J. Math. Soc. Japan
Vol. 31, No. 4, 1979

Variations of metrics on homogeneous manifolds

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(Received Jan. 24, 1978)

0. Introduction.

In [5, p. 115] Wu-Yi Hsiang conjectured the following: Of all possible Riemannian metrics on a homogeneous manifold $M=K/H$ (K compact, semi-simple), the *natural* metric, corresponding to the Cartan-Killing form of the Lie algebra of K, should admit the largest isometry group. In [1] he tested this conjecture with the second Stiefel manifold $V^{n,2}=SO(n)/SO(n-2)$, $n>20$, odd. He claimed that $\dim \mathrm{ISO}(g)\leqq\frac{1}{2}n(n-1)+1$ for all Riemannian metrics g on $V^{n,2}$, where $\mathrm{ISO}(g)$ denotes the isometry group of g, and that equality holds only when g is the natural metric. However, in this paper we will establish the following:

THEOREM. *The second Stiefel manifold* $V^{n,2}$, $n\geqq 31$, *odd, has uncountably many homothetically distinct homogeneous metrics* g, *for which* $\dim \mathrm{ISO}(g)=\frac{1}{2}n(n-1)+1$. *Note that* $\dim V^{n,2}=2n-3$.

The procedure will be to study the space of K-invariant metrics on K/H and by explicit computation of sectional curvature, distinguish different metrics by homothety type. For terminology, see Section 1.

The author would like to thank Professor Larry Mann for suggesting this problem, and for many useful conversations.

1. Background material.

In this section we collect some results on the geometry of homogeneous spaces, all of which may be found in [3, Chapter X]. We study homogeneous manifolds $M=K/H$, where K acts as isometries of some Riemannian metric on M, hence also as a group of automorphisms of the principal $O(m)$-bundle over M associated with the metric. Since H is compact, M is *reductive*; that is, the Lie algebra $\mathfrak{k}$ of K admits a vector space decomposition

$$\mathfrak{k}=\mathfrak{h}+\mathfrak{m}$$

where $\mathfrak{h}$ is the Lie algebra of H, $\mathfrak{h}\cap\mathfrak{m}=0$, and $\mathrm{ad}(H)\mathfrak{m}\subseteqq\mathfrak{m}$.

We will state the first theorem in generality. Recall that a G-structure on an m-dimensional manifold M is a principal subbundle P of $L(M)$, the bundle of linear frames over M, with structure group G.

1.1 THEOREM. *Let P be a K-invariant G-structure on a reductive homogeneous space $M=K/H$ with decomposition $\mathfrak{k}=\mathfrak{h}+\mathfrak{m}$. Then there is a one-one correspondence between the set of K-invariant connections on P and the set of linear mappings $\Lambda_{\mathfrak{m}}: \mathfrak{m}\to\mathfrak{g}$ such that*

$$\Lambda_{\mathfrak{m}}(\mathrm{ad}\,h(X))=\mathrm{ad}(\lambda(h))\Lambda_{\mathfrak{m}}(X)$$

for $X\in\mathfrak{m}$, $h\in H$, where $\mathfrak{g}$ denotes the Lie algebra of G, and λ is the linear isotropy representation of H.

We next have a formula for the curvature tensors of the connections in Theorem 1.1. Let $o=\{H\}$, the coset of H in M. Fix a frame $u_o=\{X_1, \cdots, X_m\}\in P$ at o. Identify $\boldsymbol{R}^m$ and $T_o(M)$ (the tangent space to $o\in M$) by

$$u_o: \boldsymbol{R}^m\to T_o(M)$$

where $u_o(e_i)=X_i$, and e_i is a standard basis element in $\boldsymbol{R}^m$. Identify $\mathfrak{m}$ with $T_o(M)$ as follows: For $X\in\mathfrak{m}$, evaluate X as a vector field on M at o. Thus $\Lambda_{\mathfrak{m}}(X)\in\mathfrak{g}$ may be regarded as a linear transformation of the subspace $\mathfrak{m}$. Let $[\ ,\ ]$ be the Lie bracket in $\mathfrak{k}$ and set

$$[\ ,\ ]=[\ ,\ ]_{\mathfrak{h}}+[\ ,\ ]_{\mathfrak{m}}$$

where $[\ ,\ ]_{\mathfrak{h}}\in\mathfrak{h}$ and $[\ ,\ ]_{\mathfrak{m}}\in\mathfrak{m}$.

