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The subsequentiality of product spaces

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§ 1. Introduction.

A space is said to be a *subsequential space* if it can be embedded as a subspace of a sequential space. The closed image of a metric space is shortly said to be a *Lašnev space* (cf. [4], [5]).

Professor K. Nagami posed the following two problems.

1. Can each Lašnev space be embedded in a countably compact sequential regular space?

2. Is finite (or countable) product of Lašnev spaces subsequential?

This paper gives a negative answer to the first problem and a partial answer to the second as follows:

1. Any Lašnev space, which is not metrizable, cannot be embedded in any countably compact regular space with countable tightness.

2. Assuming the continuum hypothesis (CH), there exist regular Fréchet spaces X and Y such that $X\times Y$ is not subsequential.

Each Fréchet space is subsequential. Therefore the second result shows that even a finite product of subsequential spaces is not subsequential (cf. [8, p. 179]).

In this paper spaces are assumed to be T_1 and maps to be continuous onto.

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§ 2. Theorems.

DEFINITION 1 ([1, p. 954]). A space X has countable tightness if it has the following property: If $A\subset X$ and $x\in Cl_X A$, then $x\in Cl_X B$ for some countable $B\subset A$.

Let $R=\{0\}\cup\{1/n\,;n\in\omega_0\}$ be a convergent sequence. Let S be the disjoint union of a sequence $\{R(n)\,;n\in\omega_0\}$ of copies of R, let $A=\{0(n)\in R(n)\,;0(n)=0,$ $n\in\omega_0\}$, and let $T=S/A$, the quotient space obtained from S by identifying A to a point q.

THEOREM 1. *T cannot be embedded in any countably compact regular space with countable tightness.*

PROOF. Suppose there exists a countably compact regular space X with countable tightness such that $X=Cl_X T$. Let U be an arbitrary open neighborhood of q in X. Let V be an open set in X such that

$$q\in V\subset Cl_X V\subset U .$$

Pick $s(n)\in V\cap(R(n)-0(n))$ for each n. Then, since X is countably compact,

$$Cl_X\{s(n);n\in\omega_0\}-\{s(n);n\in\omega_0\}\neq\emptyset ,$$

$$Cl_X\{s(n);n\in\omega_0\}-\{s(n);n\in\omega_0\}\subset Cl_X V\cap(X-T)\subset U .$$

This shows $q\in Cl_X(X-T)$. Since X has countable tightness, there exists a sequence $\{x(n);n\in\omega_0\}\subset X-T$ such that

$$q\in Cl_X\{x(n);n\in\omega_0\} .$$

Let $\{U(n);n\in\omega_0\}$ be a sequence of open neighborhoods of q in X such that

$$Cl_X U(n+1)\subset U(n) ,$$

$$x(n)\notin Cl_X U(n) .$$

Put

$$A(n)=U(n)\cap(R(n)-\{0(n)\}) .$$

Then $\cup\{A(n);n\in\omega_0\}\cup\{q\}$ is an open neighborhood of q in T. Let W be an open neighborhood of q in X such that

$$W\cap T=\cup\{A(n);n\in\omega_0\}\cup\{q\} .$$

We will show that $\{x(n);n\in\omega_0\}\cap W=\emptyset$, which will contradict the fact that $q\in Cl_X\{x(n);n\in\omega_0\}$. By construction of W,

$$W\cap T=\cup\{A(n);n\in\omega_0\}\cup\{q\}$$

$$=\bigcup_{i=1}^{n-1}\{A(i)-U(n)\}\cup(U(n)\cap T) .$$

Here $A(i)\cup\{q\}$ is a convergent sequence and $q\in U(n)$. Therefore $A(i)-U(n)$ is a finite set for each $i\leqq n-1$. Since $x(n)\notin Cl_X U(n)$, $x(n)\notin Cl_X(W\cap T)$. This shows $\{x(n);n\in\omega_0\}\cap W=\emptyset$ since T is dense in X. Now our proof is completed.

