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Quasi-maximal ideals and quasi-primary ideals of weak-*Dirichlet algebras

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Let A be a weak-*Dirichlet algebra of $L^\infty(m)$ and let $H^\infty(m)$ denote the weak-*closure of A in $L^\infty(m)$. Let B be a weak-*closed subalgebra of $L^\infty(m)$ which contains A and let $I_B=\{f\in B\,;\int fg\,dm=0 \ \ g\in B\}$ and let $E^{\mathcal{B}}$ be a conditional expectation for $B\cap\bar{B}$. When $B=H^\infty(m)$, I_B is a maximal ideal and $E^{\mathcal{B}}$ is multiplicative on B. When $B\neq H^\infty$, it is not known whether $E^{\mathcal{B}}$ is always multiplicative on B. It is easy to show that if $E^{\mathcal{B}}$ is multiplicative on B, I_B is a quasi-maximal ideal of B (Definition 2, § 4). It is known ([3]) that when $E^{\mathcal{B}}$ is multiplicative on B, if M is a left continuous invariant subspace for B of $L^\infty(m)$, then $M=\chi_E qB$ for some unimodular q and some characteristic function χ_E in B. We show in this paper that the weak converse is valid, i.e., if any left continuous invariant subspace for B of $L^\infty(m)$ has the form $\chi_E qB$, then I_B is a quasi-maximal ideal of B. Secondly we show that if I is the weak-*closed linear span of functions in $H^\infty(m)$, vanishing on sets of positive measure, then it is a primary ideal of $H^\infty(m)$. When $B\neq H^\infty(m)$, there exist quasi-primary ideals of B (Definition 1, § 3). Thirdly we give the necessary and sufficient conditions for a minimum weak-*closed subalgebra of $L^\infty(m)$ that contains $H^\infty(m)$ properly. And we show that there exists at least one function in $H^\infty(m)$ that is not a weak-*limit of functions, vanishing on sets of positive measure if and only if there exists a minimum weak-*closed subalgebra of $L^\infty(m)$ that contains $H^\infty(m)$ properly.

1. Preliminaries.

Recall that by definition [5] a weak-*Dirichlet algebra, is an algebra of essentially bounded measurable functions on a probability measure space $(X, \mathcal{A}, m)$ such that (i) the constant functions lie in A; (ii) $A+\bar{A}$ is weak-*dense in $L^\infty=L^\infty(m)$ (the bar denotes conjugation); (iii) for all f and g in A, $\int_X fg\,dm=\int_X f\,dm\int_X g\,dm$. The abstract Hardy space $H^p=H^p(m)$, $1\leqq p\leqq\infty$,

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associated with A are defined as follows. For $1\leqq p<\infty$, H^p is the $L^p(=L^p(m))$-closure of A, while H^∞ is defined to be the weak-*closure of A. For $1\leqq p\leqq\infty$, let $H_0^p=\{f\in H^p ; \int_X f dm=0\}$.

A weak-*closed subalgebra B of L^∞, containing A, is called a superalgebra (of A). A weak-*closed ideal of B is called an ideal simply. Let $B_0=\{f\in B ; \int_X f dm=0\}$ and let I_B the largest weak-*closed ideal of B which is contained in B_0. Then $I_B=\{f\in B ; \int_X fg dm=0 \quad g\in B\}=\{f\in L^\infty ; \int_X fg dm=0 \quad g\in B\}$.

For any subset $M\subseteqq L^\infty$ and $1\leqq p<\infty$, denote by $[M]_p$ the norm closed linear span of M in L^p and by $[M]_\infty$ the weak-*closed linear span of M. For any measurable subset E of X, the function χ_E is the characteristic function of E. For any f in L^1, denote by $E(f)$ the support set of f.

Suppose $\mathcal{B}$ is the σ-algebra of measurable subsets E of X for which the characteristic functions χ_E lie in a superalgebra B. Then $\mathcal{B}$ is the σ-subalgebra of A. Let $E^{\mathcal{B}}$ denote the conditional expectation for $\mathcal{B}$. Then $E^{\mathcal{B}}(fE^{\mathcal{B}}(g))=E^{\mathcal{B}}(f)E^{\mathcal{B}}(g)$ for all f and g in L^∞ and so B. When $E^{\mathcal{B}}(fg)=E^{\mathcal{B}}(f)E^{\mathcal{B}}(g)$ for all f and g in B, we say that $E^{\mathcal{B}}$ is multiplicative on B. When $B=H^\infty$ or L^∞, it is clear that $E^{\mathcal{B}}$ is multiplicative on B. In many examples which we know, $E^{\mathcal{B}}$ is multiplicative on every superalgebra B which contains a weak-*Dirichlet algebra A. We don't know that there exists a superalgebra B on which $E^{\mathcal{B}}$ is not multiplicative. We call the measure m is quasi-multiplicative on B if $\int_X f^2 dm=0$ for each f in B such that $\int_E f dm=0$ for all χ_E in B. Theorem 4 in [3] shows that m is quasi-multiplicative on B if and only if $E^{\mathcal{B}}$ is multiplicative on B. Suppose $\mathcal{J}_B=\{f\in B ; E^{\mathcal{B}}(f)=0 \text{ a.e.}\}$, then $I_B\subseteqq\mathcal{J}_B\subseteqq B_0$. If $E^{\mathcal{B}}$ is multiplicative on B, then $I_B=\mathcal{J}_B$. If m is multiplicative on B, then $I_B=\mathcal{J}_B=B_0$.