1.2. THEOREM. *For an invariant connection, the curvature tensor at $o\in M$ is given by*

$$R(X, Y)_o=[\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)]-\Lambda_{\mathfrak{m}}([X, Y]_{\mathfrak{m}})-\lambda([X, Y]_{\mathfrak{h}}) \qquad X, Y\in\mathfrak{m}.$$

REMARK. $\lambda([X, Y]_{\mathfrak{h}})Z=[[X, Y]_{\mathfrak{h}}, Z]$ since the linear isotropy representation λ of H corresponds to the adjoint representation of $\mathfrak{h}$ in $\mathfrak{k}$.

Using the identification of $\mathfrak{m}$ and $T_o(M)$, we have the following relationship between the $\mathrm{ad}(H)$-invariant forms on $\mathfrak{m}$ and H-invariant forms on $T_o(M)$:

1.3. PROPOSITION. *If $M=K/H$ is reductive with $\mathrm{ad}(H)$-invariant decomposition $\mathfrak{k}=\mathfrak{h}+\mathfrak{m}$, then there is a one-one correspondence between H-invariant forms $\langle X, Y\rangle_o$ on $T_o(M)$ and $\mathrm{ad}(H)$-invariant forms B on $\mathfrak{m}$. The correspondence is given by*

$$B(X, Y)=\langle X, Y\rangle_o \qquad X, Y\in\mathfrak{m}.$$

REMARK. Since an H-invariant forms on $T_o(M)$ give rise to K-invariant metrics on M, Proposition 1.3 establishes a one-one correspondence between invariant metrics on M and invariant forms on $\mathfrak{m}$.

The next theorem allows us to compute the Riemannian connection on $M=K/H$ associated with an invariant metric on M, or equivalently, any invariant form on $\mathfrak{m}$. Thus it will allow us to compute the curvature tensor, given in Theorem 1.2, explicitly.

1.4. THEOREM. *Let $M=K/H$ be a reductive homogeneous space with an* ad(H)*-invariant decomposition $\mathfrak{k}=\mathfrak{h}+\mathfrak{m}$, and let B denote an* ad(H)*-invariant positive definite symmetric bilinear form on $\mathfrak{m}$. Let $\langle\,,\rangle$ be the corresponding K-invariant metric on M. Then the Riemannian connection on M associated with the metric is given by*

$$\Lambda_{\mathfrak{m}}(X)Y=\frac{1}{2}[X, Y]_{\mathfrak{m}}+U(X, Y)$$

where U: $\mathfrak{m}\times\mathfrak{m}\rightarrow\mathfrak{m}$ is the symmetric bilinear map defined by

$$2B(U(X, Y), Z)=B(X, [Z, Y]_{\mathfrak{m}})+B([Z, X]_{\mathfrak{m}}, Y) \qquad X, Y, Z\in\mathfrak{m}.$$

Thus a study of possible ad(H)*-invariant forms on the subspace $\mathfrak{m}$ of $\mathfrak{k}$ will provide us with information about the geometry of $M=K/H$.*

Finally, recall, that a map f: $M\rightarrow M'$ between Riemannian manifolds is called a homothety if

$$\langle f_*X, f_*Y\rangle'_{f(x)}=c^2\langle X, Y\rangle_x \qquad X, Y\leqq T_x(M)$$

where c is a constant, and that in this case, corresponding sectional curvatures are related by $K'=\frac{1}{c^2}K$.

Note also that if f: $M\rightarrow M$ is an isometry of M with respect to a metric $\langle\,,\rangle$, then it is also an isometry with respect to the metric $c^2\langle\,,\rangle$ (c constant). In Sections 2 and 3 it will be important to distinguish manifolds by homothety type.

2. Varying metrics on $M=K/H$.

In this section we apply the results of Section 1 to a special situation: Suppose $M=K/H$ is a reductive homogeneous manifold where the linear isotropy action on $T_o(M)$ is *reducible*, fixing a one dimensional subspace and acting invariantly on a complementary subspace. Assume K acts as isometries of some metric on M, and let B denote the corresponding ad(H)-invariant form on $\mathfrak{m}$ in $\mathfrak{k}$ from Proposition 1.3. Let $\{X_1, \cdots, X_m\}$ be an orthonormal basis for $\mathfrak{m}$ with respect to B so that X_1 spans the stable line, and $\{X_2, \cdots, X_m\}$ span the complementary subspace. Define a new invariant form by

$$\begin{aligned} &B_t(X_1, X_1)=t>0 \\ &B_t(X_i, X_i)=1 \qquad i\geqq 2 \\ &B_t(X_i, X_j)=0 \qquad i\neq j. \end{aligned}$$

This is a one parameter variation of the form on $\mathfrak{m}$ and gives rise to a variation of the metric on $M=K/H$. What we will do is distinguish by homothety type the manifolds that arise from a variation of this type.

