\binom{i}{i-1}d_1d_{i-1}+d_i\,,\\
&g_c(u_i)=(n+2-2^i)k_1d_1^{i-1}+\binom{i}{2}k_1d_1^{i-3}d_2+\cdots+\binom{i}{i-1}k_1d_{i-1}+k_i\,,\\
&g_p(s_i)=(n+1)d_1^i+\binom{i}{2}d_1^{i-2}d_2+\cdots+\binom{i}{i-1}d_1d_{i-1}+d_i\,,\\
&g_p(u_i)=(n+1)k_1d_1^{i-1}+\binom{i}{2}k_1d_1^{i-3}d_2+\cdots+\binom{i}{i-1}k_1d_{i-1}+k_i\,.
\end{aligned}\tag{7.10}$$

Then we have

PROPOSITION 7.2. *The following diagram is commutative:*

$$\begin{array}{ccc}
H^*(W_n) & \cong & H^*(\widetilde{W}(\mathfrak{gl}(n\,;\boldsymbol{R})))\\
\big\downarrow (g_s)_* & & \big\downarrow (\tilde{f}_s)_*\\
H^*(SW_n) & \cong & H^*(\widetilde{W}(\mathfrak{l}))\,,
\end{array}$$

where $S=C$ or P (resp. $s=c$ or p and $\mathfrak{l}=\mathfrak{co}(n+1,1)$ or $\mathfrak{sl}(n+1\,;\boldsymbol{R})$).

PROOF. In general, g_s does not coincide with $\tilde{f}_s$. However using the fact $\tilde{H}^*(W(\mathfrak{l})/I')=0$ (which can be proved similarly as Theorem 4.1), it is easy to see that $g_s(u_i)-\tilde{f}_s(u_i)=dv_i$ for some $v_i\in\widetilde{W}(\mathfrak{l})$. Now by Vey (see [**9**]), we know that the classes of $u_Is_J=u_{i(1)}\cdots u_{i(s)}s_{j(1)}\cdots s_{j(t)}$ with $i(1)<\cdots<i(s)$, $j(1)\leqq\cdots\leqq j(t)$, $i(1)\leqq j(1)$ and $i(1)+|J|=i(1)+\sum j(l)>n$ form a basis for $H^*(W_n)$. Then the assertion is proved by checking that $g_s(u_Is_J)-\tilde{f}_s(u_Is_J)=dv_{I,J}$ for some $v_{I,J}$. q.e.d.

The Vey basis $\{[u_Is_J]\}$ ($[u_Is_J]$ is the class of u_Is_J) is divided into two classes, namely (i) $[u_Is_J]$ with $i(1)+|J|=n+1$ and (ii) $[u_Is_J]$ with $i(1)+|J|>n+1$. The classes in (ii) are called rigid classes and they are invariants of a connected component of the space of foliations (see [**9**]).

Now using the Vey basis, we can state

THEOREM 7.3. *The homomorphism $(\tilde{f}_s)_*\colon H^*(\widetilde{W}(\mathfrak{gl}(n\,;\boldsymbol{R})))\to H^*(\widetilde{W}(\mathfrak{l}))$ is given as follows.*

(I) *Conformal case:* (i) *If $i(1)+|J|=n+1$, then*

$$\begin{aligned}
(\tilde{f}_c)_*([u_Is_J])&=0, \quad \text{if at least one of } u_{i(2)},\cdots,u_{i(s)} \text{ is odd},\\
&=c(I,J)[k_1k_{i(2)}\cdots k_{i(s)}d_1^n], \quad \text{otherwise},
\end{aligned}$$

where $c(I,J)=(n+2-2^{i(1)})\prod(n+2-2^{j(l)})$. Moreover $[k_1k_{i(2)}\cdots k_{i(s)}d_1^n]=0$ if n is even and $i(2)>n/4$.

(ii) *If $i(1)+|J|>n+1$, then $(\tilde{f}_c)_*([u_Is_J])=0$.*

(II) *Projective case:* (i) *If $i(1)+|J|=n+1$, then*

$$(\tilde{f}_p)_*([u_I s_J])=(n+1)^{t+1}[k_1 k_{i(2)}\cdots k_{i(s)}d_1^n].$$

(ii) *If* $i(1)+|J|>n+1$, *then* $(\tilde{f}_p)_*([u_I s_J])=0$.