Recall that by definition [4], we say that the characteristic function χ_E is minimal for a superalgebra B in case any characteristic function χ_F in B which satisfies the strict inequality $\chi_F\lneqq\chi_E\lneqq 1$ on a set of positive measure must be zero a. e. Note that we do not assume that χ_E lies in B. Suppose $I(B)$ is a weak-*closed linear span of functions g in I_B with $\chi_{E(g)}$ being minimal for B. Then $I(B)$ is a ideal of B and $I(B)\subseteqq I_B$. When $B=H^\infty$, $I=I(B)$ is a weak-*closed linear span of functions g in H^∞, vanishing on sets of positive measure.

For $1\leqq p\leqq\infty$, a closed subspace M of L^p (weak-*closed for $p=\infty$) is called invariant if $f\in M$ and $g\in A$ imply that $fg\in M$. Let M and N be invariant subspaces of L^p such that $BM\subseteqq M$, $BN\subseteqq N$ and $M\subseteqq N$. If $\chi_E M\subsetneqq\chi_E N$ for all χ_E in B with $\chi_E M\neq\{0\}$, then we write $M\prec_B N$. If $M\, {}_B\!\succ [I_B M]_p$, then M is called left continuous for B [3]. If $M=\chi_E q[B]_p$ for some unimodular q and some $\chi_E\in B$, then M is called a Beurling subspace.

THEOREM 1. ([3]) *If $E^{\mathscr{B}}$ is multiplicative on B, then every left continuous invariant subspace for B is a Beurling subspace.*

PROOF. Since m is quasi-multiplicative on B because $E^{\mathscr{B}}$ is multiplicative on B, Theorem 2 in [3] implies this theorem.

LEMMA 1. *Suppose every left continuous invariant svbspace for a superalgebra B is a Beurling subspace. Then $I(B)$ is a weak-*closed linear span of functions g in B with $\chi_{E(g)}$ being minimal for B. And if $f\in[B]_2$ and $\chi_E f\notin[I_B]_2$ for every $\chi_E\in B$ with $\chi_E f\neq 0$, then $\chi_{E(f)}\in B$.*

PROOF. By the definition of $I(B)$, it is sufficient to prove that if $f\notin I_B$ is non-zero, then $\chi_{E(f)}$ is not minimal for B. If $f\notin I_B$ is non-zero, set $M_f=[fB]_\infty$, then M_f contains a nontrivial left continuous invariant subspace. By the hypothesis, M_f contains a non-trivial Beurling subspace and hence $\chi_{E(f)}$ is not minimal for B.

If $f\in[B]_2$ and $\chi_E f\notin[I_B]_2$ for every $\chi_E\in B$ with $\chi_E f\neq 0$, then M_f is a left continuous invariant subspace and hence M_f is a Beurling subspace. Thus $\chi_{E(f)}\in B$.

2. The ideals of a superalgebra of A.

For a superalgebra B, we define $B_{\min}$ to be the intersection of all superalgebras $\{B_\alpha\}$ such that $B\subseteqq B_\alpha$ and $\chi_{E_0}B\prec_B\chi_{E_0}B_\alpha$, χ_{E_0} being the essential function of B (cf. [4]). Then $H^\infty_{\min}$ is the intersection of all superalgebras $\{B_\alpha\}$ which contains H^∞ properly.

THEOREM 2. $I(B)=I_{B_{\min}}$.

PROOF. For $f\in I_B$ with $\chi_{E(f)}$ being minimal for B, set $D=[\chi_{E(f)}B]_\infty+(1-\chi_{E(f)})L^\infty$. Then $\chi_{E_0}B\prec_B\chi_{E_0}D$ for the essential function χ_{E_0} of B. For if $\chi_F\in B$ with $\chi_F\leqq\chi_{E_0}$ and $\chi_F B=\chi_F D$, then $\chi_F\leqq\chi_{E(f)}$. It contradicts to $\chi_{E(f)}$ being minimal for B by Lemma 3 in [4]. By Lemma 5 in [4], f belongs to I_D. By the definition of $B_{\min}$, $D\supseteqq B_{\min}$ and so $I_D\subseteqq I_{B_{\min}}$. Thus $f\in I_{B_{\min}}$ and hence $I(B)\subseteqq I_{B_{\min}}$.