NOTATION. Let M_t denote the homogeneous manifold with the invariant metric corresponding to the form B_t. Let $\langle\, ,\rangle^t$ denote the invariant metric and let $R_t(\, ,)$ denote the curvature tensor of M_t. Let $\Lambda^t_{\mathfrak{m}}(\ )$ denote the connection on M_t in Theorem 1.4, and let $K_t\{X, Y\}$ denote the sectional curvature of the plane determined by $\{X, Y\}$ in $T_o(M_t)$.

Define the "structure" constants of the subspace m as follows:

$$[X_i, X_j]_{\mathfrak{m}}=\sum_{k=1}^{m} c_k^{ij} X_k \qquad (\text{note that } c_k^{ii}=0 \text{ and } c_k^{ij}=-c_k^{ji}).$$

To compute the curvature tensor, we will first need to compute $\Lambda^t_m(\ )$ in four cases, using Theorem 1.4. All computations are straightforward and we simply list the results:

$$\Lambda^t_{\mathfrak{m}}(X_1)X_1=\sum_{\mathfrak{l}=2}^{m} t c_1^{\mathfrak{l}1} X_{\mathfrak{l}} \tag{2.1}$$

$$\Lambda^t_{\mathfrak{m}}(X_1)X_j=c_1^{1j}X_1+\frac{1}{2}\sum_{\mathfrak{l}=2}^{m}(c_{\mathfrak{l}}^{1j}+t c_1^{\mathfrak{l}j}+c_j^{\mathfrak{l}1})X_{\mathfrak{l}}. \qquad (j\neq 1) \tag{2.2}$$

$$\Lambda^t_{\mathfrak{m}}(X_j)X_1=\frac{1}{2}\sum_{\mathfrak{l}=2}^{m}(c_{\mathfrak{l}}^{j1}+t c_1^{\mathfrak{l}j}+c_j^{\mathfrak{l}1})X_{\mathfrak{l}}. \qquad (j\neq 1) \tag{2.3}$$

$$\Lambda^t_{\mathfrak{m}}(X_i)X_j=\frac{1}{2}\sum_{\mathfrak{l}=2}^{m}(c_{\mathfrak{l}}^{ij}+c_i^{\mathfrak{l}j}+c_j^{\mathfrak{l}i})X_{\mathfrak{l}}+\frac{1}{2t}(t c_1^{ij}+c_i^{1j}+c_j^{1i})X_1. \qquad (i,\ j\neq 1). \tag{2.4}$$

We will now compute the sectional curvature for all planes determined by pairs $\{X_i, X_j\}$. We will consider two cases: $K_t\{X_i, X_j\}_{i,j\neq 1}$ and $K_t\{X_1, X_j\}$.

Computations for $K_t\{X_i, X_j\}$, $i,\ j\neq 1$.

$$K_t\{X_i, X_j\}=\frac{\langle R_t(X_i, X_j)X_j, X_i\rangle^t}{A_t(X_i, X_j)}.$$

Here $A_t(X_i, X_j)=\langle X_i, X_i\rangle^t\langle X_j, X_j\rangle^t-(\langle X_i, X_j\rangle^t)^2$, hence $A_t(X_i, X_j)=1$, and since $\langle X_i, X_i\rangle^t=1$ for $i\geqq 2$ we need only compute this X_i coefficient of $R_t(X_i, X_j)X_j$. Now, from 1.2

$$R_t(X_i, X_j)X_j=[\Lambda^t_{\mathfrak{m}}(X_i), \Lambda^t_{\mathfrak{m}}(X_j)](X_j)-\Lambda^t_{\mathfrak{m}}([X_i, X_j]_{\mathfrak{m}})(X_j)-[[X_i, X_j]_{\mathfrak{h}}, X_j].$$

Repeated application of 2.1-2.4 yields

$$K_t\{X_i, X_j\}=\sum_{\mathfrak{l}=2}^{m} c_j^{\mathfrak{l}j}\cdot c_i^{i\mathfrak{l}}+\frac{1}{t}c_j^{1j}\cdot c_i^{i1}$$

$$-\frac{1}{4t}(tc_1^{ij}+c_i^{1j}+c_j^{1i})(c_i^{1j}+tc_1^{ij}+c_j^{i1})$$

$$\tag{2.5} -\frac{1}{4}\sum_{\mathfrak{l}=2}^{m}(c_{\mathfrak{l}}^{ij}+c_i^{\mathfrak{l}j}+c_j^{\mathfrak{l}i})(c_i^{j\mathfrak{l}}+c_j^{i\mathfrak{l}}+c_{\mathfrak{l}}^{ij})$$

$$-\frac{1}{2}c_1^{ij}(c_i^{1j}+tc_1^{ij}+c_j^{i1})-\frac{1}{2}\sum_{\mathfrak{l}=2}^{m}c_{\mathfrak{l}}^{ij}(c_i^{\mathfrak{l}j}+c_{\mathfrak{l}}^{ij}+c_j^{i\mathfrak{l}})$$

$$-\langle[[X_i,\, X_j]_{\mathfrak{h}},\, X_j],\, X_i\rangle^t\,.$$

Computations for $K_t\{X_1, X_j\}$.