THEOREM 2. *Let X be a proper Lašnev space, i.e. a Lašnev space which is not metrizable. Then X contains a closed set which is a copy of T.*

PROOF. Let $f:M\to X$ be a closed map where M is a metric space. By Morita-Hanai-Stone's theorem [7] there exists a point $p\in X$ such that $\partial f^{-1}(p)$

is not compact. Let $\{q(n); n\in\omega_0\}$ be a discrete set of points in $\partial f^{-1}(p)$ and $\{U(n); n\in\omega_0\}$ a discrete open collection of M with $q(n)\in U(n)$ for each $n\in\omega_0$. Let $Q(n)=\{q(n, m); m\in\omega_0\}$ be a convergent sequence of points in $U(n)-f^{-1}(p)$ whose limit point is $q(n)$. The sequence $\{f(Q(n)); n\in\omega_0\}$ has the following property: For each $k\in\omega_0$ there exists $n\ (>k)$ such that

$$f(Q(n))-\bigcup_{i=1}^{k} f(Q(i)) \quad \text{is infinite.}$$

Assume contrary, i.e. there exists some $k\in\omega_0$ such that

$$f(Q(n))-\bigcup_{i=1}^{k} f(Q(i)) \quad \text{is finite for each } n>k\,.$$

Then

$$f^{-1}(\bigcup_{i=1}^{k} f(Q(i)))\cap Q(n) \quad \text{is infinite for each } n>k\,.$$

Therefore there exists $q(n, m(n))\in f^{-1}(\bigcup_{i=1}^{k} f(Q(i)))\cap Q(n)$ such that $f(q(n, m(n)))\neq f(q(j, m(j)))$ for $n\neq j$. The set $\{q(n, m(n)); n\in\omega_0\}$ is closed in M but $p\in Cl_X\{f(q(n, m(n))); n\in\omega_0\}$, which is a contradiction.

Put

$$L(n)=f(Q(n))-\bigcup_{i=1}^{n-1} f(Q(i))\,.$$

Put

$$n_1=\min\{n>1; f(Q(n))-f(Q(1)) \quad \text{is infinite}\}\,.$$

Then

$$L(n_1)=\{f(Q(n_1))-f(Q(1))\}-\bigcup_{k=2}^{n_1-1}\{f(Q(k))-f(Q(1))\}\,.$$

Since $\bigcup_{k=2}^{n_1-1}\{f(Q(k))-f(Q(1))\}$ is finite, $L(n_1)$ is infinite. Put

$$n_2=\min\{n>n_1; f(Q(n))-\bigcup_{i=1}^{n_1} f(Q(i)) \quad \text{is infinite}\}\,.$$

Continuing in this manner, we obtain a sequence $\{L(n_k); k\in\omega_0\}$ such that $L(n_k)$ is an infinite set for each $k\in\omega_0$ and such that

$$L(n_k)\cap L(n_j)=\emptyset \quad \text{for } k\neq j.$$

Put

$$L=\bigcup\{L(n_k); k\in\omega_0\}\cup\{p\}\,.$$

Note that every point of $L(n_k)$ is isolated in L for each $k\in\omega_0$. Now it is easy to show that the set L is closed and homeomorphic to T. The proof is completed.

COROLLARY 1. *Let X be a proper Lašnev space. Then X cannot be embedded in a countably compact regular spaces with countable tightness.*

PROOF. Suppose X can be embedded in a countably compact regular space Y with countable tightness. Let L be a copy of T contained in X. Then $Cl_Y L$ is a countably compact regular space with countable tightness which contradicts Theorem 1. The proof is completed.