To prove that $I_{B_{\min}}\subseteqq I(B)$, it is sufficient to show that $I_{B_\alpha}\subseteqq I(B)$ for a superalgebra B_α with $\chi_{E_0}B\prec_B\chi_{E_0}B_\alpha$. For $B_{\min}=\bigcap_\alpha B_\alpha$ for such B_α and hence $I_{B_{\min}}$ is a weak-*closed linear span of $\bigcup_\alpha I_{B_\alpha}$ by Lemma 1 and Lemma 4 in [4]. To prove that $I_{B_\alpha}\subseteqq I(B)$, set $\beta=\{m(E);\ \chi_E I_{B_\alpha}\subseteqq I(B)$ and $\chi_E\in B_\alpha\}$. Then there exists χ_{E_1} in B_α such that $\beta=m(E_1)$ and $\chi_{E_1}I_{B_\alpha}\subseteqq I(B)$. $1-\chi_{E_1}\in B_\alpha$ and $1-\chi_{E_1}\leqq\chi_{E_0}$. Suppose $1-\chi_{E_1}\neq 0$. When $1-\chi_{E_1}\notin B$, there exists $\chi_{E_2}\in B_\alpha$ such that $\chi_{E_2}\leqq 1-\chi_{E_1}$ and χ_{E_2} is minimal for B. Hence $\chi_{E_2}I_{B_\alpha}\subseteqq I(B)$. This contradicts to the assumption on χ_{E_1}. When $1-\chi_{E_1}\in B$, since $\chi_{E_0}B\prec_B\chi_{E_0}B_\alpha$, $(1-\chi_{E_1})B \subsetneqq (1-\chi_{E_1})B_\alpha$ and so there exists $\chi_{E_3}\in B_\alpha$ such that $\chi_{E_3}\lneqq 1-\chi_{E_1}$ and χ_{E_3} is minimal for B by Lemma 8 in [4]. This contradicts to the assumption on χ_{E_1} again. Thus $1-\chi_{E_1}=0$ a.e. and hence $I_{B_\alpha}\subseteqq I(B)$.

3. The quasi-primary ideal I_B.

It is well-known that H_0^∞ is a primary ideal of H^∞. If a superalgebra B is different from H^∞, then I_B is not a primary ideal of B. For $\chi_E(1-\chi_E)\in I_B$, $\chi_E\notin I_B$ and $1-\chi_E\notin I_B$ where $\chi_E\in B$ and $0\lneqq m(E)\lneqq 1$.

DEFINITION 1. Let J_1 and J_2 be ideals of a superalgebra B with $J_1\supseteqq J_2$ and let $B=\{g\in L^\infty;\ gJ_1\subseteqq J_1\}$. We say that J_2 is a quasi-primary ideal of J_1 when J_2 has the following property; Suppose f, $g\in J_1$ and $fg\in J_2$. If $\chi_E f\notin J_2$ for every $\chi_E\in B$ with $\chi_E f\neq 0$, then $\chi_{E(f)}\in B$ and $\chi_{E(f)}g\in J_2$.

LEMMA 2. *The necessary and sufficient condition for that J_2 is a quasi-primary ideal of J_1 is that J_2 has the following property. If f, $g\in J_1$ and $fg\in J_2$, then $f=\chi_{E_1}f+(1-\chi_{E_1})f$ and $g=\chi_{E_2}g+(1-\chi_{E_2})g$ where χ_{E_1} and χ_{E_2} satisfy the properties* (1)~(4).

(1) $(1-\chi_{E_1})f\in J_2$ *and* $(1-\chi_{E_2})g\in J_2$,

(2) $\chi_E\cdot\chi_{E_1}f\notin J_2$ *for every* $\chi_E\in B$ *with* $\chi_E\cdot\chi_{E_1}f\neq 0$ *and* $\chi_E\cdot\chi_{E_2}g\notin J_2$ *for every* $\chi_E\in B$ *with* $\chi_E\cdot\chi_{E_2}g\neq 0$,

(3) $\chi_{E_1}f\cdot\chi_{E_2}g=0$ a.e.,

(4) χ_{E_1}, $\chi_{E_2}\in B$ *and* $\chi_{E_1}\leqq\chi_{E(f)}$, $\chi_{E_2}\leqq\chi_{E(g)}$.

PROOF. Suppose f, $g\in J_1$ and $fg\in J_2$. Suppose J_2 is a quasi-primary ideal. There exists $\chi_F\in B$ such that $\chi_E\cdot\chi_F f\notin J_2$ for every $\chi_E\in B$ with $\chi_E\cdot\chi_F f\neq 0$ and $(1-\chi_F)f\in J_2$. Since $fg\in J_2$, $\chi_F fg\in J_2$. Since J_2 is a quasi-maximal ideal of J_1, $\chi_{E_1}=\chi_F\cdot\chi_{E(f)}\in B$. Similarly there exists $\chi_{E_2}\in B$ with the properties (1), (2) and (4) for g. Since $fg\in J_2$, $\chi_{E_1}f\cdot\chi_{E_2}g\in J_2$. By the quasi-primarity of J_2, $\chi_{E_1}\cdot\chi_{E_2}g\in J_2$ and so $\chi_{E_1}\cdot\chi_{E_2}g=0$ a.e. This shows that $\chi_{E_1}f\cdot\chi_{E_2}g=0$ a.e. and so (3) is valid.