$$K_t\{X_1,\, X_j\}=\frac{\langle R_t(X_1,\, X_j)X_j,\, X_1\rangle^t}{A_t(X_1,\, X_j)}\,.$$

Here $A_t(X_1, X_j)=t$, but $\langle X_1, X_1\rangle^t=t$ so we need only find the X_1 coefficient of $R_t(X_1, X_j)X_j$ to compute $K_t\{X_1, X_j\}$. Now

$$R_t(X_1,\, X_j)X_j=[\Lambda_{\mathfrak{m}}^t(X_1),\, \Lambda_{\mathfrak{m}}^t(X_j)](X_j)-\Lambda_{\mathfrak{m}}^t([X_1,\, X_j]_{\mathfrak{m}})(X_j)-[[X_1,\, X_j]_{\mathfrak{h}},\, X_j]\,.$$

From 2.1-2.4, we obtain

$$K_t\{X_1,\, X_j\}=\sum_{\mathfrak{l}=2}^{m}c_j^{\mathfrak{l}j}\cdot c_1^{1\mathfrak{l}}-\frac{1}{4t}\sum_{\mathfrak{l}=2}^{m}(c_{\mathfrak{l}}^{1j}+tc_1^{\mathfrak{l}j}+c_j^{\mathfrak{l}1})(tc_1^{j\mathfrak{l}}+c_j^{1\mathfrak{l}}+c_{\mathfrak{l}}^{1j})$$

$$\tag{2.6} -(c_1^{1j})^2-\frac{1}{2t}\sum_{\mathfrak{l}=2}^{m}c_{\mathfrak{l}}^{1j}(tc_1^{\mathfrak{l}j}+c_{\mathfrak{l}}^{1j}+c_j^{1\mathfrak{l}})$$

$$-\frac{1}{t}\langle[[X_1,\, X_j]_{\mathfrak{h}},\, X_j],\, X_1\rangle^t\,.$$

REMARK. While these formulas do not offer great insight into the geometry of the homogeneous manifold M_t, we can notice the following fact: This variation technique is uniformly continuous in t for a fixed plane and t in some closed interval $[a, b]$ with $0<a<b$. This fact will be useful for the following discussion. Let $G_2(T_o(M_t))$ denote the Grassmann of two-planes in $T_o(M_t)$. Since $G_2(T_o(M_t))$ is compact, and since the sectional curvature is continuous on $G_2(T_o(M_t))$, we may define:

DEFINITION. Let

$a(t)$=minimum sectional curvature in $G_2(T_o(M_t))$,

$b(t)$=maximum sectional curvature in $G_2(T_o(M_t))$,

$h(t)=a(t)/b(t)$, if $b(t)\neq 0$.

From the discussion at the end of Section 1 we see that $h(t)$ is a homothety invariant. The fact that the variation technique is uniformly continuous in t, together with the fact that the Grassmann is compact, gives us the following:

PROPOSITION. *$h(t)$ is continuous in t.*

Thus for a family of manifolds M_t obtained via this variation technique, if we can determine that $h(t)\neq h(1)$ for some t, we will conclude that there are uncountably many homothetically distinct homogeneous manifolds in this family. This will be done in the next section for the second Stiefel manifold $V^{n,2}=SO(n)/SO(n-2)$.

3. Proof of the theorem.

In the previous section we developed the technique of "varying" metrics on homogeneous manifolds under the assumption that the linear isotropy action was reducible, fixing a one dimensional line and acting invariantly on a complementary subspace. In this section, using formulas 2.5 and 2.6, we will compute the sectional curvatures of two-planes in $T_o(M_t)$ for the second Stiefel manifold $V^{n,2}=SO(n)/SO(n-2)$, and using the homothety invariant $h(t)$, we will show that there are *uncountably many* homothetically distinct homogeneous metrics on $V^{n,2}$ having $SO(2)\times SO(n)$ as full isometry group, thus disproving a theorem due to Hsiang [**1**], and forcing as alteration in a conjecture, also due to Hsiang, [**5**].