Let N denote the natural numbers. A countable space with one non-isolated point will be denoted by $N\cup\{\mathfrak{G}\}$. Here $\{\mathfrak{G}\}$ is the non-isolated point, and its filter of neighborhoods restricted to N is the elements of $\mathfrak{G}$. We denote by βN the Stone-Čech compactification of N. For a filter $\mathfrak{G}=\{G_\alpha;\ \alpha\in A\}$, we denote $G=\cap\{Cl_{\beta N}G_\alpha;\ \alpha\in A\}$ and say G is the realization of $\mathfrak{G}$. For each $M\subset N$, we denote $M^*=Cl_{\beta N}M-M$.

We recall some information on βN.

LEMMA 1 ([**9**, p. 414]). *A set U is open-closed in N^* if and only if there exists $M\subset N$ for which $U=M^*$.*

LEMMA 2 ([**9**, p. 414]). *$G^*\subset H^*$ if and only if $G-H$ is a finite set, where G and H are subsets of N.*

DEFINITION 2. Let X be a space. A point $x\in X$ is said to be a *P-point* of X, if the intersection of each sequence of neighborhoods of x contains a neighborhood of x.

LEMMA 3 ([**9**, p. 415], CH). *There exist P-points in N^*.*

DEFINITION 3 ([**2**, p. 376]). A space X is said to be an *F-space* if each disjoint two cozero sets of X are completely separated in X.

LEMMA 4 ([**2**, p. 376]). *N^* is an F-space.*

Lemmas 5, 6 and 7 below are well-known and easy to prove, so we omit the proofs.

LEMMA 5. *Let G be a closed subset of N^*. Then there exists a filter $\mathfrak{G}$ on N whose realization is G.*

LEMMA 6. *Let $\mathfrak{G}=\{G_\alpha;\alpha\in A\}$ be a free filter whose realization is G. Then $\{G_\alpha^*;\ \alpha\in A\}$ is a neighborhood base of G in N^*.*

Let $\mathfrak{G}$ be a filter. Then we say that $\mathfrak{G}$ *determines an ultrafilter* if the realization of $\mathfrak{G}$ is a singleton in N^*.

LEMMA 7. *Let $\mathfrak{G}$ be a filter on N. Then the following are equivalent:*

i) *$\mathfrak{G}$ determines an ultrafilter.*

ii) *There exists an ultrafilter $\mathfrak{H}$ such that for each $H\in\mathfrak{H}$ there exists $G\in\mathfrak{G}$ such that $G-H$ is finite.*

DEFINITION 4 ([**3**]). A space X is said to be *Fréchet* if, whenever $x\in Cl_X A$ for some $A\subset X$, there exists a sequence $\{x(n);\ n\in\omega_0\}\subset A$ such that $\lim_{n\to\infty}x(n)=x$.

LEMMA 8 ([**6**, Theorem 1]). *Let $\mathfrak{G}$ be a free filter on N and let $\mathfrak{G}$ be the realization of $\mathfrak{G}$. Then $N\cup\{\mathfrak{G}\}$ is a Fréchet space if and only if $G=Cl_{\beta N}(\mathrm{Int}_{N^*}G)$.*

LEMMA 9 (CH). *Let p be a P-point of N^*. Then there exists a filter $\{V_\alpha;\ \alpha\in\omega_1\}$ on N such that*

i) $V_\alpha^* \subsetneqq V_\beta^*$ *for* $\alpha \gneqq \beta$,

ii) $\{V_\alpha^* ; \alpha \in \omega_1\}$ *is a neighborhood base of* p *in* N^*.

PROOF. Let $\mathfrak{U} = \{U_\alpha ; \alpha \in \omega_1\}$ be the filter on N such that the realization of $\mathfrak{U}$ is p.

Put

$$V_0 = U_0 .$$

Assume $\{V_\beta ; \beta < \alpha\}$ is already constructed as follows:

$$V_\gamma^* \subsetneqq V_\delta^* \quad \text{for any } \delta < \gamma < \alpha,$$

$$V_\gamma^* \subset U_\gamma^* \quad \text{for any } \gamma < \alpha.$$

Since p is a P-point,

$$p \in \operatorname{Int}_{N^*}(\cap \{V_\beta^* ; \beta < \alpha\}) \cap U_\alpha^*.$$

Take $V_\alpha \subset N$ such that

$$p \in V_\alpha^* \subsetneqq U_\alpha^* \cap \operatorname{Int}_{N^*}(\cap \{V_\beta ; \beta < \alpha\}).$$

It is easy to show that $\{V_\alpha ; \alpha \in \omega_1\}$ satisfies the conditions i) and ii). The proof is completed.