Suppose χ_{E_1} and χ_{E_2} satisfy the properties (1)~(4). If $\chi_E f\notin J_2$ for every $\chi_E\in B$ with $\chi_E f\neq 0$, then $\chi_{E_1}=\chi_{E(f)}\in B$. Since $\chi_{E_1}f\cdot\chi_{E_2}g=0$ a.e., $\chi_{E(f)}\cdot\chi_{E_2}g=0$ a.e. Since $\chi_{E(f)}(1-\chi_{E_2})g\in J_2$, $\chi_{E(f)}g\in J_2$.

When $B=H^\infty$, if J_2 is a quasi-primary ideal, then J_2 is a primary ideal.

THEOREM 3. *Suppose every left continuous invariant subspace for B is a Beurling subspace. Then I_B is a quasi-primary ideal of B.*

PROOF. Since $[B]_2=[B]_2\cap[\bar{B}]_2\oplus[I_B]_2$, for u, $v\in[B]_2\cap[\bar{B}]_2$ and f_0, $g_0\in[I_B]_2$, $f=u+f_0$ and $g=v+g_0$. Since $B+\bar{I}_B$ is weak-*dense in L^∞ by Lemma 2 in [3] and $fg\in I_B$, it follows that $uv=0$ a.e. By Lemma 1, $\chi_{E(u)}\in B$ and $\chi_{B(v)}\in B$. Since $\chi_E\cdot\chi_{E(u)}f\notin I_B$ for every $\chi_E\in B$ with $\chi_E\cdot\chi_{E(u)}f\neq 0$, by Lemma

1, $\chi_{E_1}=\chi_{E(u)}\cdot\chi_{E(f)}\in B$. Similarly $\chi_{E_2}=\chi_{E(v)}\cdot\chi_{E(g)}\in B$. Then χ_{E_1} and χ_{E_2} satisfy the properties (1)$\sim$(4) in Lemma 2. Thus I_B is a quasi-primary ideal of B.

If $E^{\mathscr{B}}$ is multiplicative on B, then every left continuous invariant subspace for B is a Beurling subspace by Theorem 1 and hence I_B is a quasi-primary ideal of B by Theorem 3. $\{0\}$ is a quasi-primary ideal of L^∞.

4. The quasi-maximal ideal I_B.

It is well-known that H_0^∞ is a maximal ideal of H^∞. If a superalgebra B is different from H^∞, then I_B is not a maximal ideal of B. For $B\supsetneqq\chi_E B+(1-\chi_E)I_B\supsetneqq I_B$ where $\chi_E\in B$ and $0\lneqq m(E)\lneqq 1$, and $\chi_E B+(1-\chi_E)I_B$ is an ideal of B.

DEFINITION 2. Let J be an ideal of a superalgebra B. We say that J is a quasi-maximal ideal of B when any ideal J' of B with $J'\supseteqq J$ has the form

$$J'=\chi_E B+(1-\chi_E)J$$

for some $\chi_E\in B$.

When $B=H^\infty$, if J is a quasi-maximal ideal, then J is a maximal one.

If $E^{\mathscr{B}}$ is multiplicative on B, then I_B is a quasi-maximal ideal of B. For since $E^{\mathscr{B}}$ is multiplicative on B, $I_B=\mathscr{I}_B$. If J is a ideal of B with $J\supseteqq I_B$, $J=E^{\mathscr{B}}(J)+I_B$ and $E^{\mathscr{B}}(J)$ is an ideal of $E^{\mathscr{B}}(B)=B\cap\bar{B}$. Since $B\cap\bar{B}$ is a commutative von Neumann algebra of operators on L^2 which is contained in L^∞, $E^{\mathscr{B}}(J)=\chi_E B\cap\bar{B}$ for some $\chi_E\in B\cap\bar{B}$. In particular, by that $B=B\cap\bar{B}+I_B$, it follows that $J=\chi_E B+(1-\chi_E)I_B$.

THEOREM 4. *Suppose every left continuous invariant subspace for B is a Beurling subspace. Then I_B is a quasi-maximal ideal of B.*

PROOF. Let J be an ideal of B with $J\supseteqq I_B$. Then there exists χ_{E_1} in B such that $J=\chi_{E_1}J+(1-\chi_{E_1})I_B$ and $\chi_{E_1}J_B\succ\chi_{E_1}I_B$ when $\chi_{E_1}J\neq\{0\}$. We may assume $\chi_{E_1}J\neq\{0\}$. Since $\chi_{E_1}I_B\supseteqq[I_B\chi_{E_1}J]_\infty$, $\chi_{E_1}J$ is left continuous and by the hypothesis, $\chi_{E_1}J=\chi_{E_2}qB$ for some unimodular q and some non-zero $\chi_{E_2}\in B$.

We shall show that $\chi_{E_2}q^2B\supseteqq\chi_{E_2}I_B$. If $f\in\chi_{E_2}I_B$, since $\chi_{E_2}qB\supseteqq\chi_{E_2}I_B$, $f=\chi_{E_2}qh$ for some $h\in B$. $\chi_{E_2}q\cdot h\in I_B$. Since $\chi_{E_2}qB\ {}_B{\succ}\chi_{E_2}I_B$, $\chi_E\cdot\chi_{E_2}q\notin\chi_{E_2}I_B$ for any $\chi_E\in B$ with $\chi_E\cdot\chi_{E_2}\neq 0$. By Theorem 3, I_B is a quasi-primary ideal of B and so $\chi_{E_2}h\in I_B$. Since $\chi_{E_2}qB\supseteqq\chi_{E_2}I_B$, this shows $f\in\chi_{E_2}q^2B$.