PROOF. (I) By Proposition 7.2, we have

$$\begin{aligned}(\tilde{f}_c)_*([u_I s_J])&=(g_c)_*([u_I s_J])\\ &=[(g_c)_*(u_I)\Pi\{(n+2-2^{j(l)})d_1^{j(l)}+\binom{j(l)}{2}d_1^{j(l)-2}d_2+\cdots+d_{j(l)}\}]\\ &=\Pi(n+2-2^{j(l)})[(g_c)_*(u_I)d_1^{|J|}]+Q,\end{aligned}$$

where Q is the class of a linear combination of terms like

$$Q_{k,J'}=(g_c)_*(u_I)d_1^{|J|-k}d_{J'}, \qquad \text{with } |J'|=k.$$

Now if $k=|J|$, then $Q_{k,J'}=0$ since $k>n/2$ and $l(d_{J'})=2k$. If $k<|J|$, then $Q_{k,J'}=(1/n)d\{(g_c)_*(u_1 u_I)d_1^{|J|-k-1}d_{J'}\}$, hence $Q=0$ and

$$(\tilde{f}_c)_*([u_I s_J])=\Pi(n+2-2^{j(l)})[(g_c)_*(u_I)d_1^{|J|}].$$

On the other hand, considering the length we have

$$\begin{aligned}(g_c)_*(u_I)d_1^{|J|}&=\Pi\{(n+2-2^{i(l)})k_1 d_1^{i(l)-1}+\cdots+k_{i(l)}\}d_1^{|J|}\\ &=(n+2-2^{i(1)})k_1 k_{i(2)}\cdots k_{i(s)}d_1^{i(1)+|J|-1}+k_{i(1)}\cdots k_{i(s)}d_1^{|J|}\\ &=(n+2-2^{i(1)})k_1 k_{i(2)}\cdots k_{i(s)}d_1^{i(1)+|J|-1}+d(k_1 k_{i(1)}\cdots k_{i(s)}d_1^{|J|-1}).\end{aligned}$$

This proves (I). (II) can be proved similarly. q. e. d.

The map $\tilde{f}_s$: $\widetilde{W}(\mathfrak{gl}(n\,;\boldsymbol{R}))\to\widetilde{W}(\mathfrak{l})$ commutes with the actions of $O(n)$ and $SO(n)$. Therefore it induces $\tilde{f}_s$: $\widetilde{W}(\mathfrak{gl}(n\,;\boldsymbol{R}))_{G(n)}\to\widetilde{W}(\mathfrak{l})_{G(n)}$, where $G=O$ or SO. By a similar argument as above, we obtain the following. Let $\{[u_I s_J],[s_J]\}$ be the Vey basis for $H^*(WO_n)\cong H^*(\widetilde{W}(\mathfrak{gl}(n\,;\boldsymbol{R}))_{O(n)})$ (see [9]).

THEOREM 7.4. *The homomorphism* $(\tilde{f}_s)_*$: $H^*(\widetilde{W}(\mathfrak{gl}(n\,;\boldsymbol{R}))_{O(n)})\to H^*(\widetilde{W}(\mathfrak{l})_{O(n)})$ *is given as follows.*

(I) *Conformal case:* (i) *If* $i(1)+|J|=n+1$, *then*

$$\begin{aligned}(\tilde{f}_c)_*([u_I s_J])&=0, && \textit{for } s>2,\\ &=c(I,J)[k_1 d_1^n], && \textit{for } s=1.\end{aligned}$$

(ii) *If* $i(1)+|J|>n+1$, *then* $(\tilde{f}_c)_*([u_I s_J])=0$.

(iii) $(\tilde{f}_c)_*([s_J])=0$, *for* $|J|>n/2$,

$=[d_J]$, *for* $|J|\leqq n/2$.