LEMMA 10 (CH). *There exist two filters* $\mathfrak{F}$ *and* $\mathfrak{G}$ *such that*

i) $N \cup \{\mathfrak{F}\}$ *and* $N \cup \{\mathfrak{G}\}$ *are Fréchet spaces.*

ii) $\mathfrak{H} = \{F \cap G ; F \in \mathfrak{F}, G \in \mathfrak{G}\}$ *determines the ultrafilter.*

PROOF. Let p be a P-point of N^* and let $\{V_\alpha ; \alpha \in \omega_1\}$ be the filter in Lemma 9.

For any $\alpha \in \omega_1$, we choose $W_{\alpha 1}$ and $W_{\alpha 2}$, subsets of N, such that

$$W_{\alpha 1}^* \neq \emptyset, \ W_{\alpha 2}^* \neq \emptyset,$$

$$W_{\alpha 1}^* \cap W_{\alpha 2}^* = \emptyset,$$

$$W_{\alpha 1}^* \cup W_{\alpha 2}^* \subset V_\alpha^* - V_{\alpha+1}^*.$$

$\{W_{\alpha 1}^* ; \alpha \in \omega_1\}$ and $\{W_{\alpha 2}^* ; \alpha \in \omega_1\}$ have the following properties:

$$Cl_{\beta N}(\cup \{W_{\beta 1}^* ; \beta < \alpha\}) \cap V_\alpha^* = \emptyset, \quad \alpha \in \omega_1, \tag{1}$$

$$Cl_{\beta N}(\cup \{W_{\beta 2}^* ; \beta < \alpha\}) \cap V_\alpha^* = \emptyset, \quad \alpha \in \omega_1, \tag{2}$$

$$p \in Cl_{\beta N}(\cup \{W_{\alpha 1}^* ; \alpha \in \omega_1\}) \cap Cl_{\beta N}(\cup \{W_{\alpha 2}^* ; \alpha \in \omega_1\}). \tag{3}$$

Put

$$F = Cl_{\beta N}(\cup \{W_{\alpha 1}^* ; \alpha \in \omega_1\}), \tag{4}$$

$$G = Cl_{\beta N}(\cup \{W_{\alpha 2}^* ; \alpha \in \omega_1\}). \tag{5}$$

Let $\mathfrak{F} = \{F_\xi ; \xi \in A\}$ and $\mathfrak{G} = \{G_\eta ; \eta \in B\}$ be two filters whose realizations are F

and G, respectively. Then $N\cup\{\mathfrak{F}\}$ and $N\cup\{\mathfrak{G}\}$ are both Fréchet by Lemma 8. We will show that $\mathfrak{H}=\{F_\xi\cap G_\eta;\xi\in A, \eta\in B\}$ determines an ultrafilter. Let $D\in p$ be any element of the ultrafilter p. Then we will show that there exist $F_\xi\in\mathfrak{F}$ and $G_\eta\in\mathfrak{G}$ such that

$$F_\xi\cap G_\eta-D \quad \text{is finite,}$$

$$p\in F_\xi^*\cap G_\eta^*.$$

Since D^* is open in N^* containing p, then there exists $V_\gamma\subset N$ such that

$$p\in V_\gamma^*\subset D^*. \tag{6}$$

$\cup\{W_{\beta 1}^*;\beta<\gamma\}$ and $\cup\{W_{\beta 2}^*;\beta<\gamma\}$ are cozero sets in N^*. Therefore, by Lemma 4,

$$Cl_{\beta N}(\cup\{W_{\beta 1}^*;\beta<\gamma\})\cap Cl_{\beta N}(\cup\{W_{\beta 2}^*;\beta<\gamma\})=\emptyset.$$