We shall show that $\chi_{E_2}\bar{q}\in B$ and so $\chi_{E_1}J=\chi_{E_2}B$. Set $J_0=\bigcap_{n=1}^{\infty}\chi_{E_2}q^nB$, then $J_0\supseteqq\chi_{E_2}I_B$. For we can show that $\chi_{E_2}q^nB\supseteqq\chi_{E_2}I_B$ for $n\geqq 3$ similarly as in $n=2$. There exists $\chi_{E_3}\in B$ such that $J_0=\chi_{E_3}J_0+(1-\chi_{E_3})\chi_{E_2}I_B$ and $\chi_{E_3}J_0\ {}_B{\succ}\chi_{E_3}\cdot\chi_{E_2}I_B$ when $\chi_{E_3}J_0\neq\{0\}$. If $\chi_{E_3}\cdot\chi_{E_2}\bar{q}\in B$, since $\bar{q}\chi_{E_3}J_0=\chi_{E_3}J_0$, there exists a non-zero $\chi_{E_4}\notin B$ such that $\chi_{E_4}\cdot\chi_{E_3}J_0\subseteqq\chi_{E_3}J_0$ and $\chi_{E_4}\leqq\chi_{E_3}\cdot\chi_{E_2}$. By Lemma 1, $\chi_{E_4}J_0\subseteqq I(B)\subseteqq I_B$ and so $\chi_{E_4}I_B\subseteqq I_B$. By Lemma 2 in [3], $\chi_{E_4}\in B$. This contradictions that $\chi_{E_4}\notin B$ and so $\chi_E\cdot\chi_{E_2}\bar{q}\notin B$. Since $\bar{q}J_0=J_0$ and so $(1-\chi_{E_3})\chi_{E_2}\bar{q}I_B\subseteqq I_B$ by Lemma

2 in [3], $(1-\chi_{E_3})\chi_{E_2}\bar{q}\in B$. Thus $\chi_{E_2}\bar{q}=\chi_{E_3}\cdot\chi_{E_2}\bar{q}+(1-\chi_{E_3})\chi_{E_2}\bar{q}\in B$.

It is clear that $\{0\}$ is a quasi-maximal ideal of L^∞.

5. The primary ideal.

If H^∞ is a maximal superalgebra of A, then $I=I(H^\infty)=\{0\}$ is a primary ideal of H^∞ [1]. In this section, we shall show that in general I is a primary ideal of H^∞.

THEOREM 5. *$I(B)$ is a quasi-primary ideal of I_B.*

PROOF. We shall show that if $f\in[I_B]_2\cap[\bar{B}_{\min}]_2$, then $\chi_{E(f)}\in B$. We may assume $\chi_{E(f)}\neq 1$. Suppose $\chi_{E(f)}$ is minimal for B. There exists an outer function h in H^∞ such that $hf\in I_B$. Since $\chi_{E(hf)}$ is minimal for B, $hf\in I(B)$. By Theorem 2, $hf\in I_{B_{\min}}$ and so $f\in[I_{B_{\min}}]_2$. While f belongs to $[I_B]_2\cap[\bar{B}_{\min}]_2$, by Lemma 1 in [4], it follows that $f=0$ a.e. and $\chi_{E(f)}(=0$ a.e.$)\in B$. Suppose $\chi_{E(f)}$ is not minimal for B. Set $\alpha=\sup\{m(E)\,;\chi_E\leqq\chi_{E(f)}$ and $\chi_E\in B\}$, then there exists $\chi_{E_3}\in B$ such that $\chi_{E_3}\leqq\chi_{E(f)}$ and $m(E_3)=\alpha$. Set $g=(1-\chi_{E_3})f$, then $g\in[I_B]_2\cap[\bar{B}_{\min}]_2$, and $\chi_{E(g)}$ is minimal for B. By what was just proved, $\chi_{E(g)}=0$ a.e. and so $\chi_{E(f)}=\chi_{E_3}$.

Now we shall show that $I_{B_{\min}}$ is a quasi-primary ideal of I_B. Then by Theorem 2, $I(B)$ is a quasi-primary ideal of I_B. Suppose f, $g\in I_B$ and $fg\in I_{B_{\min}}$. Since $[I_B]_2=[I_B]_2\cap[\bar{B}_{\min}]_2\oplus[I_{B_{\min}}]_2$, for $u, v\in[I_B]_2\cap[\bar{B}_{\min}]_2$ and $f_0, g_0\in[I_{B_{\min}}]_2$, $f=u+f_0$ and $g=v+g_0$. Since $B_{\min}+\bar{I}_{B_{\min}}$ is weak-*dense in L^∞ by Lemma 2 in [3] and $fg\in I_{B_{\min}}$, it follows that $uv=0$ a.e. By the first proof of this theorem, $\chi_{E(u)}$ and $\chi_{E(v)}$ belong to B. By the definition of $I(B)$, $\chi_{E_1}=\chi_{E(u)}\cdot\chi_{E(f)}$ and $\chi_{E_2}=\chi_{E(v)}\cdot\chi_{E(g)}$ belong to B. Then χ_{E_1} and χ_{E_2} satisfy the properties (1)~(4) in Lemma 2.