(II) *Projective case:* (i) *If* $i(1)+|J|=n+1$, *then*

$$(\tilde{f}_p)_*([u_I s_J])=(n+1)^{t+1}[k_{I'} d_1^n], \qquad where \quad I'=I-\{i(1)\}\ .$$

(ii), (iii) *The same formulas as in the conformal case.*

We omit the oriented case since the formulas for that case are now almost clear. As for the rigid classes of smooth foliations, it follows from Theorems 7.3 and 7.4 that

COROLLARY 7.5. *The rigid classes of smooth foliations are all zero on conformal or projective foliations.*

REMARK 7.6. (i) The statement (iii) in Theorem 7.4 is nothing but the strong vanishing theorem of Nishikawa and Sato [**22**] mentioned in the Introduction.

(ii) Combining the results in §5 and Theorems 7.3 and 7.4, we can read off the characteristic classes (of smooth foliations) of Examples 5.1–5.4. They have been systematically calculated first by Kamber and Tondeur [**11**] [**13**] (see also [**28**] [**29**]).

(iii) Yamato [**29**] has proved Corollary 7.5 under conditions "local homogeneity" and the normal bundle is trivial.

(iv) Recently, Fuchs [**7**] has announced that the characteristic classes defined by Bott-Haefliger and Bernstein-Rosenfeld are all non-trivial. In particular, the rigid classes are non-zero. Corollary 7.5 shows that they are obstructions for a smooth foliation to be conformal or projective.

(v) In principle, the construction and computation in §§3–7 could be done also for foliations associated with those second order G-structures classified by Kobayashi and Nagano [**15**], provided the existence and uniqueness of the normal Cartan connection are established (cf. [**24**]). In fact, recently Takeuchi [**26**] has developed a detailed study of such foliations and in particular he has proved Corollary 7.5 for a wide class of them. We also mention that Nishikawa and Takeuchi [**23**] have proved the strong vanishing theorem for them.

8. Continuous variation.

In this section we study how our characteristic classes behave under deformations of conformal or projective foliations. Let us call an element $x\in H^*(\widetilde{W}(\mathfrak{l}))$ "rigid" if for any differentiable one-parameter family F_t of conformal or projective foliations with trivial normal bundles on a smooth manifold M, $\Phi(x)(F_0)=\Phi(x)(F_1)$ holds. For elements of $H^*(\widetilde{W}(\mathfrak{l})_{SO(n)})$, we define similarly. Now by the same method as in Heitsch [**9**] (see also [**6**]), we have

THEOREM 8.1. *Let $k_I d_J\in\widetilde{W}(\mathfrak{l})$ or $\widetilde{W}(\mathfrak{l})_{SO(n)}$ be a cocycle as in Proposition 4.3 or 4.5 and assume that $1\notin I$, $1\notin J$ and $2(i(1)+|J|)>n+2$. Then the class $[k_I d_J]$ is rigid.*

If I contains 1, then $[k_I d_J]$ may be non-rigid. For example the class $[k_1 d_1^n]$, which is the Godbillon-Vey invariant, may vary continuously under deformations of conformal or projective foliations (except the case when the foliation is conformal of even codimension with trivial normal bundle, in which case we know that the Godbillon-Vey invariant is zero). Thurston has proved that it can vary under deformations of smooth foliations (unpublished). However his construction does not seem to yield examples in conformal nor projective context.

In [**17**], Lazarov and Pasternack have shown that any non zero class in $H^{4m-1}(RW_{4m-2})$ which does not contain h_χ vary continuously, by using a residue formula for isolated zero points of Killing vector fields. As a corollary they obtained a surjective homomorphism

$$\pi_{4m-1}(BR\Gamma_n)\longrightarrow R^{d(m)}\longrightarrow 0\,, \tag{8.1}$$

where $d(m)=\dim H^{4m}(BO(4m-2))$. Since these characteristic classes can be defined already in the conformal or projective context (Theorem 6.2), (8.1) holds for R replaced by C or P.

REMARK 8.2. Recently, Nishikawa [**21**] has obtained a residue formula for projective vector fields and shown that some of the classes in $H^{2m-1}(PWO_{2m-1})$ vary continuously.

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Shigeyuki MORITA
Department of Mathematics
Osaka City University

Present address
Department of Mathematics
College of General Education
University of Tokyo
Meguro, Tokyo 153
Japan