HSIANG'S CONJECTURE. Let $M=K/H$, where K is compact, semi-simple. There is a "natural" form on $\mathfrak{k}$ which is ad(K)-invariant (hence ad(H)-invariant), namely, the Cartan-Killing form, given by

$$\phi(X, Y)=\text{trace ad}(X)\,\text{ad}(Y) \qquad X, Y\in\mathfrak{k}\,.$$

Since K is compact, semi-simple, ϕ is negative definite. Thus we may define

$$B(X, Y)=-\phi(X, Y)\,.$$

Restricting to the subspace $\mathfrak{m}$ in $\mathfrak{k}$ gives us, as Hsiang says, the most "natural" ad(H)-invariant form on $\mathfrak{m}$, hence the most "natural" metric on the homogeneous space $M=K/H$.

To state Hsiang's conjecture in his terms we need to introduce the following: The *degree of symmetry* of a differentiable manifold M is the maximum dimension of all isometry groups of all possible Riemannian metrics on M. In case M is *compact*, this agrees with the maximum dimension of all compact subgroups of Diff(M).

CONJECTURE. ([**5**]) The natural metric on a homogeneous manifold $M=K/H$ is the most symmetric metric.

To test his conjecture, Hsiang used the second Stiefel manifold $V^{n,2}=SO(n)/SO(n-2)$. For $n\geqq 31$, odd, the largest connected transitive compact group of motions of $V^{n,2}$ is $SO(2)\times SO(n)$ (cf. [2]). In [**1**] Hsiang claims that the natural metric on $V^{n,2}$ *alone* (up to scalar factor) has $SO(2)\times SO(n)$ as the connected component of the identity of the full isometry group. We will show that this is not true: In fact, there are *uncountably many* homothetically distinct homogeneous metrics on $V^{n,2}$ having $SO(2)\times SO(n)$ as the identity component of the full isometry group.

REMARKS. Hsiang is in error in [**1**] when he claims the equivalence of his Theorem and Theorem′. Hsiang's conjecture may be valid in the following revised form: The degree of symmetry of a homogeneous manifold of a compact, semi-simple Lie group equals the dimension of the isometry group of the natural metric, but there may be homothetically distinct metrics having the same isometry group.

We now begin the proof of the theorem stated in the introduction. From [**2**] we know that the group $SO(2)\times SO(n)$ acts on $V^{n,2}$ as follows: We may regard $V^{n,2}$ as $n\times 2$ matrices with columns of norm 1. For $(A, B)\in SO(2)\times SO(n)$ and $V\in V^{n,2}$ let

$$(A, B)V=BVA^{-1}.$$

Define multiplication in $SO(2)\times SO(n)$ pointwise

$$(A, B)(A', B')=(AA', BB'). \tag{*}$$

Then

$$\begin{aligned}(AA', BB')V=(BB')V(AA')^{-1}&=B(B'V(A')^{-1})A^{-1}\\&=B((A', B')V)A^{-1}\\&=(A, B)((A', B')V).\end{aligned}$$

Thus this is an action. To apply our results of Section 2 we embed $SO(2)\times SO(n)$ in $SO(n+2)$ as follows:

$$(A,\ B)\longrightarrow \begin{array}{cc} \left[\begin{array}{c|c} A & \\ \hline & B \end{array}\right] & \begin{array}{c} 2 \\ n \end{array} \\ \begin{array}{cc} 2 & n \end{array} & \end{array}.$$

Notice that matrix multiplication is exactly * multiplication.

We now fix a distinguished point in $V^{n,2}$ (the "origin"), namely

$$V_0=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} n$$

and compute the stabilizer of $SO(2)\times SO(n)$ at V_0. That is, we seek (A, B) such that

$$(A, B)V_0=V_0\,.$$

Straightforward computation shows that the stabilizer must consist of matrices of the form

$$\begin{array}{cc} \left[\begin{array}{c|c|c} C & 0 & 0 \\ \hline 0 & C & 0 \\ \hline 0 & 0 & D \end{array}\right] & \begin{array}{l} 2 \\ 2 \\ n-2 \end{array} \\ \begin{array}{ccc} 2 & 2 & n-2 \end{array} & \end{array} \qquad C\in SO(2),\quad D\in SO(n-2)\,.$$

Thus we may write

$$V^{n,2}=\frac{SO(2)\times SO(n)}{SO(2)\times SO(n-2)}$$

where $H=SO(2)\times SO(n-2)$ is embedded in $SO(2)\times SO(n)$ in a twisted fashion.