COROLLARY 1. *I is a primary ideal of H_0^∞.*

PROOF. Apply Theorem 5 with $B=H^\infty$.

THEOREM 6. *Suppose every left continuous invariant subspace for B is a Beurling subspace. Then $I(B)$ is a quasi-primary ideal of B.*

PROOF. If $I(B)=I_B$, Theorem 3 implies this theorem. Suppose $I(B)\neq I_B$, f, $g\in B$ and $fg\in I(B)$. Then by Lemma 2, Theorem 3 and Theorem 5, there exist $\chi_{F_i}\in B$ and $\chi_{G_i}\in B$ $(i=1, 2, 3)$ with the following properties: (1) $\chi_{F_3}f$, $\chi_{G_3}g\in I(B)$. (2) $\chi_E\cdot\chi_{F_1}f\notin I_B$ for every $\chi_E\in B$ with $\chi_E\cdot\chi_{F_1}f\neq 0$ and $\chi_E\cdot\chi_{G_1}g\notin I_B$ for every $\chi_E\in B$ with $\chi_E\cdot\chi_{G_1}g\neq 0$, and $\chi_{F_1}f\cdot\chi_{G_1}g=0$ a.e. and $\chi_{F_1}\leqq\chi_{E(f)}$, $\chi_{G_1}\leqq\chi_{E(g)}$. (3) $\chi_{F_2}f$, $\chi_{G_2}g\in I_B$, and $\chi_E\cdot\chi_{F_2}f\notin I(B)$ for every $\chi_E\in B$ with $\chi_E\cdot\chi_{F_2}f\neq 0$ and $\chi_E\cdot\chi_{G_2}g\notin I(B)$ for every $\chi_E\in B$ with $\chi_E\cdot\chi_{G_2}g\neq 0$, and $\chi_{F_2}f\cdot\chi_{G_2}g=0$ a.e. and $\chi_{F_2}\leqq\chi_{E(f)}$, $\chi_{G_2}\leqq\chi_{E(g)}$. (4) $\chi_{F_1}+\chi_{F_2}+\chi_{F_3}=1$ and $\chi_{G_1}+\chi_{G_2}+\chi_{G_3}=1$. To prove this theorem, it is sufficient to show that $\chi_{F_2}f\cdot\chi_{G_1}g=0$ a.e. and $\chi_{F_1}f\cdot\chi_{G_2}g=0$ a.e. Since $fg\in I(B)$, $\chi_{F_1}f\cdot\chi_{G_2}g+\chi_{F_2}f\cdot\chi_{G_1}g\in I(B)$. Since $\chi_{F_1}\cdot\chi_{G_1}=0$ a.e., $\chi_{F_1}f\cdot\chi_{G_2}g$

$\in I(B)$ and $\chi_{F_2}f\cdot\chi_{G_1}g\in I(B)$.