By Lemmas 1 and 6, there exist K and L such that

$$Cl_{\beta N}(\cup\{W_{\beta 1}^*;\beta<\gamma\})\subset K^*\subset N^*-V_\gamma^*, \tag{7}$$

$$Cl_{\beta N}(\cup\{W_{\beta 2}^*;\beta<\gamma\})\subset L^*\subset N^*-V_\gamma^*, \tag{8}$$

$$K^*\cap L^*=\emptyset. \tag{9}$$

By (1), (4) and (7),

$$F=Cl_{\beta N}(\cup\{W_{\beta 1}^*;\beta<\gamma\})\cup Cl_{\beta N}(\cup\{W_{\beta 1}^*;\beta\geqq\gamma\})\subset K^*\cup V_\gamma^*.$$

Similarly

$$G\subset L^*\cup V_\gamma^*.$$

By Lemma 6, there exist $F_\xi\in\mathfrak{F}$ and $G_\eta\in\mathfrak{G}$ such that

$$F\subset F_\xi^*\subset K^*\cup V_\gamma^*,$$

$$G\subset G_\eta^*\subset L^*\cup V_\gamma^*.$$

Then, by (9),

$$p\in F\cap G\subset F_\xi^*\cap G_\eta^*\subset V_\gamma^*\subset D^*.$$

Therefore $F_\xi\cap G_\eta-D$ is finite by Lemma 2. The proof is completed.

DEFINITION 5 ([3, p. 109]). Let X be a space. A subset U of X is said to be *sequentially open* if each sequence in X converging to a point in U is eventually in U. X is said to be a *sequential space* if each sequentially open subset of X is open.

LEMMA 11. *Let $\mathfrak{G}$ be an ultrafilter on N. Then $N\cup\mathfrak{G}$ is not subsequential.*

PROOF. Let X be a sequential space such that

$$N\cup\{\mathfrak{G}\}\subset X,\ N\cup\{\mathfrak{G}\} \text{ is dense in } X.$$

$\{\mathfrak{G}\} \in Cl_X(X-(N\cup\{\mathfrak{G}\}))$ implies that there exists a sequence $\{x(n); n\in\omega_0\}$ such that $\lim_{n\to\infty} x(n)=\{\mathfrak{G}\}$. Let $\{U(n); n\in\omega_0\}$ be a sequence of open sets in X such that

$$U(n)\cap U(m)=\emptyset \text{ for } n\neq m,\ x(n)\in U(n) \text{ for each } n\in\omega_0.$$

Put

$$A=\cup\{U(2n)\cap N;\ n=1,2,\cdots\},$$

$$B=\cup\{U(2n+1)\cap N;\ n=0,1,\cdots\}.$$

Then $A\in\mathfrak{G}$ and $B\in\mathfrak{G}$, which is impossible since $A\cap B=\emptyset$. The proof is completed.

THEOREM 3 (CH). *There exist Fréchet spaces X and Y such that $X\times Y$ is not subsequential.*

PROOF. Let p be a P-point of N^*. Let $X=N\cup\{\mathfrak{F}\}$ and $Y=N\cup\{\mathfrak{G}\}$ be Fréchet spaces in Lemma 10. We define $f: N\cup\{p\}\to X\times Y$ such that

$$f(n)=(n,n),$$

$$f(p)=\{\mathfrak{F}\}\times\{\mathfrak{G}\}.$$

Then f is an embedding since

$$f^{-1}((F_\xi\times G_\eta)\cap\Delta)=F_\xi\cap G_\eta,$$

where $\Delta=\{(n,n); n\in N\}$.

Each subspace of subsequential space is subsequential. Therefore Lemma 11 implies that $X\times Y$ is not subsequential. The proof is completed.

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