From Section 2, we know that to study the geometry of $V^{n,2}$ it will suffice to study the relationship between the Lie algebra $\mathfrak{so}(2)+\mathfrak{so}(\mathfrak{n})$ of $SO(2)\times SO(n)$ and the Lie subalgebra $\mathfrak{so}(2)+\mathfrak{so}(n-2)$. We will write the full algebra and subalgebra as follows:

$$\mathfrak{so}(2)+\mathfrak{so}(n)=\left[\begin{array}{c|c} \mathfrak{so}(2) & \\ \hline & \mathfrak{so}(n) \end{array}\right]$$

$$\mathfrak{so}(2)+\mathfrak{so}(n-2)=\left[\begin{array}{c|c|c} \mathfrak{c} & & \\ \hline & \mathfrak{c} & \\ \hline & & \mathfrak{d} \end{array}\right] \qquad \mathfrak{c}\in\mathfrak{so}(2),\quad \mathfrak{d}\in\mathfrak{so}(n-2)\,.$$

All of our computations (in Section 2) are carried out in terms of basis vectors for the algebra and subalgebra. It will be convenient to designate a basis for the subalgebra and extend it to the full algebra. Recall that

$$\mathfrak{so}(n+2)=\{\mathfrak{a}\mid \mathfrak{a}+\mathfrak{a}^t=0,\ \mathfrak{a}\text{ is }(n+2)\times(n+2),\text{ real}\}\,.$$

Set

$$\begin{array}{cc} E_1=\left[\begin{array}{cc|cc|c} & 1 & & & \\ -1 & & & & \\ \hline & & & 1 & \\ & & -1 & & \\ \hline & & & & \end{array}\right] & \begin{array}{l} 2 \\ 2 \\ n-2 \end{array} \\ \begin{array}{ccc} 2 & 2 & n-2 \end{array} & \end{array} \qquad \text{(zeroes elsewhere)}$$

$$E_{ij}=\begin{array}{cc} & \begin{array}{ccc} \qquad & \qquad & \quad j \quad i \end{array}\\ \begin{array}{c}\\ \\ j\\ i\end{array} & \left[\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & & \begin{array}{cc} & 1\\ -1 & \end{array}\end{array}\right]\end{array} \qquad \begin{array}{l}\text{where } 3\leqq j\leqq n, \quad j<i\\ \text{(zeroes elsewhere)}\end{array}$$

(For convenience, the index ij starts at the third row and column.)

To extend to a basis for the full algebra we set

$$X_1=\left[\begin{array}{c|c|c} & & \\ \hline & \begin{array}{cc} & 1\\ -1 & \end{array} & \\ \hline & & \end{array}\right] \qquad \text{(zeroes elsewhere)}$$

$$X_1=\begin{array}{cc} & \begin{array}{ccc} \qquad & \quad 1 & \qquad i \end{array}\\ \begin{array}{c}\\ 1\\ \\ i\end{array} & \left[\begin{array}{c|c|c} & & \\ \hline & & \begin{array}{c}1\\ \cdots\cdots\end{array} \\ \hline & -1\ \vdots & \end{array}\right]\end{array}$$

$$\tilde{X}_1=\begin{array}{cc} & \begin{array}{ccc} \qquad & \quad 2 & \qquad i \end{array}\\ \begin{array}{c}\\ 2\\ \\ i\end{array} & \left[\begin{array}{c|c|c} & & \\ \hline & & \begin{array}{c}\cdots\cdots\\ 1\end{array} \\ \hline & \vdots\ -1 & \end{array}\right]\end{array} \qquad \begin{array}{l}3\leqq i\leqq n\\ \text{(zeroes elsewhere)}\end{array}$$

DEFINITION. Let

$$\mathfrak{m}_o=\operatorname{span}\{X_1\}$$

$$\mathfrak{m}_1=\operatorname{span}\{X_3, \cdots, X_n\}$$

$$\tilde{\mathfrak{m}}_1=\operatorname{span}\{\tilde{X}_3, \cdots, \tilde{X}_n\}\,.$$

We claim that $\mathfrak{m}_o$ is the "stable" line and $\mathfrak{m}_1+\tilde{\mathfrak{m}}_1$ is ad(H)-invariant. Since H is connected, it suffices to establish that $[\mathfrak{h}, \mathfrak{m}_o]=0$, and $[\mathfrak{h}, \mathfrak{m}_1+\tilde{\mathfrak{m}}_1]\subseteqq\mathfrak{m}_1+\tilde{\mathfrak{m}}_1$. The following are easily derived:

3.1. *Computations.*

a) $[E_1, X_1]=0$

b) $[E_{ij}, X_1]=0$

c) $[E_1, X_i]=-\tilde{X}_i \qquad 3\leqq i\leqq n$

d) $[E_1, \tilde{X}_i]=X_i \qquad 3\leqq i\leqq n$

$$\text{e)}\quad [E_{jk}, X_i]=\begin{cases} 0 & \text{if } j, k\neq i \\ X_k & \text{if } j=i \\ -X_j & \text{if } k=i \end{cases}$$

$$\text{f)}\quad [E_{jk}, \tilde{X}_i]=\begin{cases} 0 & \text{if } j, k\neq i \\ \tilde{X}_k & \text{if } j=i \\ -\tilde{X}_j & \text{if } k=i. \end{cases}$$

Notice that a and b establish that $\mathfrak{m}_o$ is stable, and that c through f establish the invariance of $\mathfrak{m}_1+\tilde{\mathfrak{m}}_1$.

To use formulas 2.5 and 2.6, we need also to compute $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}}$ and $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}$ where $\mathfrak{m}=\mathfrak{m}_o+\mathfrak{m}_1+\tilde{\mathfrak{m}}_1$, and $[\ ,\ ]_{\mathfrak{m}}$ and $[\ ,\ ]_{\mathfrak{h}}$ denote the projections of the bracket into $\mathfrak{m}$ and $\mathfrak{h}$ respectively. The following computations are straightforward:

3.2. *Computations.*

a) $[X_1, X_1]=0$

b) $[X_1, X_i]=-\tilde{X}_i$ hence $[X_1, X_i]_{\mathfrak{h}}=0$

c) $[X_1, \tilde{X}_i]=X_i$ hence $[X_1, \tilde{X}_i]_{\mathfrak{h}}=0$

d) $[X_i, X_j]=E_{ij}$ $j<i$ hence $[X_i, X_j]_{\mathfrak{m}}=0$

e) $[\tilde{X}_i, \tilde{X}_j]=E_{ij}$ $j<i$ hence $[\tilde{X}_i, \tilde{X}_j]_{\mathfrak{m}}=0$

f) $[X_i, \tilde{X}_j]=0$ if $i\neq j$

g) $[X_i, \tilde{X}_i]=-X_1$ hence $[X_i, \tilde{X}_i]_{\mathfrak{h}}=0$.

Finally, we need to define bilinear forms B_t on $\mathfrak{m}$ for each $t>0$ and show that they are ad(H)-invariant.

DEFINITION. Define B_t by

$$\begin{array}{llll} B_t(X_1, X_1)=t & & B_t(X_i, X_j)=0 & i\neq j \\ B_t(X_i, X_i)=1 & 3\leqq i\leqq n & B_t(\tilde{X}_i, \tilde{X}_j)=0 & i\neq j \\ B_t(\tilde{X}_i, \tilde{X}_i)=1 & 3\leqq i\leqq n & B_t(X_i, \tilde{X}_j)=0. & \end{array}$$

Claim. B_t is ad(H)-invariant.

Since H is connected, it suffices to verify that

$$B_t([Z, X], Y)+B_t(X, [Z, Y])=0$$

for all $Z\in\mathfrak{h}$, and $X, Y\in\mathfrak{m}$. This is straightforward.

We are now in a position to compute the sectional curvature of two-planes in $T_o(M_t)$ determined by pairs of basis vectors in $\{X_1, X_3, \cdots, X_n, \tilde{X}_3, \cdots, \tilde{X}_n\}$. To apply formulas 2.5 and 2.6 we need the structure constants c_k^{ij} defined by

$$[X_i, X_j]_{\mathfrak{m}} = \sum_{k=1}^{m} c_k^{ij} X_k \qquad (m = \dim \mathfrak{m}).$$

From 3.2, we see that the only non-zero structure constants are

$$c_{\tilde{i}}^{1i} = -1, \qquad c_i^{1\tilde{i}} = 1, \qquad c_1^{i\tilde{i}} = -1 .$$

(We have used the index $\tilde{i}$ for the basis vector $\tilde{X}_i$.) Thus the computations of formulas 2.5 and 2.6 are greatly simplified.

Computations using formula 2.5. We compute the following for $3 \leqq j < i \leqq n$.

A) $K_t\{X_i, X_j\}$

B) $K_t\{\tilde{X}_i, \tilde{X}_j\}$

C) $K_t\{X_i, \tilde{X}_j\}$ $(i \neq j)$

D) $K_t\{X_i, \tilde{X}_i\}$.

A) $K_t\{X_i, X_j\}$. In formula 2.5 the only contribution is from the last term

$$-\langle [[X_i, X_j]_{\mathfrak{h}}, X_j], X_i \rangle^t .$$

But from 3.2, $[X_i, X_j]_{\mathfrak{h}} = E_{ij}$, and $[E_{ij}, X_j] = -X_i$ from 3.1, hence the last term is $-\langle -X_i, X_i \rangle^t = 1$ since $i \neq 1$.