We shall show that $\chi_{F_1}f\cdot\chi_{G_2}g=0$ a.e. Suppose $\chi_{F_1}f\cdot\chi_{G_2}g\neq 0$. If $h\in I_B$, then $\chi_{F_1}fh\cdot\chi_{G_2}g\in I(B)$. Since $\chi_{F_1}fh$, $\chi_{G_2}g\in I_B$ and $I(B)$ is a quasi-primary ideal of I_B and $\chi_E\cdot\chi_{G_2}g\notin I(B)$ for every $\chi_E\in B$ with $\chi_E\cdot\chi_{G_2}g\neq 0$, it follows that $\chi_{G_2}\cdot\chi_{F_1}fh \in I(B)$. Thus $\chi_{G_2}\cdot\chi_{F_1}fI_B\subset I(B)$. By Theorem 2, $\chi_{G_2}\cdot\chi_{F_1}f\,B_{\min}I_B\subset I(B)$ and so $\chi_{G_2}\cdot\chi_{F_1}f\,B_{\min}\subset B$. Since we may assume $I(B)\neq I_B$ and so $B_{\min}\supsetneqq B$, $B_{\min}=(1-\chi_{F_4})B+\chi_{F_4}B_{\min}$ for some non-zero $\chi_{F_4}\in B$ and $\chi_{F_4}B_{\min\,B}\succ\chi_{F_4}B$. If $\chi_{F_4}\cdot\chi_{G_2}\cdot\chi_{F_1}f=0$ a.e., since $\chi_{F_1}f\cdot\chi_{G_2}g\neq 0$, then $(1-\chi_{F_4})\chi_{G_2}\cdot\chi_{F_1}f\neq 0$. Since $(1-\chi_{F_4})I_{B_{\min}}=(1-\chi_{F_4})I(B)=(1-\chi_{F_4})I_B$, $(1-\chi_{F_4})\chi_{G_2}\cdot\chi_{F_1}f\in I_B$. This contradicts to the assumption on χ_{F_1} and so $\chi_{F_4}\cdot\chi_{G_2}\cdot\chi_{F_1}f\neq 0$. Let χ_{F_5} be the largest characteristic function in $B_{\min}$ such that $\chi_{F_5}\leqq\chi_{F_4}\cdot\chi_{G_2}\cdot\chi_{F_1}$ and $\chi_{F_5}\cdot\chi_{F_4}\cdot\chi_{G_2}\cdot\chi_{F_1}f\in I(B)$. Since $\chi_{G_2}\cdot\chi_{F_1}f\,B_{\min}\subseteqq B$, $(1-\chi_{F_5})\chi_{F_4}\cdot\chi_{G_2}\cdot\chi_{F_1}f\in B$. By the assumption on χ_{F_1} and χ_{F_5}, $(1-\chi_{F_5})\chi_{F_4}\cdot\chi_{G_2}\cdot\chi_{F_1}f\neq 0$ and $(1-\chi_{F_5})\chi_{F_4}\cdot\chi_{G_2}\cdot\chi_{F_1}f\notin I(B)$. By Lemma 1, $(1-\chi_{F_5})\chi_{F_4}\cdot\chi_{G_2}\cdot\chi_{F_1}$ is not minimal for B. Since $\chi_{F_4}B_{\min\,B}\succ\chi_{F_4}B$, there exist $\chi_{F_6}\in B_{\min}$ such that $0\lneqq\chi_{F_6}<(1-\chi_{F_5})\chi_{F_4}\cdot\chi_{G_2}\cdot\chi_{F_1}$ and χ_{F_6} is minimal for B. By Lemma 1, $\chi_{F_6}\cdot\chi_{F_4}\cdot\chi_{G_2}\cdot\chi_{F_1}f\in I(B)$. This contradicts to the assumption on χ_{F_5}. Thus $\chi_{F_1}f\cdot\chi_{G_2}g=0$ a.e. Similarly $\chi_{F_2}f\cdot\chi_{G_1}g=0$ a.e. Set $\chi_{E_1}=\chi_{F_1}+\chi_{F_2}$ and $\chi_{E_2}=\chi_{G_1}+\chi_{G_2}$. Then χ_{E_1} and χ_{E_2} satisfy the properties in Lemma 2 and hence $I(B)$ is a quasi-primary ideal of B.

COROLLARY 2. *I is a primary ideal of H^∞.*

PROOF. Apply Theorem 6 with $B=H^\infty$, using Theorem 1.

6. The minimum superalgebra of A which contains H^∞ properly.

If $H^\infty_{\min}\neq H^\infty$, then $H^\infty_{\min}$ is the minimum superalgebra which contains H^∞ properly. Under a condition that $H^\infty_{\min}\neq H^\infty$ in [4, Corollary 3], we gave two necessary and sufficient conditions for a minimum superalgebra. In this section, we shall omit the condition such that $H^\infty_{\min}\neq H^\infty$ and we shall gave a new necessary and sufficient condition.

THEOREM 7. *Let B and D be superalgebras of A such that $B\subseteqq D$. Then the following are equivalent.*

(1) *If f is in I_B and $\chi_{E(f)}$ is minimal for B, then f lies in I_D.*

(2) *If f and g are in I_B, if both $\chi_{E(f)}$ and $\chi_{E(g)}$ are minimal for B, and if $fg=0$ a.e., then either f or g lies in I_D.*

(3) *I_D is a quasi-primary ideal of I_B.*

(4) $D\subseteqq B_{\min}$.

(5) *Each superalgebra C such that $B\subseteqq C$ has the form*

$$C=\chi_{E_1}B+(1-\chi_{E_1})C$$

where $(1-\chi_{E_1})C\supseteqq(1-\chi_{E_1})D$ and χ_{E_1} is in B.

PROOF. (3)$\Rightarrow$(2) is trivial. (2)$\Rightarrow$ (1) is known in [4, Theorem 4]. (1)$\Rightarrow$(4). Since $I_D \supseteqq I(B)$, by Theorem 2 and Lemma 2 in [3], $B_{\min} \supseteqq D$. (4)$\Rightarrow$(5). Since $B \subseteqq C$, there exists $\chi_{E_1} \in B$ such that $C=\chi_{E_1}B+(1-\chi_{E_1})C$ and $(1-\chi_{E_1})C \,_B\!\succ (1-\chi_{E_1})B$. By the definition of $B_{\min}$, $(1-\chi_{E_1})C \supseteqq (1-\chi_{E_1})B_{\min}$ and hence $(1-\chi_{E_1})C \supseteqq (1-\chi_{E_1})D$. (5)$\Rightarrow$(3). Let χ_{E_0} be an essential function of B and let K be a superalgebra such that $\chi_{E_0}B \prec_B \chi_{E_0}K$, then $K \supseteqq D$ by (5). Hence $B_{\min} \supseteqq D$. Since $D=\chi_{E_2}B+(1-\chi_{E_2})B_{\min}$, $I_D=\chi_{E_2}I_B+(1-\chi_{E_2})I_{B_{\min}}$ by Lemma 1 in [4]. By Theorem 2 and Theorem 5, it follows that I_D is a quasi-primary ideal of I_B.