B) $K_t\{\tilde{X}_i, \tilde{X}_j\} = 1$ by similar computation.

C) $K_t\{X_i, \tilde{X}_j\}$ with $i \neq j$. Again, the only possible contribution in formula 2.5 comes from the last term

$$-\langle [[X_i, \tilde{X}_j]_{\mathfrak{h}}, \tilde{X}_j], X_i \rangle^t .$$

but from 3.2 we have $[X_i, \tilde{X}_j]_{\mathfrak{h}} = 0$, therefore $K_t\{X_i, \tilde{X}_j\} = 0$.

D) $K_t\{X_i, \tilde{X}_i\}$. Any part of formula 2.5 with $\mathfrak{l} \geqq 2$ gives zero contribution by the simplicity of the c_k^{ij}'s. Hence

$$K_t\{X_i, \tilde{X}_i\} = -\frac{1}{4t}(t c_1^{i\tilde{i}} + c_i^{1\tilde{i}} + c_{\tilde{i}}^{1i})(c_i^{\tilde{i}1} + t c_1^{i\tilde{i}} + c_{\tilde{i}}^{i1})$$

$$-\frac{1}{2} c_1^{i\tilde{i}} (c_i^{1\tilde{i}} + t c_1^{i\tilde{i}} + c_{\tilde{i}}^{i1}) - \langle [[X_i, \tilde{X}_i]_{\mathfrak{h}}, \tilde{X}_i], X_i \rangle^t .$$

Evaluating this, using the fact that $[X_i, \tilde{X}_i]_{\mathfrak{h}} = 0$, yields

$$K_t\{X_i, \tilde{X}_i\} = 1 - \frac{3}{4} t .$$

Computations using formula 2.6. We compute $K_t\{X_1, X_j\}$. The computation for $K_t\{X_1, \tilde{X}_j\}$ is similar. The only contribution in formula 2.6 arises when $\mathfrak{l}=\tilde{j}$. Then

$$K_t\{X_1, X_j\}=-\frac{1}{4t}(c^{1j}_{\tilde{j}}+t\,c^{\tilde{j}j}_{1}+c^{\tilde{j}1}_{j})(t\,c^{j\tilde{j}}_{1}+c^{1\tilde{j}}_{j}+c^{1j}_{\tilde{j}})$$

$$-\frac{1}{2t}c^{1j}_{\tilde{j}}(t\,c^{\tilde{j}j}_{1}+c^{1j}_{\tilde{j}}+c^{1\tilde{j}}_{j})-\frac{1}{t}\langle[[X_1, X_j]_{\mathfrak{h}}, X_j], X_1\rangle^t .$$

Evaluating this yields $K_t\{X_1, X_j\}=\frac{t}{4}$. Similarly, $K_t\{X_1, \tilde{X}_j\}=\frac{t}{4}$.

REMARKS. A decomposition $\mathfrak{k}=\mathfrak{h}+\mathfrak{m}$ together with an ad(H)-invariant form B is called *naturally reductive* if

$$B(X, [Z, Y]_{\mathfrak{m}})+B([Z, X]_{\mathfrak{m}}, Y)=0$$

for $X, Y, Z\in\mathfrak{m}$. Our decomposition is naturally reductive when $t=1$, and the resulting Stiefel manifold has non-negative curvature.

Let $V^{n,2}_t$ denote the second Stiefel manifold with the homogeneous metric corresponding to the form B_t. Recall, from Section 3, that $h(t)=a(t)/b(t)$ is a homothety invariant, where $a(t)$ and $b(t)$ are, respectively, the minimum and maximum sectional curvatures in $G_2(T_o(V^{n,2}_t))$. From the computations and the above discussion we have the following:

$$\begin{array}{ll} b(t)\geqq 1 \quad \text{for all} \quad t>0, & \text{hence } h(t) \text{ is continuous}, \\ a(1)=0, & \text{hence } h(1)=0, \\ a(t)<0 \quad \text{for} \quad t>\frac{4}{3}, & \text{hence } h(t)<0 \quad \text{for } t>\frac{4}{3}. \end{array}$$

Continuity of $h(t)$ now establishes the existence of uncountably many homothetically distinct homogeneous metrics on the second Stiefel manifold, and thus proves the Theorem.

REMARK. Similar computation was used in [4] to establish the existence of uncountably many homothetically distinct homogeneous metrics on the spheres $S^{2n-1}=U(n)/U(n-1)$ in the course of a classification of homogeneous Riemannian manifolds.

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