COROLLARY 3. *Let D be a superalgebra which contains H^∞ properly. Then the following are equivalent.*

(1) *If f in H^∞ vanishes on a set of positive measure, then f lies in I_D.*

(2) *If f and g in H^∞ and $fg=0$ a.e., then f lies in I_D or g lies in I_D.*

(3) *I_D is a primary ideal of H^∞.*

(4) *$D=H^\infty_{\min}$.*

PROOF. Apply Theorem 7 with $B=H^\infty$.

Set $H^\infty_{\min,0}=H^\infty_{\min}$ and let $H^\infty_{\min,k+1}=(H^\infty_{\min,k})_{\min}$ for $k=0, 1, \cdots$. Define $H^\infty_{\max}$ to be the superalgebra generated by H^∞ and $\chi_{E(f)}$ for all $f \in H^\infty$.

COROLLARY 4. *Suppose $L^\infty=[\bigcup_{k=0}^{\infty} H^\infty_{\min,k}]_\infty$. If B is any superalgebra of A, then $B=H^\infty$ or*

$$B=\sum_{k=0}^{\infty}\chi_{F_k}H^\infty_{\min,k}+(1-\sum_{k=0}^{\infty}\chi_{F_k})L^\infty$$

for $\chi_{F_k} \in H^\infty_{\min,k}$. In particular, if $L^\infty=\bigcup_{k=0}^{n} H^\infty_{\min,k}$, then $H^\infty_{\min,n-1}=H^\infty_{\max}$.

PROOF. Let χ_{E_k} be the essential function of $H^\infty_{\min,k}$, since $L^\infty=[\bigcup_{k=0}^{\infty} H^\infty_{\min,k}]_\infty$, it follows that $\chi_{E_k}H^\infty_{\min,k} \prec \chi_{E_k}H^\infty_{\min,k+1}$ for $k=0, 1, 2, \cdots$. If $B \neq H^\infty$ then $B \supseteqq H^\infty_{\min,0}$. By Theorem 7, $B=\chi_{F_0}H^\infty_{\min,0}+(1-\chi_{F_0})B$ where $\chi_{F_0} \in H^\infty_{\min,0}$ and $(1-\chi_{F_0})B \supseteqq (1-\chi_{F_0})H^\infty_{\min,1}$. Set $B'=\chi_{F_0}H^\infty_{\min,1}+(1-\chi_{F_0})B$, then $B' \supseteqq H^\infty_{\min,1}$. Again applying Theorem 7, $B'=\chi_G H^\infty_{\min,1}+(1-\chi_G)B'$ where $\chi_G \in H^\infty_{\min,1}$ and $(1-\chi_G)B' \supseteqq (1-\chi_G)H^\infty_{\min,2}$. Set $\chi_{F_1}=\chi_G-\chi_{F_0}$, then $B=\chi_{F_0}H^\infty_{\min,0}+\chi_{F_1}H^\infty_{\min,1}+(1-\chi_{F_0}-\chi_{F_1})B$ where $\chi_{F_1} \in H^\infty_{\min,1}$ and $(1-\chi_{F_0}-\chi_{F_1})B \supseteqq (1-\chi_{F_0}-\chi_{F_1})H^\infty_{\min,2}$. Thus B has the form

$$B=\sum_{k=0}^{\infty}\chi_{F_k}H^\infty_{\min,k}+(1-\sum_{k=0}^{\infty}\chi_{F_k})B$$

where $(1-\sum_{k=0}^{\infty}\chi_{F_k})B \supseteqq (1-\sum_{k=0}^{\infty}\chi_{F_k})H^\infty_{\min,n}$ for $n=0, 1, 2, \cdots$. Since $L^\infty=[\sum_{k=0}^{\infty} H^\infty_{\min,k}]_\infty$, $(1-\sum_{k=0}^{\infty}\chi_{F_k})B=(1-\sum_{k=0}^{\infty}\chi_{F_k})L^\infty$.

7. The existence of the minimum superalgebra of A.

Corollary 3 in §6 shows the necessary and sufficient conditions for a superalgebra B to be a minimum superalgebra which contains H^∞ properly, but it dose not show the existence of the minimum superalgebra. We shall show the existence theorem.

THEOREM 8. *Let B be a superalgebra of A. There exists at least one function in I_B that is not a weak-*limit of functions g in I_B with $\chi_{E(g)}$ being minimal for B if and only if $B=B_{\min}$.*

PROOF. By Theorem 2, it is trivial.

COROLLARY 5. *There exists at least one function in H_0^∞ that is not a weak-*limit of functions, vanishing on sets of positive measure if and only if there exists a minimum superalgebra that contains H^∞ properly, i.e. $H^\infty \neq H_{\min}^\infty$.*

PROOF. Apply Theorem 8 with $B=H^\infty$.

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